

The Closest Known Attempted Proof of the Twin Prime Conjecture

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ABSTRACT

In Part 1, using a primitive formulation of the circle method, we present a sufficiency condition for the twin prime conjecture that misses by just an epsilon. In Part 2, using the standard formulation of the circle method for the twin prime conjecture, we show that the Hardy-Littlewood minor arc sufficiency condition implies the m -prime conjecture for every even integer m . We show exactly where the proof breaks down, when m is an odd integer.

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Introduction

Part 1

We employ arguably the most primitive formulation possible of the circle method applied to the twin prime conjecture to expose its fundamental components and concepts.

We then present a sufficiency condition, from which the twin prime conjecture follows from our formulation of the circle method.

We then show that sufficiency condition is false.

Let

$$g(x, v) = \sum_{2 \leq m \leq v} \frac{\epsilon(mx)}{\log m} (v \geq 2), g(x, v) = 0 (v < 2);$$

$$l_s[v] = \sum_{2 \leq m \leq v} \frac{1}{\log m} = g(0, v) \sim \left(\frac{v}{\log v} \right), l_s[0] = l_s[1] = 0;$$

$$x_0 = \left(\frac{\log^{2+\epsilon} n}{n} \right) \quad \text{Part 1} \quad x_0 = \left(\frac{\log^{15} n}{n} \right) \quad \text{Part 2}$$

$$f(x, v) = \sum_{p \leq v} \epsilon(px), \text{ where } \epsilon(z) = e^{2\pi iz} (v \geq 0);$$

$$r(n, m) = \text{number of } (p_i, p_j), p_i \leq n, p_j \leq n, p_i - p_j = m.$$

We assume the sufficiency condition:

There exist infinitely many n such that for each such n for all m such that

$$\frac{n}{\log^{2+\epsilon} n} \leq m \leq n \quad \text{we have}$$

$$r(n, m) \leq \frac{n}{\log^{2+\epsilon} n}$$

From the sufficiency condition and the assumption that the twin prime conjecture is false we are able to prove for n in that infinite sequence, $I_4(n) = o\left(\frac{n}{\log^2 n}\right)$ (cf. page 8), which, from our

primitive formulation of the circle method construction, immediately implies a contradiction. So the twin prime conjecture must be true.

Counterexample to the Sufficiency Condition

Assume $n \geq N_0$. By the PNT the number of primes in $\left[\frac{n}{2}, n\right]$ is asymptotic to $\frac{n}{2 \log n}$ and the number of primes in $\left[\frac{n}{8}, \frac{n}{4}\right]$ is asymptotic to $\frac{n}{8 \log n}$.

$$\text{Let } S = \left\{ m = p_i - p_j \mid \frac{n}{2} \leq p_i \leq n, \frac{n}{8} \leq p_j \leq \frac{n}{4} \right\}$$

$|S|$ is (asymptotically) equal to $\frac{n^2}{16 \log^2 n}$ and $\frac{n}{4} \leq m \leq \frac{7n}{8}$.

But by the assumption we have $|S| \leq \frac{7n^2}{8 \log^{2+\epsilon} n}$.

Remark: There exists unconditionally an infinite number of

$$r(n, m) \geq \frac{n}{\log^{2+\epsilon} n}.$$

So our sufficiency condition misses by just an epsilon. We believe, after a very careful search of the literature, that we have produced the closest known attempted proof of the twin prime conjecture.

Previously, the two most important attempted proofs of the twin prime conjecture are due to Chen Junrun (1966) and Roger Heath-Brown (1983)

Chen Junrun: Define Chen primes: $p+2$ is either a prime or a product of two primes. There are infinitely many Chen primes.

Roger Heath-Brown: At least one of the following two statements are true:

(Twin Prime conjecture) There are infinitely many primes p such that $p+2$ is also prime.

(No Siegel zeros) There exists a constant $c > 0$ such that for every real Dirichlet character χ of conductor $q > 1$, the associated L -function $s \rightarrow L(s, \chi)$ has no zeros in the interval

$$\left[1 - \frac{c}{\log q}, 1 \right].$$

These results involve very deep and very complicated mathematical concepts and arguments. In very sharp contrast we obtain a truly remarkable attempted proof by a very elementary construction and mathematical argument. We believe our result is closer to twin primes than theirs is.

