

## Research Article

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## The Multipartite Property and the Cycle Structure of the Graph Associated with the Elements of Unequal Order in the Group $(\mathbb{Z}_n, \oplus)$

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### ABSTRACT

The first two authors have introduced a new class of graphs  $\Gamma(\mathbb{Z}_n)$  associated with the elements of unequal order in the group  $(\mathbb{Z}_n, \oplus)$  of residue classes modulo  $n \geq 1$ , an integer. It is established that it is a connected graph and the vertex degree and size of the graph  $\Gamma(\mathbb{Z}_n)$  are determined. In this paper, it is shown that  $\Gamma(\mathbb{Z}_n)$  is Hamiltonian if  $n$  is even. Further, it is proved that it is not a complete graph and it is a complete multipartite graph. The formula for the number of triangles in  $\Gamma(\mathbb{Z}_n)$  is determined.

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**Received:** August 05, 2026; **Accepted:** August 11, 2026; **Published:** September 24, 2026

**Keywords:** Complete Graph, Component of a Graph, Multipartite Graph, Hamiltonian Property, Triangle

**AMS Subject Classification:** 05C07, 05C30, 05C40, 20B05, 68R10.

### Introduction

The interplay of algebraic concepts from number theory, abstract algebra and graph theory has been the most creative area of research, leading to a branch of modern mathematics called Algebraic graph theory. Frucht [10] showed that all finite groups can be represented as the automorphism group of a connected algebraic graph. In 1878, Cayley was the pioneer in studying graphs arising out of finite groups, which led to the theory of Cayley graphs.

Nathanson [17] introduced the concept of congruences in number theory into graph theory and initiated the study of number theoretic graphs. In this way many researchers [3, 5, 6, 7, 8, 18] have studied the graphs associated with various concepts in algebra.

For a given positive integer  $n$  and an element  $\bar{a}$  of  $\mathbb{Z}_n$ , the order of  $\bar{a}$  is denoted by  $o(\bar{a})$  and is the least positive integer  $k$  less than or equal to  $n$ , such that  $k\bar{a} = 0$ . In [18], the first two authors introduced the graph  $\Gamma(\mathbb{Z}_n)$ , whose vertex set is  $\mathbb{Z}_n$  and whose edge set is  $E = \{(\bar{a}, \bar{b})/\bar{a}, \bar{b} \in \mathbb{Z}_n \text{ and } o(\bar{a}) \neq o(\bar{b})\}$ .

For an integer  $n \geq 1$  and a divisor  $d$  of  $n$ , let  $D(n) = \{d/d \text{ is a positive integer that divides } n\}$ . Then  $|D(n)| = d(n)$ , the number of divisors of  $n$ , and let

$$D_d = \{\bar{a} \in \mathbb{Z}_n / o(\bar{a}) = d\}.$$

The following results were established in [18].

**Theorem 1.1** (Th 2.2, [18]): For any divisor  $d$  of  $n$ , let  $\bar{l}$  be a positive integer less than  $n$ , such that  $\bar{l} \in D_d$ . Then for any positive integer  $r$  less than  $n$ ,  $\bar{r} \in D_d$  if, and only if,  $(r, d) = 1$ .

**Corollary 1.2** (Cor 2.3, [18]): For any divisor  $d$  of  $n$ ,

$$D_d = \{\bar{r}\bar{l} / \bar{l} \in D_d \text{ and } (r, d) = 1\}.$$

**Corollary 1.3** (Cor 2.4, [18]): For any divisor  $d$  of  $n$ ,  $|D_d| = \varphi(d)$ , where  $\varphi$  is the Euler totient function.

**Theorem 1.4** (Th 2.5, [18]): For the integer  $n > 1$ ,  $\mathbb{Z}_n$  is the disjoint union of  $D_{d_1}, D_{d_2}, \dots, D_{d_{d(n)}}$ , where  $d_1 < d_2 < \dots < d_{d(n)}$  are the distinct divisors of  $n$ .

