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Approximations of Confluent Hypergeometric Functions in Continued Fractions

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ABSTRACT

Confluent hypergeometric functions, often designated by the symbol 1F1 are an important class of special functions that appear in various fields of mathematics and physics. The aim of this article is to present the continued fraction expansions of the confluent hypergeometric functions and to provide an efficient method for calculating the generalized hypergeometric series. Furthermore, we examine some numerical examples which illustrate our theoretical results.

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1. Introduction and motivation

The fundamental idea behind the theory of continued fractions, concerning real numbers, is to provide an approximation of various real numbers using rational numbers. A continued fraction is an expression derived through an iterative process, representing a number as the sum of its integer part and the reciprocal of another number, then expressing this other number as the sum of its integer part and its reciprocal, and so forth. One of the primary reasons why continued fractions are so useful in computation is that they often offer a representation of transcendental functions that is much more generally valid than the classical representation, such as through power series. Furthermore, in the case of convergence, continued fraction expansions have the advantage of converging more rapidly than other numerical algorithms.

Recently, the extension of the theory of continued fractions from real numbers to the case of matrices has seen several developments and interesting applications (see [1,5,6,8]). Since computations involving functions with matrix arguments can be performed using large computers, it will be worthwhile to attempt to further develop this matrix theory. In this direction, and generally in a Banach space, some convergence results on non-commutative continued fractions have been established, see ([4,5]).

Confluent hypergeometric functions are an important class of special functions, they are solutions of the second-order linear differential equation known as the Kummer equation, which takes the following form

$$x\frac{d^2y}{dx^2} + (b-x)\frac{dy}{dx} - ay = 0$$

where a and b are real parameters. This equation is widely used to model physical phenomena such as diffusion, vibrations and growth processes.
Kummer function ${}_1F_1$ is defined as follows for $a, x \in \mathbb{R}$, $b \in \mathbb{R}/\mathbb{Z}^-$

$${}_1F_1(a,b;x) = \sum_{n=0}^{+\infty} \frac{a^{(n)}}{b^{(n)}} \frac{x^n}{n!},$$

where, the Pochhammer symbol $a^{(j)}$ is defined as $a^{(0)} = 1$, and $a^{(j)} = a(a+1)(a+2)...(a+j-1)$ for $j > 0$.
In [12], two theorems are presented by Wynn, covering various aspects of continued fractions theory, particularly focusing on elements that do not commute under multiplication. Extensions of Worpitsky's work in Banach space have been established by Haydan [4].
Moreover, in [11], several convergence criteria are outlined for noncommutative continued fractions, where the arguments are $m \times m$ matrices represented in the forms $K(B_n/A_n)$.
Confluent hypergeometric functions are characterized by their Taylor or Puiseux series, converging around the point $x = 0$. They generalize the hypergeometric functions ${}_2F_1$ which are a special case when a and b are integers [9].
These functions are closely related to special functions such as Bessel functions, Laguerre functions, and Hermite functions. Their study and use are widespread in many fields, including mathematical analysis, theoretical physics, statistics and engineering.

2. Preliminary and notations

Throughout this paper, we denote by $\mathcal{M}_m$ the set of $m \times m$ real matrices.
For any $A, B \in \mathcal{M}_m$ with B invertible, we write $\frac{A}{B} = B^{-1}A$. It is clear that for any invertible matrix C, we have $\frac{CA}{CB} = \frac{A}{B}$.

Definition 2.1. Let $(A_n)_{n\geq 0}, (B_n)_{n\geq 1}$ be two nonzero sequences of $\mathcal{M}_m$. The continued fraction of (A_n) and (B_n) denoted by $K(B_n/A_n)$, is the quantity

$$A_0 + \cfrac{B_1}{A_1 + \cfrac{B_2}{A_2 + \ldots + \cfrac{B_n}{A_n}}} = \left[A_0; \frac{B_1}{A_1}, \frac{B_2}{A_2}, ..., \frac{B_n}{A_n}\right],$$

where we suppose that all the inverses of the matrices in the above expression exist.
Sometimes, we use briefly the notation $\left[A_0; \frac{B_k}{A_k}\right]_{k=1}^{n}$. The fraction $\frac{P_n}{Q_n} := \left[A_0; \frac{B_k}{A_k}\right]_{k=1}^{n}$ is called the n^{th} convergent of the continued fraction $K(B_n/A_n)$. When $B_n = I$ for all $n \geq 1$, then $K(I/A_n)$ is called an ordinary continued fraction.