Part 2

In 1921 Hardy and Littlewood stated, under the assumption of GRH, if one could establish the estimate $o\left(\frac{n}{\log^2 n}\right)$ on the minor

arc, the twin prime conjecture would follow. In [7] we removed the assumption of GRH. In Part 2 we show that the $o\left(\frac{n}{\log^2 n}\right)$ estimate on the minor arc implies the m -prime conjecture for even integer m .

We now address the result in Part 1.

Theorem 1 The sufficiency condition and assumption that the twin prime conjecture is false are not comparable.

Proof

We assume $n > C_0$.

Clearly,

$$\begin{aligned} r(n, 2) &= \int_{x_0}^{1+x_0} |f(x, n)|^2 \epsilon(-2x) dx \\ &= \int_{x_0}^{1-x_0} |f(x, n)|^2 \epsilon(-2x) dx + \int_{1-x_0}^{1+x_0} |f(x, n)|^2 \epsilon(-2x) dx \\ &= I(\alpha) + I(\beta) \end{aligned}$$

Estimate of $I(\beta)$

Basic Theorem [3] Let $m \geq 3$, $k \leq \log^u n$ and $(k, l) = 1$

$$\left| \pi(m, k, l) - \frac{l_s m}{\phi(k)} \right| \leq C m \exp\left(\frac{-\log^{1/2} n}{200}\right)$$

This is proved by standard analytic number theory techniques. (cf. Theorem 55 in [3])

Corollary 1

$$\left| \pi([v]; q, l) - \frac{\text{ls}[v]}{\phi(q)} \right| < n \log^{-100} n \text{ for } (0 \leq v \leq n), (q, n) = 1, n > C_0.$$

This follows immediately from the Basic Theorem by considering case 1 $0 \leq v \leq n^{1/2}$ and case 2 $n^{1/2} < v \leq n$.

Lemma 1

$$\sum_{\substack{0 < l \leq q \\ (l, q) = 1 \\ (h, q) = 1}} e\left(\frac{lh}{q}\right) = \mu(q).$$

By Theorem 272 in [5] we have

$$\sum_{\substack{0 < l \leq q \\ (l, q) = 1}} e\left(\frac{lh}{q}\right) = \frac{u(N)\phi(q)}{\phi(N)} \text{ where } q = aN, a = (h, q).$$

But $(h, q) = 1$

Lemma 2 [3]

$$\begin{aligned} f(x_1 + x_2, v) &= e(vx_2)f(x_1, v) - 2\pi i x_2 \int_0^v e(ux_2)f(x_1, u) du. \\ e(vx_2) - e(px_2) &= \int_p^v 2\pi i x_2 e(ux_2) du \\ f(x_1 + x_2, v) &= \sum_{p \leq v} e(px_1)e(px_2) \\ &= \sum_{p \leq v} \left\{ e(vx_2) - 2\pi i x_2 \int_p^v e(ux_2) du \right\} \\ &= e(vx_2) \sum_{p \leq v} e(px_1) - 2\pi i x_2 \int_0^v e(ux_2) \sum_{p \leq u} e(px_1) du \\ &= e(vx_2)f(x_1, v) - 2\pi i x_2 \int_0^u e(ux_2)f(x_1, u) du. \end{aligned}$$

Lemma 3 [3]

$$g(x_1 + x_2, v) = e(vx_2)g(x_1, v) - 2\pi i x_2 \int_0^v e(ux_2)g(x_1, u) du.$$

This can be proved in the same way as the proof of Lemma 2.

Lemma 4 [3]

$$f\left(\frac{h}{q} + y, n\right) = e(ny)f\left(\frac{h}{q}, n\right) - 2\pi i y \int_0^n e(uy)f\left(\frac{h}{q}, u\right) du.$$

This follows from Lemma 2 with $x_1 = \frac{h}{q}$, $x_2 = y$, $v = n$.

Lemma 5 [3]

$$g(y, n) = e(ny)g(0, n) - 2\pi i y \int_0^n e(uy)g(0, u) du.$$

This follows from Lemma 3 with $x_1 = 0$, $x_2 = y$, $v = 0$.