**Remark 1.5:** For any divisor  $d$  of  $n$ , let  $l_d$  denote the least positive integer less than  $n$  such that  $\bar{l}_d \in D_d$  and let

$$\Phi(d) = \{r/1 \leq r < d \text{ and } (r, d) = 1\}.$$

Clearly  $|\Phi(d)| = \varphi(d)$ , where  $\varphi(d)$  is the Euler Totient function, and

$$D_d = \{r\bar{l}_d : r \in \Phi(d)\}.$$

**Definition 1.6:** Let  $n \geq 1$  be an integer. Then the graph  $\Gamma(Z_n)$  whose vertex set  $V$  is the set  $Z_n$ , and the edge set  $E$  is  $E = \{(\bar{a}, \bar{b})/\bar{a}, \bar{b} \in Z_n \text{ and } o(\bar{a}) \neq o(\bar{b})\}$ .

Clearly,  $\Gamma(Z_n)$  is a simple undirected graph without loops. The graphs  $\Gamma(Z_n)$  for  $n = 9, 11$  and  $12$  are given below.

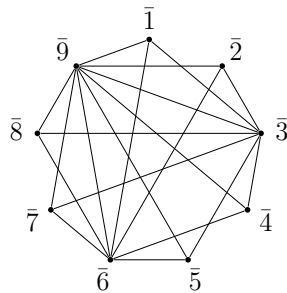


**Figure. 1.1** The graph  $\Gamma(Z_9)$

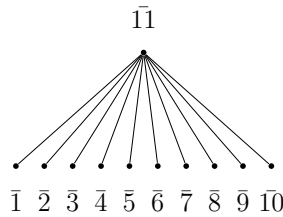


**Figure. 1.2** The graph  $\Gamma(Z_{11})$

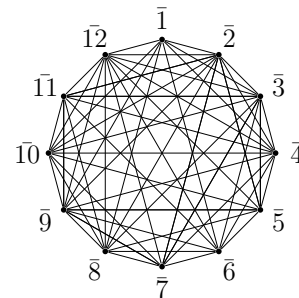


**Figure. 1.3** The graph  $\Gamma(Z_{12})$

### Hamiltonian Property of the Graph $\Gamma(Z_n)$

**Definition 2.1:** A Hamilton cycle in a graph  $G$  is a cycle that contains every vertex exactly once of  $G$ . A graph  $G$  is called Hamiltonian if it contains a Hamilton cycle.

For the primes 9 and 11, the graphs  $\Gamma(Z_9)$  and  $\Gamma(Z_{11})$  are not Hamiltonian as they do not have Hamilton cycles. However, for the even number 12, the graph  $\Gamma(Z_{12})$  is Hamiltonian with Hamilton cycle  $(\bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}, \bar{6}, \bar{7}, \bar{8}, \bar{9}, \bar{10}, \bar{11}, \bar{12}, \bar{1})$ .

In this section, it is established that  $\Gamma(Z_n)$  is Hamiltonian if  $n > 1$  is an even integer.

**Lemma 2.2:** Let  $n \geq 1$  be an integer and let  $r$  be any integer such that  $1 \leq r < n$ . Then  $o(\bar{r}) = n$  if, and only if,  $(r, n) = 1$ .

*Proof.* Let  $n \geq 1$  be an integer and let  $r$  be any integer such that  $1 \leq r < n$ . Consider the finite cyclic group  $\langle a \rangle$  of order  $n$ , generated by  $a$ . That is,

$$\langle a \rangle = \{a^m : 0 \leq m < n \text{ and } a^n = e\}.$$

Then it is easy to see that  $\langle a \rangle \cong (Z_n, \oplus)$ , under the isomorphism  $a^r \mapsto \bar{r}$ ,  $1 \leq r < n$ .