The following proposition gives an adequate method to calculate $K(B_n/A_n)$.

Proposition 2.2. *The elements* $(P_n)_{n\geq -1}$ *and* $(Q_n)_{n\geq -1}$*of the* n^{th} *convergent of* $K(B_n/A_n)$ *are given by the relationships*

$$\begin{cases} P_{-1} = I, \quad P_0 = A_0 \\ Q_{-1} = 0, \quad Q_0 = I \end{cases} \text{and} \begin{cases} P_n = A_nP_{n-1} + B_nP_{n-2} \\ Q_n = A_nQ_{n-1} + B_nQ_{n-2} \end{cases}, n \geq 1.$$

Proof. This can be done by induction.
The proof of the next Proposition is elementary and we left it to the reader.

Proposition 2.3. *For any two matrices* C *and* D *with* C *invertible, we have*

$$C\left[A_0; \frac{B_k}{A_k}\right]_{k=1}^{n} D = \left[CA_0D; \frac{B_1D}{A_1C^{-1}}, \frac{B_2C^{-1}}{A_2}, \frac{B_k}{A_k}\right]_{k=3}^{n}.$$

The continued fraction $K(B_n/A_n)$ converges in $\mathcal{M}_m$ if the sequence $(F_n) = (\frac{P_n}{Qn}) = (Q_n^{-1}P_n)$ converges in $\mathcal{M}_m$ in the sense that there exists a matrix $F \in \mathcal{M}_m$ such that
$lim_{n\to+\infty}||F_n - F|| = 0$. In the other case, we say that $K(B_n/A_n)$ is divergent. It is clear that

$$\frac{P_n}{Q_n} = A_0 + \sum_{k=1}^{n} (\frac{P_k}{Q_k} - \frac{P_{k-1}}{Q_{k-1}}). \tag{1}$$

From (1), we see that the continued fraction $K(B_n/A_n)$ converges in $\mathcal{M}_m$ if and only if the series $\sum_{k=1}^{n} (\frac{P_k}{Q_k} - \frac{P_{k-1}}{Q_{k-1}})$ converge in $\mathcal{M}_m$.

Definition 2.4. Let (A_n), (B_n), (C_n) and (D_n) be four sequences of matrices. We say that the continued fractions $K(B_n/A_n)$ and $K(D_n/C_n)$ are equivalent if we have $F_n = G_n$ for all $n \geq 1$, where F_n and G_n are the n^{th} convergent of $K(B_n/A_n)$ and $K(D_n/C_n)$ respectively.

Remarque 2.5. Let (A_n) and (B_n) be two sequences of $\mathcal{M}_m$ and I the m^{th} its identity matrix. We notice that we can write the first convergents of the continued fraction $K(B_n/A_n)$ by

$$F_1 = A_0 + A_1^{-1}B_1 = A_0 + (B_1^{-1}A_1)^{-1},$$
$$F_2 = A_0 + (A_1 + A_2^{-1}B_2)^{-1}B_1 = A_0 + (B_1^{-1}A_1 + (B_2^{-1}A_2B_1)^{-1})^{-1}.$$

If we put, $A_1^* = B_1^{-1}A_1$ and $A_2^* = B_2^{-1}A_2B_1$, we have

$$F_1 = A_0 + \frac{I}{A_1^*}, F_2 = A_0 + \frac{I}{A_1^* + \frac{I}{A_2^*}}.$$

Generally, we prove by a recurrence that if we put for all $k \geq 1$, (see [11])

$$A_{2k}^* = (B_{2k}...B_2)^{-1}A_{2k}B_{2k-1}...B_1$$

and

$$A^*_{2k+1} = (B_{2k+1}...B_1)^{-1} A_{2k+1} B_{2k}...B_2,$$

then the continued fractions $K(B_n/A_n)$ and $K(I/A^*_n)$ are equivalent.

In order to simplify the statements on some partial quotients of continued fractions with matrix arguments, we need the following proposition which is an example of equivalent continued fractions.

Proposition 2.6. **[11]** Let $\left[A_0; \frac{B_k}{A_k}\right]_{k=1}^{+\infty}$ be a given continued fraction. Then

$$\frac{P_n}{Q_n} = \left[A_0; \frac{B_k}{A_k}\right]_{k=1}^{n} = \left[A_0; \frac{X_k B_k X_{k-2}^{-1}}{X_k A_k X_{k-1}^{-1}}\right]_{k=1}^{n},$$

where $X_{-1} = X_0 = I$ and $X_1, X_2, ..., X_n$ are arbitrary invertible matrices.