Theorem 2 Let [3]

$$\begin{aligned} q &\leq \log^{15} n, \\ |y| &\leq x_0, \\ (q, n) &= 1. \end{aligned}$$

$$\text{Then } \left| f\left(\frac{h}{q} + y, n\right) - \frac{\mu(q)}{\phi(q)} g(y, n) \right| \leq \log^{-80+\epsilon} n.$$

Proof.

$$\left| f\left(\frac{h}{q}, v\right) - \sum_{\substack{p \leq v \\ p \nmid q}} e\left(\frac{ph}{q}\right) \right| \leq \sum_{p \nmid q} 1 < q.$$

Clearly,

$$\sum_{\substack{p \leq v \\ p \nmid q}} e\left(\frac{ph}{q}\right) = \sum_{\substack{0 < l \leq q \\ (l, q) = 1}} e\left(\frac{lh}{q}\right) \sum_{\substack{p \leq v \\ p \equiv l \pmod{q}}} 1 = \sum_{\substack{0 < l \leq q \\ (l, q) = 1}} e\left(\frac{lh}{q}\right) \pi([v]; q, l).$$

Hence by Lemma 1 and Corollary 1 we have

$$\begin{aligned}
 \left| f\left(\frac{h}{q}, v\right) - \frac{\mu(q)}{\phi(q)} g(0, v) \right| &< q + \left| \sum_{\substack{p \leq v \\ p \nmid q}} e\left(\frac{ph}{q}\right) - \frac{\mu(q)}{\phi(q)} \text{ls}[v] \right| \\
 &= q + \left| \sum_{\substack{0 < l \leq q \\ (l, q) = 1}} e\left(\frac{lh}{q}\right) \left\{ \pi([v]; q, l) - \frac{\text{ls}[v]}{\phi(q)} \right\} \right| \\
 &= q + qn \log^{-100} n < 2n \log^{-85} n \quad (0 \leq v \leq n).
 \end{aligned}$$

$$\begin{aligned}
 \left| f\left(\frac{h}{q} + y, n\right) - \frac{\mu(q)}{\phi(q)} g(y, n) \right| &= \left| e(ny) \left\{ f\left(\frac{h}{q}, n\right) - \frac{\mu(q)}{\phi(q)} g(0, n) \right\} \right. \\
 &\quad \left. - 2\pi i y \int_0^n e(vy) \left\{ f\left(\frac{h}{q}, v\right) - \frac{\mu(q)}{\phi(q)} g(0, v) \right\} dv \right| \\
 &\leq \left| f\left(\frac{h}{q}, n\right) - \frac{\mu(q)}{\phi(q)} g(0, n) \right| + 2\pi x_0 \int_0^n \left| f\left(\frac{h}{q}, v\right) - \frac{\mu(q)}{\phi(q)} g(0, v) \right| dv \\
 &\leq 2n \log^{-85} n (1 + 2\pi x_0 n) < n \log^{-80+\epsilon} n,
 \end{aligned}$$

By Theorem 2 we have for $|y| \leq x_0$

$$|f(1 + y, n) - g(y, n)| \leq (n \log^{-80+\epsilon} n)$$

By the trivial inequalities $|f(x, n)| \leq n$ and $|g(y, n)| \leq n$ and the fact that if $|a| \leq n$ and $|b| \leq n$, then

$|a^2 - b^2| \leq 2n|a - b|$ with $a = f(1 + y, n)$ and $b = g(y, n)$ and $b = g(y, n)$ we have for $|y| \leq x_0$

$$|f^2(1 + y, n) - g^2(y, n)| < C_2 n^2 \log^{-80+\epsilon} n.$$

Immediately we have

$$(1 + y, n)^2 - |g(y, n)|^2 \leq |f^2(1 + y, n) - g^2(y, n)| \leq C_2 n^2 \log^{-80+\epsilon} n.$$

By the change of variable $y = (x - 1)$ we have

$$I(\beta) = \int_{-x_0}^{x_0} |f(1 + y, n)|^2 \epsilon(-2y) dy;$$

so that

$$\begin{aligned}
 \left| \int_{-x_0}^{x_0} |f(1 + y, n)|^2 \epsilon(-2y) dy - \int_{-x_0}^{x_0} |g(y, n)|^2 \epsilon(-2y) dy \right| \\
 \leq \int_{-x_0}^{x_0} |f^2(1 + y, n) - g^2(y, n)| dy < C_6 n \log^{-78} n.
 \end{aligned}$$

Let

$$T_1(n) = \int_{-x_0}^{x_0} |g(y, n)|^2 \epsilon(-my) dy.$$

Then

$$|I(\beta) - T_1(n)| \leq C_6 n \log^{-78} n.$$

Let

$$T(n) = \sum_{\substack{m_1 m_2 \\ m_1 \geq 2, m_2 \geq 2 \text{ and } m_2 - m_1 = 2 \\ m_1 \leq n, m_2 \leq n}} \log^{-1} m_1 \log^{-1} m_2.$$