Since  $a^r$  is a generator of the cyclic group  $\langle a \rangle$  if, and only if,  $(r, n) = 1$ , it follows that  $\bar{r}$  is a generator of the group  $(Z_n, \oplus)$  if, and only if,  $(r, n) = 1$ . That is,  $o(\bar{r}) = n$  if, and only if,  $(r, n) = 1$ .

**Theorem 2.3:** If  $n > 1$  is an even integer, then the graph  $\Gamma(Z_n)$  is Hamiltonian. In fact,  $(\bar{1}, \bar{2}, \bar{3}, \bar{4}, \dots, \bar{n}, \bar{1})$  is a Hamilton cycle in  $\Gamma(Z_n)$ .

*Proof.* Let  $n > 1$  be an even integer. For any integer  $r$ ,  $1 \leq r \leq n$ , consider  $\bar{r}$  and  $\overline{r+1}$ .

If possible, assume that

$$\begin{aligned} o(\bar{r}) &= o(\overline{r+1}) = d. \text{ Then } d \text{ is the least positive integer such that} \\ d(\bar{r}) &= \bar{0} \text{ and } d(\overline{r+1}) \\ \Rightarrow d(\bar{r}) &= \bar{0} \text{ and } d(\bar{r}) \\ \Rightarrow d(\bar{r}) &= \bar{0} \text{ and } d(\bar{r}) \\ \Rightarrow d &= \bar{0} \Rightarrow d = n. \end{aligned}$$

Clearly, one of  $r$  and  $r+1$  is even and the other is odd. Suppose that  $r$  is even. Then  $r$  is even and  $n$  is even, so  $(r, n) \geq 2$ . Hence  $(r, n) \neq 1$ . By the Lemma 2.2,  $o(\bar{r}) \neq n$ , and consequently  $d < n$ , which is a contradiction to  $d = n$ .

On the other hand, if  $r$  is odd, then  $r+1$  is even. By a similar argument,  $o(\overline{r+1}) < n$ , which again gives a contradiction to  $d = n$ .

So, our assumption that  $o(\bar{r}) = o(\overline{r+1})$  is wrong, so that  $o(\bar{r}) \neq o(\overline{r+1})$  for  $1 \leq r < n$ . This shows that  $(\bar{r}, \overline{r+1})$ ,  $1 \leq r < n$  is an edge of  $\Gamma(Z_n)$  and  $(\bar{1}, \bar{2}, \bar{3}, \dots, \bar{n}, \bar{1})$  is a cycle of length  $n$  in  $\Gamma(Z_n)$  and which is a Hamilton cycle in  $\Gamma(Z_n)$ , so that  $\Gamma(Z_n)$  is Hamiltonian.

**Example 2.4:** The graph  $\Gamma(Z_n)$  and one of its Hamilton cycles  $(\bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}, \bar{6}, \bar{7}, \bar{8}, \bar{9}, \bar{10}, \bar{11}, \bar{12}, \bar{1})$  are given below.

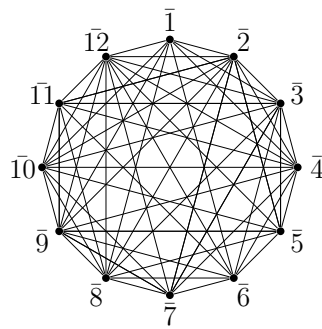


**Figure. 2.1** The graph  $\Gamma(Z_{12})$

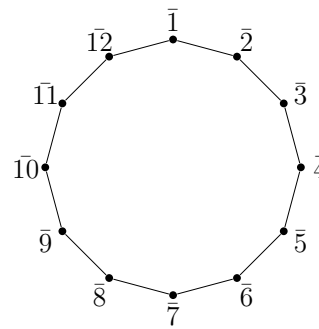


**Figure. 2.2** A Hamilton cycle in the graph  $\Gamma(Z_{12})$

### The Multipartite Property of the Graph $\Gamma(Z_n)$

Using the Theorem 1.4, we now show that  $\Gamma(Z_n)$  is a complete multipartite graph.