To end this section, we give the following Propositions.

Proposition 2.7. **[2]** Let r be a strictly positive real number, $x_0 \in \mathbb{R}$, if the real function $f(x)$ can be expanded in a power series in $]x_0 - r, x_0 + r[$ by

$$f(x) = \sum_{k=0}^{+\infty} a_k (x - x_0)^k,$$

then expansion cited above remains valid when the scalar argument x is replaced by a matrix A whose eighenvalues lie within to the interval of convergence of f..

The next theorem gives sufficient conditions which ensure the convergence of a continued fraction $K(B_n/A_n)$ in o$\mathcal{M}_m$.

Theorem 2.8. **[11]** *Let (A_n), (B_n) be two sequences of $\mathcal{M}_m$, where A_n and B_n are invertible matrices. If*

$$\| (B_{2k-2}...B_2)^{-1} A_{2k-1}^{-1} B_{2k-1}...B_1 \| \leq \alpha$$

and

$$||(B_{2k-1}...B_1)^{-1} A_{2k}^{-1} B_{2k}...B_2|| \leq \beta$$

for all $k \geq 1$, where $0 < \alpha < 1, 0 < \beta < 1$ and $\alpha\beta \leq 1/4$, then the continued fraction $K(B_n/A_n)$ converges in $\mathcal{M}_m$.

3. Main results

3.1. The real case

An important example of a confluent hypergeometric functions is the Kummer function, also known as the generalized confluent hypergeometric function or function $M(a, b, x)$.
We also note that Kummer function is also defined by

$${}_1F_1(a, b; x) = M(a, b, x) = \frac{1}{\Gamma(a)\Gamma(b-a)} \int_0^1 e^{xu} u^{a-1} (1-u)^{b-a-1} du,$$

This function appears in many fields of mathematics and physics, including the solution of differential equations, probability theory, statistics and other scientific and technical applications. It is widely used because of its simple analytical form and rich mathematical properties. We recall that many elementary and special functions can be expressed in terms of hypergeometric functions.

Examples

$$
\begin{gathered}
{}_1F_1(0,b;x)=1,\\
{}_1F_1(a,a;x)=e^x,\quad {}_1F_1(1,2;2x)=\frac{e^x}{x}\sinh(x),\\
{}_1F_1(1/2,3/2;-x^2)=\frac{\sqrt{\pi}}{2x}erf(x),\\
{}_1F_1(1,a+1;x)=ax^{-a}e^x\gamma(a,x),\\
{}_1F_1(a+1,b;x)=\frac{2a-b+x}{a}\,{}_1F_1(a,b;x)+\frac{b-a}{a}\,{}_1F_1(a-1,b;x),\\
{}_1F_1(a,b+1;x)=\frac{b(b-1+x)}{(b-a)x}\,{}_1F_1(a,b;x)-\frac{b(b-1)}{(b-a)x}\,{}_1F_1(a,b-1;x).
\end{gathered}
$$

Our first result is given in the next Theorem.

Theorem 3.1. *Let x be a real number such that $\mid x \mid< 1$, a and b two strictly positive real numbers, a continued fraction representation of the Kummer hypergeometric function is given by*

$$
{}_1F_1(a,b;x)=\left[1;\frac{a}{b}x,\frac{-bx}{2\dfrac{b+1}{a+1}+x},\ \frac{-\dfrac{b+n-2}{a+n-2}(n-1)x}{\dfrac{b+n-1}{a+n-1}n\left(1+\dfrac{a+n-1}{b+n-1}\dfrac{x}{n}\right)}\right]_{n=3}^{+\infty}.
$$

The following lemma characterizes equivalence of real continued fractions.

Lemma 3.2. **[5]** . Let (r_n) be a non-zero sequence of real numbers. Then, one has

$$
\left[a_0;\frac{b_1}{a_1},\frac{b_2}{a_2},...,\frac{b_n}{a_n},...\right]=\left[a_0;\frac{r_1b_1}{r_1a_1},\frac{r_2r_1b_2}{r_2a_2},...,\frac{r_nr_{n-1}b_n}{r_na_n},...\right].
$$

In order to prove Theorem 3.1, we recall the following Lemma, from the development of a the function given by the Taylor series, we give the development in continued fractions of the series that was established by Euler.

lemma 3.3, [6] . Let f be a real function which has the Taylor series development $f(x)=\sum\limits_{n=0}^{+\infty}c_nx^n$ in $J\subset\mathbb{R}$. Then, the development in continued fraction of $f(x)$ is

$$
f(x)=\left[c_0;\frac{c_1x}{1},\frac{-c_2x}{c_1+c_2x},\frac{-c_1c_3x}{c_2+c_3x},...\frac{-c_{n-2}c_nx}{c_{n-1}+c_nx},...\right].
$$