Clearly,

$$T(n) = \int_{-\frac{1}{2}}^{\frac{1}{2}} |g(y, n)|^2 \epsilon(-2y) dy.$$

Also, the number of terms on the right hand side is $(n-3)$ and each term is greater than $\log^{-1} n$ and less than 1 so that

$$(n-3) \log^{-2} n < T(n) < n$$

$$\left| \sum_{m=2}^{m_1} e(my) \right| \leq \frac{1}{|\sin \pi y|} \leq \frac{1}{2y} \quad \left(m_1 \geq 2, 0 < |y| \leq \frac{1}{2} \right)$$

Hence by definition of $g(y, n)$ and Abel's lemma we have

$$|g(y, n)| < |y|^{-1} \quad \left(0 < |y| < \frac{1}{2} \right);$$

so that

$$|T(n) - T_1(n)| \leq 2 \int_{x_0}^{\frac{1}{2}} y^{-2} dy \leq C_4 n \log^{-2+\epsilon} n.$$

Hence

$$|I(\beta) - T(n)| \leq C_6 n \log^{-96+2\epsilon} n + C_4 n \log^{-(2+\epsilon)} n.$$

$$\text{So } I(\beta) = T(n) + o\left(\frac{n}{\log^2 n}\right) \quad (1)$$

Estimate of $I(\alpha)$

Clearly,

$$I(\alpha) = \int_{x_0}^{1-x_0} |f(x, n)|^2 \cos 4\pi x dx$$

$$|f(x, n)|^2 = \pi(n) + 2 \sum_{\substack{2 \leq m \leq n \\ p_i \leq n, p_j \leq n \\ m = p_i - p_j}} r(n, m) \cos(4\pi m x)$$

Hence,

$$\begin{aligned} \int_{x_0}^{1-x_0} |f(x, n)|^2 \cos(4\pi x) dx &= \int_{x_0}^{1-x_0} \pi(n) \cos(4\pi x) dx + \int_{x_0}^{1-x_0} r(n, 2) \cos^2(4\pi x) dx \\ &\quad + 2 \sum r(n, m) \int_{x_0}^{1-x_0} \cos(2\pi m x) \cos(4\pi x) dx \\ &\quad \quad \quad 2 < m < \frac{n}{\log^{4+\epsilon} n} \\ &\quad + 2 \sum r(n, m) \int_{x_0}^{1-x_0} \cos(2\pi m x) \cos(4\pi x) dx \\ &\quad \quad \quad \frac{n}{\log^{4+\epsilon} n} \leq m \leq n \\ &= I_1(n) + I_2(n) + I_3(n) + I_4(n). \end{aligned}$$

Lemma 6.

$$\frac{\sin \theta}{\theta} < \frac{2 + \cos \theta}{3} \quad (0 < \theta \leq 2\pi) \quad [6, \text{ p. } 107].$$

Lemma 7. If $m > 2$

$$2 \int_{x_0}^{1-x_0} \cos(2\pi m x) \cos(4\pi x) dx = \frac{-\sin(2\pi(m-2)x_0)}{\pi(m-2)} - \frac{\sin(2\pi(m+2)x_0)}{\pi(m+2)}$$

Proof.

If $m > 2$

$$\begin{aligned}
 2 \int_{x_0}^{1-x_0} \cos(2\pi mx) \cos(4\pi x) dx &= \frac{\sin(2\pi(m-2)x)}{2\pi(m-2)} + \frac{\sin(2\pi(m+2)x)}{2\pi(m+2)} \Big|_{x_0}^{1-x_0} \\
 &= \frac{\sin(2\pi(m-2)(1-x_0))}{2\pi(m-2)} + \frac{\sin(2\pi(m+2)(1-x_0))}{2\pi(m+2)} \\
 &\quad - \left(\frac{\sin(2\pi(m-2)x_0)}{2\pi(m-2)} + \frac{\sin(2\pi(m+2)x_0)}{2\pi(m+2)} \right) \\
 &= \frac{\sin(2\pi(m-2)(-x_0))}{2\pi(m-2)} + \frac{\sin(2\pi(m+2)(-x_0))}{2\pi(m+2)} \\
 &\quad - \left(\frac{\sin(2\pi(m-2)x_0)}{2\pi(m-2)} + \frac{\sin(2\pi(m+2)x_0)}{2\pi(m+2)} \right) \\
 &= -\frac{\sin(2\pi(m-2)x_0)}{2\pi(m-2)} - \frac{\sin(2\pi(m+2)x_0)}{2\pi(m+2)} \\
 &\quad - \frac{\sin(2\pi(m-2)x_0)}{2\pi(m-2)} - \frac{\sin(2\pi(m+2)x_0)}{2\pi(m+2)} \\
 &= -\frac{\sin(2\pi(m-2)x_0)}{\pi(m-2)} - \frac{\sin(2\pi(m+2)x_0)}{\pi(m+2)}
 \end{aligned}$$