**Theorem 3.1:** For  $n > 1$  an integer, the graph  $\Gamma(Z_n)$  is a complete  $d(n)$ -partite graph and  $\Gamma(Z_n) \cong K_{\varphi(d_1), \varphi(d_2), \dots, \varphi(d_{d(n)})}$ , where  $d_1, d_2, \dots, d_{d(n)}$  are the distinct divisors of  $n$ .

*Proof.* Let  $n > 1$  be an integer. By the Theorem 1.4,  $Z_n$  is the disjoint union of the sets  $D_{d_1}, D_{d_2}, \dots, D_{d_{d(n)}}$ , where  $d_1, d_2, \dots, d_{d(n)}$  are all the distinct divisors of  $n$  and  $D_{d_i} = \{\bar{a} \in Z_n / o(\bar{a}) = d_i\}$ .

Suppose  $\bar{a}$  and  $\bar{b}$  are any two distinct vertices of  $\Gamma(Z_n)$ , such that,  $\bar{a}, \bar{b} \in D_{d_i}$  for any  $i, 1 \leq i \leq d(n)$ . Then  $o(\bar{a}) = o(\bar{b}) = d_i$ .

So, there is no edge between  $\bar{a}$  and  $\bar{b}$ , which shows that the graph  $\Gamma(Z_n)$  is **not a complete graph**.

Further, if  $\bar{a}$  and  $\bar{b}$  are any two distinct vertices of  $\Gamma(Z_n)$ , such that  $\bar{a} \in D_{d_i}$  and  $\bar{b} \in D_{d_j}$ , for  $1 \leq i, j \leq d(n)$  and  $i \neq j$ , then  $o(\bar{a}) = d_i \neq d_j = o(\bar{b})$ . So there is an edge between  $\bar{a}$  and  $\bar{b}$ .

These show that  $D_{d_1}, D_{d_2}, \dots, D_{d_{d(n)}}$  are disjoint subsets of the vertex set of the graph  $\Gamma(Z_n)$ , such that, there is no edge between the vertices belonging to  $D_{d_i}$  for  $1 \leq i \leq d(n)$ .

From these, it follows that the partitions  $(D_{d_1}, D_{d_2}, \dots, D_{d_{d(n)}})$  of the graph  $\Gamma(Z_n)$  is a  $d(n)$ -partition of the vertex set of the graph  $\Gamma(Z_n)$  and  $\Gamma(Z_n)$  is a complete  $d(n)$ -partite graph. By the Corollary 1.3,  $|D_{d_i}| = \varphi(d_i)$ , for  $1 \leq i \leq d(n)$ , so that

$$\Gamma(Z_n) \cong K_{\varphi(d_1), \varphi(d_2), \dots, \varphi(d_{d(n)})}.$$

In the following theorem, it is shown that  $\Gamma(Z_n)$  is a complete bipartite graph and a star graph if, and only if,  $n$  is a prime.

**Theorem 3.2:** For  $n > 1$ , an integer, the graph  $\Gamma(Z_n)$  is a complete bipartite graph if, and only if,  $n$  is a prime.

*Proof.* For  $n > 1$ , an integer, assume that  $\Gamma(Z_n)$  is a complete bipartite graph with bipartition  $(A, B)$ . Then  $Z_n = A \cup B$ ,  $A \neq \emptyset$ ,  $B \neq \emptyset$ ,  $A \cap B = \emptyset$ , and no two vertices in  $A$ , as well as, in  $B$ , are adjacent and there is an edge between every vertex in  $A$  and every vertex in  $B$ .

Since  $o(\bar{0}) = 1$  and  $o(\bar{1}) = n$ , one of  $\bar{0}$  and  $\bar{1}$  belongs to  $A$  and the other to  $B$ , say,  $\bar{0} \in A$  and  $\bar{1} \in B$ .

Further, for any  $\bar{a} \in Z_n$ ,  $\bar{a} \neq \bar{0}$ , we have  $o(\bar{a}) > 1$ . We claim that  $\bar{a} \notin A$ . For if  $\bar{a} \in A$ , then by the nature of  $A$ ,  $o(\bar{a}) = o(\bar{0}) = 1$ , a contradiction. So  $\bar{a} \notin A$ , for all  $\bar{a} \in Z_n$  and  $\bar{a} \neq \bar{0}$ , so that  $\bar{a} \in B$ , for all  $\bar{a} \neq \bar{0}$ . From this it follows that  $A = \{\bar{0}\}$  and  $B = \{\bar{1}, \bar{2}, \dots, \overline{n-1}\}$ .

Since  $o(\bar{1}) = n$ , it follows that  $o(\bar{a}) = n$ , for all  $\bar{a} \in B$ . That is,  $o(\bar{1}) = o(\bar{2}) = \dots = o(\overline{n-1}) = n$ , so that  $n$  is a prime number.

Conversely, assume that  $n$  is a prime number. Then  $n$  has only two divisors, namely, 1 and  $n$ . But  $o(\bar{0}) = 1$  implies that  $D_1 = \{\bar{0}\}$  and  $o(\bar{r}) = n$ , for  $1 \leq r \leq n-1$ , implies that  $D_n = \{\bar{1}, \bar{2}, \dots, \overline{n-1}\}$ . Further,

$$Z_n = D_1 \cup D_n, \text{ (disjoint union).}$$

So, by the Theorem 3.1,  $\Gamma(Z_n)$  is a complete bipartite graph with bipartition  $(D_1, D_n)$ .

**Corollary 3.3:** For  $n > 1$ ,  $\Gamma(Z_n) \cong K_{1,n-1}$  and hence is a star graph if, and only if,  $n$  is a prime.

**Proof.** Let  $n > 1$  be an integer. By the Theorem 3.2,  $\Gamma(Z_n)$  is a complete bipartite graph with bipartition  $(D_1, D_n)$  if, and only if,  $n$  is a prime.

Since  $|D_1| = \varphi(1)$  and  $|D_n| = \varphi(n) = n - 1$ , by the Theorem 3.1, it follows that  $\Gamma(Z_n) \cong K_{1,n-1}$ , if, and only if,  $n$  is a prime.

Further,  $K_{1,n-1}$  is a star graph, which implies that  $\Gamma(Z_n)$  is a star graph if, and only if,  $n$  is a prime.

The following corollary is immediate from the Corollary 3.3.

**Corollary 3.4:** If  $n$  is a prime, then  $\Gamma(Z_n)$  is acyclic and hence it is a tree.

**Example 3.5:** The graph  $\Gamma(Z_8)$  as a 4-partite complete graph with 4-partition  $(D_1, D_2, D_4, D_8)$ , where  $D_1 = \{\bar{8}\}$ ,  $D_2 = \{\bar{4}\}$ ,  $D_4 = \{\bar{2}, \bar{6}\}$ , and  $D_8 = \{\bar{1}, \bar{3}, \bar{5}, \bar{7}\}$ . The graph  $\Gamma(Z_8)$  as a 4-partite complete graph is shown below.

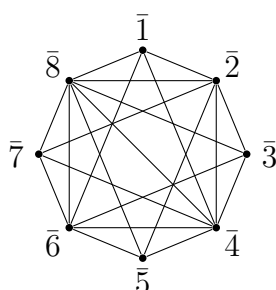


**Figure. 3.1** The graph  $\Gamma(Z_8)$

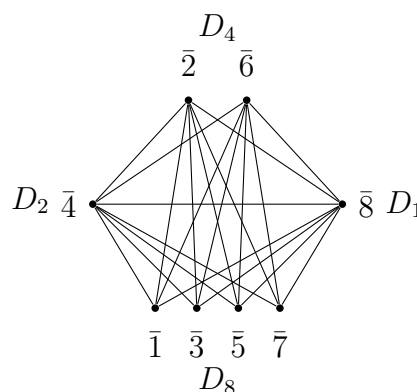


**Figure. 3.2** The graph  $\Gamma(Z_8)$  as a 4-partite graph

**Remark:** Observer that 8 is an even number.

**Example 3.6:** The graph  $\Gamma(Z_{11})$  is a complete bipartite graph with bipartition  $(D_1, D_{11})$ , where  $D_1 = \{\bar{1}\}$  and  $D_{11} = \{\bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}, \bar{6}, \bar{7}, \bar{8}, \bar{9}, \bar{10}\}$ .