To show $I_1(n) = o\left(\frac{n}{\log^2 n}\right)$.

$$\int_{x_0}^{1-x_0} \pi(n) \cos(4\pi x) dx = \frac{\pi(n)}{4\pi} \sin(4\pi x) \Big|_{x_0}^{1-x_0} = -\frac{\pi(n)}{2\pi} \sin(4\pi x_0).$$

Now apply Lemma 6.

To show $I_2(n) = o\left(\frac{n}{\log^2 n}\right)$.

This is immediate by Statement 1.

To show $I_3(n) = o\left(\frac{n}{\log^2 n}\right)$.

This follows immediately from Lemma 6, Lemma 7 and the facts that

$$2\pi(m-2)x_0 \leq 2\pi \leftrightarrow m \leq x_0^{-1} + 2 \text{ and } 2\pi(m+2)x_0 \leq 2\pi \leftrightarrow m \leq x_0^{-1} - 2,$$

$$2 < m \leq \frac{n}{\log^{4+\epsilon} n}, r(n, m) \leq (1-\epsilon) \frac{n}{\log n} \text{ and } x_0 = \frac{\log^{2+\epsilon} n}{n}.$$

To show $I_4(n) = o\left(\frac{n}{\log^2 n}\right)$ for n in $a(n)$.

This follows immediately from the sufficiency condition and the facts that

$$\begin{aligned}
 \frac{n}{\log^{4+\epsilon} n} \leq m \leq n, \left| 2 \int_{x_0}^{1-x_0} \cos(2\pi mx) \cos(4\pi x) dx \right| &\leq \frac{1}{\pi} \left(\frac{1}{m-2} + \frac{1}{m+2} \right) \text{ by Lemma 7,} \\
 \sum_{\frac{n}{\log^{4+\epsilon} n}}^n \frac{1}{m} &\leq \log n - \left[\log \frac{n}{\log^{4+\epsilon} n} \right] = (4+\epsilon) \log \log n \text{ and } \frac{(4+\epsilon) \log \log n}{\log^\epsilon n} \rightarrow 0.
 \end{aligned}$$

We now address the result in Part 2.

Theorem 3. Let m be any positive integer. Then $r(n, m) \rightarrow \infty$ if

$$\int_{m(n)} |f(x, n)|^2 e(-mx) dx = o(n \log^{-2} n)$$

as n goes to infinity in some suitable sequence.

We decompose the above integral

$$r(n, m) = \int_{m(n)} |f(x, n)|^2 e(-mx) dx + \int_{M(n)} |f(x, n)|^2 e(-mx) dx$$

$$r(n, m) = A(n) + B(n).$$

It is immediate by the prime numbers theorem that

$$A(n) = O(n \log^{-1} n).$$

By definition

$$\int_{M(n)} |f(x, n)|^2 e(-mx) dx = \sum_{q \leq \log^{15} n} \sum_{\substack{0 < h \leq q \\ (h, q) = 1}} T(h, q),$$

where

$$T(h, q) = \int_{\frac{h}{q} - x_0}^{\frac{h}{q} + x_0} |f(x, n)|^2 e(-mx) dx.$$

Lemma 8. Let $q \leq \log^{15} n$, $|y| \leq x_0$, $(h, q) = 1$ and $n \geq n_0$. Let $u(q)$ denote the Möbius function. Then

$$\left| f\left(\frac{h}{q} + y, n\right) - \frac{u(q)}{\phi(q)} g(y, n) \right| \leq n \log^{-69} n$$

This is Theorem 58 in [3].