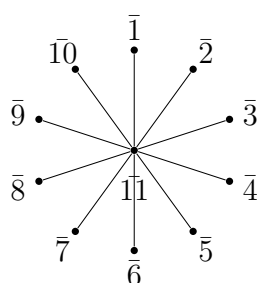


**Figure. 3.3** The graph  $\Gamma(Z_{11})$

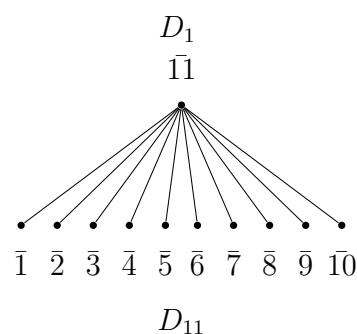


**Figure. 3.4** The graph  $\Gamma(Z_{11})$  as a 2-partite graph

**Remark:** Observer that 11 is a prime number.

### Enumeration of Triangles in $\Gamma(Z_n)$

In this section, it is established that the graph  $\Gamma(Z_n)$  has no triangles if  $n$  is a prime and has triangles if  $n$  is a composite number. Further, the number of distinct triangles is determined if  $n$  is a composite number.

**Theorem 4.1:** If  $n > 1$  is a prime number, then the graph  $\Gamma(Z_n)$  has no triangles.

*Proof.* Let  $n > 1$  be a prime. Then, by the Corollary 3.3, the graph  $\Gamma(Z_n)$  is isomorphic to the complete bipartite graph  $K_{1,n-1}$  and this is a tree. Therefore,  $\Gamma(Z_n)$  has no cycles and hence has no triangles.

**Theorem 4.2:** If  $n \geq 4$  is a composite number, then the graph  $\Gamma(Z_n)$  has triangles, and the number of distinct triangles in  $\Gamma(Z_n)$  is equal to

$$\sum_{\substack{i,j,k=1 \\ i \neq j, j \neq k, k \neq i}}^{d(n)} |D_{d_i}| |D_{d_j}| |D_{d_k}| = \sum_{\substack{i,j,k=1 \\ i \neq j, j \neq k, k \neq i}}^{d(n)} \varphi(d_i) \varphi(d_j) \varphi(d_k),$$

where  $d_1, d_2, \dots, d_{d(n)}$  are the distinct divisors of  $n$  and  $D_{d_i} = \{\bar{a} \in Z_n / o(\bar{a}) = d_i\}$ .

*Proof.* Let  $n \geq 4$  be a composite number. Then  $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_k^{\alpha_k}$ , where  $p_1, p_2, \dots, p_k$  are distinct primes,  $k > 1$ , and  $\alpha_i \geq 1$ ,  $1 \leq i \leq k$  are integers.

Consider the vertices  $\bar{1}, \bar{p}_i, \bar{n}$ , where  $1 \leq i \leq k$ . We have  $o(\bar{1}) = n$ ,  $o(\bar{p}_i) = \frac{n}{p_i}$ , and  $o(\bar{n}) = 1$ . Since  $n \neq \frac{n}{p_i}$ , it follows that there is an edge between  $\bar{1}$  and  $\bar{p}_i$ . Similarly, since  $\frac{n}{p_i} \neq 1$  and  $1 \neq n$ , there are edges between  $\bar{p}_i$  and  $\bar{n}$  and between  $\bar{n}$  and  $\bar{1}$ , so that  $(\bar{1}, \bar{p}_i, \bar{n})$  is a 3-cycle and hence a triangle in the graph  $\Gamma(Z_n)$ .

By the Theorem 1.4,  $Z_n$  is the disjoint union of  $D_{d_1}, D_{d_2}, \dots, D_{d_{d(n)}}$  or  $Z_n = \bigcup_{i=1}^{d(n)} D_{d_i}$ , where  $d_1, d_2, \dots, d_{d(n)}$  are the distinct divisors of  $n$  and  $D_{d_i} = \{\bar{a} \in Z_n : o(\bar{a}) = d_i\}$ .

Further, by the Theorem 3.1, the graph  $\Gamma(Z_n)$  is a complete  $d(n)$ -partite graph with  $d(n)$ -partition  $(D_{d_1}, D_{d_2}, \dots, D_{d_{d(n)}})$ . That is, there is no edge between any two vertices in  $D_{d_i}$  and there is an edge between any vertex of  $D_{d_i}$  and any vertex of  $D_{d_j}$ , for  $i, j = 1, 2, \dots, d(n)$  and  $i \neq j$ .

So, a vertex  $v_i \in D_{d_i}$ , a vertex  $v_j \in D_{d_j}$  and a vertex  $v_k \in D_{d_k}$  give a triangle  $(v_i, v_j, v_k)$  for  $1 \leq i < j < k \leq d(n)$  and  $i \neq j, j \neq k, k \neq i$ .

So apparently there are  $\sum_{\substack{i,j,k=1 \\ i \neq j, j \neq k, k \neq i}}^{d(n)} |D_{d_i}| |D_{d_j}| |D_{d_k}|$  number of triangles in the graph  $\Gamma(Z_n)$ .

By the Corollary 1.3,  $|D_{d_i}| = \varphi(d_i)$ , for  $1 \leq i \leq d(n)$ . So apparently there are

$$\sum_{\substack{i,j,k=1 \\ i \neq j, j \neq k, k \neq i}}^{d(n)} \varphi(d_i) \varphi(d_j) \varphi(d_k) \text{ --- (1)}$$

number of triangles in the graph  $\Gamma(Z_n)$ .

However, for  $1 \leq i, j, k \leq d(n)$ ,  $i \neq j, j \neq k, k \neq i$ , and for the vertices  $v_i \in D_{d_i}, v_j \in D_{d_j}, v_k \in D_{d_k}$ , the triads

$(v_i, v_j, v_k), (v_i, v_k, v_j), (v_j, v_i, v_k), (v_j, v_k, v_i), (v_k, v_i, v_j), (v_k, v_j, v_i)$  represent the same triangle in the graph  $\Gamma(Z_n)$ .

So, to obtain distinct triangles, it is enough to take  $1 \leq i < j < k \leq d(n)$  in summation (1). So, the number of distinct triangles in the graph  $\Gamma(Z_n)$  is

$$\sum_{\substack{i,j,k=1 \\ i < j < k}}^{d(n)} \varphi(d_i) \varphi(d_j) \varphi(d_k).$$

**Example 4.3:** Consider the graph  $\Gamma(Z_8)$ . Here  $Z_8 = \{\bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}, \bar{6}, \bar{7}, \bar{8}\}$ .

Here  $n = 8$  and the set of divisors of 8 is 1, 2, 4, 8. So  $d_1 = 1$ ,  $d_2 = 2$ ,  $d_3 = 4$ , and  $d_4 = 8$ . We have

$$\varphi(d_1) = \varphi(1) = 1, \varphi(d_2) = \varphi(2) = 1, \varphi(d_3) = \varphi(4) = 2, \text{ and } \varphi(d_4) = \varphi(8) = 4.$$