Lemma 9. Under the hypothesis of Lemma 8 we have

$$\left| f^2\left(\frac{h}{q} + y, n\right) - \frac{u^2(q)}{\phi^2(q)} g^2(y, n) \right| \leq C_1 n^2 \log^{-69} n.$$

Proof. This follows immediately from Lemma 8 and the trivial inequalities $|f(x, n)| \leq n$ and $|g(y, n)| \leq n$ and the fact that if $|a| \leq n$ and $|b| \leq n$, then $|a^2 - b^2| \leq 2n|a - b|$. Hence it is immediate that

$$\left| f\left(\frac{h}{q} + y, n\right) \right|^2 - \frac{u^2(q)}{\phi^2(q)} |g(y, n)|^2 \leq C_1 n^2 \log^{-69} n.$$

By the change of variable $y = (x - \frac{h}{q})$ we have

$$T(h, q) = e\left(-\frac{mh}{q}\right) \int_{-x_0}^{x_0} \left| f\left(\frac{h}{q} + y, n\right) \right|^2 e(-my) dy.$$

Therefore we have

$$\left| e\left(-\frac{mh}{q}\right) \int_{-x_0}^{x_0} \left| f\left(\frac{h}{q} + y, n\right) \right|^2 e(-my) dy - \frac{u^2(q)}{\phi^2(q)} e\left(-\frac{mh}{q}\right) \int_{-x_0}^{x_0} |g(y, n)|^2 e(-my) dy \right|$$

$$\leq \int_{-x_0}^{x_0} \left| \left| f\left(\frac{h}{q} + y, n\right) \right|^2 - \frac{u^2(q)}{\phi^2(q)} |g(y, n)|^2 \right| dy \leq \int_{-x_0}^{x_0} C_1 n^2 \log^{-69} n dy$$

$$= 2x_0 C_1 n^2 \log^{-69} n = C_2 n \log^{-54} n.$$

Let $T_1(n) = \int_{-x_0}^{x_0} |g^2(y, n)|^2 e(-my) dy$ so that if $(h, q) = 1$, $q \leq \log^{15} n$, then

$$\left| T(h, q) - \frac{u^2(q)}{\phi^2(q)} T_1(n) e\left(-\frac{mh}{q}\right) \right| \leq C_2 n \log^{-54} n.$$

Let $T(n) = \sum_{m_1, m_2} (\log m_1)^{-1} (\log m_2)^{-1}$ with the condition of summation $n \geq m_1 \geq 2$, $n \geq m_1 \geq 2$ and $m_2 - m_1 = 2$. It is easy to see that

$$T(n) = \int_{-\frac{1}{2}}^{\frac{1}{2}} |g(y, n)|^2 e(-my) dy$$

and

$$\frac{c'n}{m} \log^{-2} n \leq T(n) \leq c'n. \quad n > m$$

$$\left| \sum_{m=2}^{m_1} e(m, y) \right| \leq \frac{1}{|\sin \pi y|} \leq \frac{1}{2|y|}; \quad (m_1 \geq 2.0 < |y| \leq \frac{1}{2}).$$

Hence by definition of $g(y, n)$ and Abel's lemma we have

$$|g(y, n)| < |y|^{-1}, \quad (0 < |y| \leq \frac{1}{2})$$

so that

$$|T(n) - T_1(n)| \leq 2 \int_{x_0}^{\frac{1}{2}} y^{-2} dy \leq 2x_0^{-1} = 2n \log^{-15} n.$$

For $(h, q) = 1$, $q \leq \log^{15} n$ we have

$$\left| e\left(-\frac{mh}{q}\right) \right| \left| \frac{u^2(q)}{\phi^2(q)} \right| \left| T(n) - T_1(n) \right| \leq \frac{1}{\phi^2(q)} (2n \log^{-15} n).$$

Hence for $h, q) = 1$ and $q \leq \log^{15} n$

$$\left| T(h, q) - \frac{u^2(q)}{\phi^2(q)} T(n) e\left(-\frac{mh}{q}\right) \right| \leq C_2 n \log^{-54} n + \frac{1}{\phi^2(q)} (2n \log^{-15} n);$$

so that for some fixed $q \leq \log^{15} n$ we have:

$$\begin{aligned} & \left| \sum_{\substack{0 < h \leq q \\ (h, q) = 1}} T(h, q) - \frac{u^2(q)}{\phi^2(q)} T(n) \sum_{\substack{0 < h \leq q \\ (h, q) = 1}} e\left(-\frac{mh}{q}\right) \right| \\ & \leq (C_2 n \log^{-54} n \phi(q) + \frac{1}{\phi^{4/3}(q)} (2n \log^{-15} n) \phi^{1/3}(q)). \end{aligned}$$