Thus, the total number of triangles in the graph  $\Gamma(Z_8)$  is equal to

$$\begin{aligned} & \sum_{\substack{i,j,k=1 \\ i < j < k}}^{d(n)} \varphi(d_i) \varphi(d_j) \varphi(d_k) \\ &= \varphi(d_1) \varphi(d_2) \varphi(d_3) + \varphi(d_1) \varphi(d_2) \varphi(d_4) + \varphi(d_1) \varphi(d_3) \varphi(d_4) + \varphi(d_2) \varphi(d_3) \varphi(d_4) \\ &= (1)(1)(2) + (1)(1)(4) + (1)(2)(4) + (1)(2)(4) = 2 + 4 + 8 + 8 = 22. \end{aligned}$$

From the Figure 3.2, one can see that the following 22 triads of vertices give the 22 distinct triangles in the graph  $\Gamma(Z_8)$ .

(1, 2, 4), (1, 2, 8), (1, 4, 6), (1, 4, 8), (1, 6, 8);  
(2, 3, 4), (2, 3, 8), (2, 4, 5), (2, 4, 7), (2, 4, 8), (2, 5, 8), (2, 7, 8);

(3, 4, 6), (3, 4, 8), (3, 6, 8);  
 (4, 5, 6), (4, 5, 8), (4, 6, 7), (4, 6, 8), (4, 7, 8);  
 (5, 6, 8), (6, 7, 8).

## References

1. Apostol TM (1976) Introduction to Analytic Number Theory. Undergraduate Texts in Mathematics, Springer.
2. Bondy JA, Murty USR (2008) Graph Theory. Springer-Verlag London.
3. Finch EC, Jones L (2002) A curious connection between Fermat numbers and finite groups. The American Mathematical Monthly 109:517-524.
4. Cayley A (1889) On the theory of groups. American Journal of Mathematics 11:139-157.
5. Chalapathi T, Madhavi L, Venkataramana S (2013) Enumeration of triangles in a divisor Cayley graph. Momona Ethiopian Journal of Science 5:163-173.
6. Chalapathi T, Kiran Kumar RVMS (2015) The structural properties of even free graphs. International Journal of Computer Applications 121:19-21.
7. Devendra J, Madhavi L, Nagalakshumma T (2022) Enumeration of triangles and Hamiltonian property of the zero-divisor Cayley graph of the ring  $(\mathbb{Z}_n, \oplus, \odot)$ . European Journal of Mathematics and Statistics 3.
8. Farrokhi DG M, Saeedi F (2014) On the number of elements of a given order. The Extended Abstract of the 6th International Group Theory Conference, Golestan University, Gorgan, Iran:73-75.
9. Freud R, Palfy PP (1996) On the possible number of elements of given order in a finite group. Israel Journal of Mathematics 93:345-358.
10. Frucht R (1949) Graphs of degree three with a given abstract group. Canadian Journal of Mathematics 1:365-378.
11. Herstein IN (1975) Topics in Algebra. 2nd edn, University of Chicago, John Wiley & Sons.
12. Herzog M (1977) Counting group elements of order  $p$  modulo  $p^2$ . Proceedings of the American Mathematical Society 62:47-50.
13. Kumar Das A (2009) On finite groups having perfect order subsets. International Journal of Algebra 3:629-637.
14. Lanski C (2010) Concepts in Abstract Algebra. American Mathematical Society.
15. Madhavi L (2003) Studies on Domination Parameters and Enumeration of Cycles in Some Arithmetic Graphs. PhD Thesis, Sri Venkateswara University, India.
16. Nagalakshumma T, Devendra J, Madhavi L (2019) The nilpotent Cayley graph of the residue class ring  $(\mathbb{Z}_n, \oplus, \odot)$ . Journal of Computer and Mathematical Sciences 10:1244-1252.
17. Nathanson MB (1980) Components of arithmetic graphs. Monatshefte für Mathematik 29: 219-220.
18. Shaik Shahida, Madhavi L (2026) Graphs associated with the elements of unequal order in the group  $(\mathbb{Z}_n, \oplus)$ . Malaya Journal of Mathematik 14:243-251.

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