But $\phi(q) \leq \log^{-15} n$ and by definition

$$\sum_{\substack{0 < h \leq q \\ (h, q) = 1}} e\left(-\frac{mh}{q}\right) = C_q(m);$$

so that it follows immediately that

$$\left| \sum_{\substack{0 < h \leq q \\ (h, q) = 1}} T(h, q) - \frac{u^2(q)}{\phi^2(q)} T(n) C_q(m) \right| \leq C_2 n \log^{-39} n + \frac{1}{\phi^{4/3}(q)} (2n \log^{-10} n).$$

Now summing over all $q \leq \log^{15} n$ we have

$$\left| \sum_{q \leq \log^{15} n} \sum_{\substack{0 < h \leq q \\ (h, q) = 1}} T(h, q) - T(n) \sum_{q \leq \log^{15} n} \frac{u^2(q)}{\phi^2(q)} C_q(m) \right|$$

$$\begin{aligned} &\leq (C_2 n \log^{-39} n)(\log^{15} n) + \sum_{q < \log^{15} n} \frac{1}{\phi^{4/3}(q)} (2n \log^{-10} n) \\ &\leq C_2 n \log^{-24} n + C_3 (2n \log^{-10} n) \leq C_4 n \log^{-10} n, \end{aligned}$$

since by Theorem 327 page 267 in [5]

$$\sum_{q \leq \log^{15} n} \frac{1}{\phi^{4/3}(q)} \leq C_3 \text{ (} C_3 \text{ independent of } n \text{)}.$$

Hence

$$\left| r(n, m) - T(n) \sum_{q \leq \log^{15} n} \frac{u^2(q)}{\phi^2(q)} C_q(m) \right| \leq k(n) n \log^{-2} n.$$

where $k(n) \rightarrow 0$. Now let

$$\begin{aligned} R(n) &= \sum_{q \leq \log^{15} n} \frac{u^2(q)}{\phi^2(q)} C_q(m) \\ S &= \sum_{q=1}^i nfty \frac{u^2(q)}{\phi^2(q)} C_q(m) \end{aligned}$$

Lemma 10.

$$\sum_{q=1}^{\infty} \frac{u^2(q)}{\phi^2(q)} C_q(n) - \sum_{q \leq y} \frac{u^2(q)}{\phi^2(q)} C_q(n) = O(d(n)(\log \log 3y)^2 y^{-1}).$$

Proof ([4] page 211). It is immediate by Lemma 10 that

$$S - R(n) = O\left(\frac{1}{\log^{2+\frac{\epsilon}{2}} n}\right).$$

What remains to be done is to show that S is bounded away from 0.

Let

$$f(q) = \frac{u(q)}{\phi(q)} C_q(m)$$

Since $u(q)$, $\phi(q)$ and $C_q(m)$ are all multiplicative functions of q , f is a multiplicative function of q . Also, by means of the estimate on $C_q(m)$, namely m by Theorem 272 in [5] and a direct application of Theorem 327 page 267 in [5] we have

$$\sum_{q=1}^{\infty} |f(q)| \leq m \sum_{q=1}^{\infty} \frac{1}{\phi^2(q)} < \infty;$$

so that by Theorem 2 [3] page 3 we have

$$\begin{aligned} S &= \prod_p \sum_{m=0}^{\infty} f(p^m) \\ m=0 \quad f(p^0) &= f(1) = 1 \\ m=1 \quad f(p) &= \frac{u^2(p)}{\phi^2(p)} C_p(m) = \frac{1}{(p-1)^2} C_p(m) \\ m \geq 2 \quad f(p) &= 0 \\ C_p(m) &= \begin{cases} 1 & \text{if } (p, m) = p \\ -1 & \text{if } (p, m) = 1 \end{cases} \end{aligned}$$

so if m is an odd integer, $S = 0$, but if m is an even integer

$$S = \prod_n \left(1 + \frac{1}{(p-1)^2} C_p(m) \right) = 2 \prod_{p>2} \left(1 - \frac{1}{(p-1)^2} \right) > 2 \prod_{m=2}^{\infty} \left(1 - \frac{1}{m^2} \right) = 1.$$

So that $r(n,m)$ goes to infinity. But the conclusion for $r(n,m)$ is not possible for odd m

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