Proof of Theorem 3.1. We apply Lemma 3.3 to the function ${}_1F_1(a,b;x) = \sum_{n=0}^{+\infty} c_n x^n$ by putting $c_n = \dfrac{a^{(n)}}{b^{(n)}}.\dfrac{1}{n!}$ for all $n \geq 1$. So, we have

$$c_0 = 1,\ \frac{c_1 x}{1} = \frac{\frac{a}{b}x}{1},\ \frac{-c_2 x}{c_1 + c_2 x} = \frac{\frac{-a^{(2)}}{b^{(2)}}\frac{x}{2}}{\frac{2a(b+1)+a^{(2)}x}{2b^{(2)}}}.$$

For $n \geq 3$, we get

$$-c_{n-2}c_n x = -\left(\frac{a^{(n-2)}}{(n-2)!b^{(n-2)}}\right).\frac{a^{(n)}}{n!b^{(n)}}x = -\left(\frac{a^{(n-2)}}{(n-2)!b^{(n-2)}}\right)^2 \prod_{k=n-2}^{n-1}\frac{a+k}{b+k}.\frac{x}{n(n-1)}.$$

Furthermore, we have

$$c_{n-1} + c_n x = \frac{a^{(n-1)}}{b^{(n-1)}}\frac{1}{(n-1)!}\left(1 + \frac{a+n-1}{b+n-1}\frac{x}{n}\right).$$

Therefore, the continued fraction expansion of ${}_1F_1(a,b;x)$ is

$${}_1F_1(a,b;x) = \left[1; \frac{ax}{b}, \frac{\frac{-a^{(2)}x}{2b^{(2)}}}{\frac{2a(b+1)+a^{(2)}x}{2b^{(2)}}}, \frac{b_n}{a_n}\right]_{n=3}^{+\infty},$$

with

$$b_n = -\left(\frac{a^{(n-2)}}{(n-2)!b^{(n-2)}}\right)^2 \frac{(a+n-1)(a+n-2)}{(b+n-1)(b+n-2)}\frac{1}{n(n-1)}x$$

and

$$a_n = \frac{a^{(n-1)}}{b^{(n-1)}}\frac{1}{(n-1)!}\left(1 + \frac{a+n-1}{b+n-1}\frac{x}{n}\right).$$

In order to simplify the above expansion, let us define the sequence $(r_n)_{n\geq 1}$ by

$$\begin{cases} r_1 = b, \\ r_2 = 2\dfrac{b^{(2)}}{a^{(2)}}, \\ \\ r_n = \dfrac{b^{(n)}}{a^{(n)}}n!,\ for\ n \geq 3. \end{cases}$$

Then, we obtain

$$\begin{cases} \dfrac{r_1 a_1}{r_1 b_1} = \dfrac{a}{b}x, \\ \dfrac{r_1 r_2 a_2}{r_2 b_2} = \dfrac{-bx}{2\dfrac{b+1}{a+1} + x}, \\ \dfrac{r_n r_{n-1} a_n}{r_n b_n} = \dfrac{\dfrac{-(b+n-2)(n-1)}{a+n-2}x}{\dfrac{b+n-1}{a+n-1}n\left(1 + \dfrac{a+n-1}{b+n-1}\dfrac{x}{n}\right)} \quad \text{for} \quad n \geq 3. \end{cases}$$

By applying the result of Lemma 3.2 to the sequence $(r_n)_{n\geq 1}$, we found

$${}_1F_1(a,b;x) = \left[1; \frac{ax}{b}, \frac{-bx}{2\dfrac{b+1}{a+1} + x}, \frac{b'_n}{a'_n}\right]_{n=3}^{+\infty},$$

with

$$\begin{cases} b'_n = \dfrac{-(b+n-2)(n-1)}{a+n-2}x, \\ a'_n = \dfrac{b+n-1}{a+n-1}n\left(1 + \dfrac{a+n-1}{b+n-1}\dfrac{x}{n}\right) \end{cases}$$

which completes the proof of Theorem 3.1.

3.2. The matrix case

Definition 3.4. Let A be a matrix in $\mathcal{M}_m$, such that $\|A\| < 1$, we define the Generalized hypergeometric function of a matrix A by the expression

$${}_1F_1(a,b;A) = \sum_{n=0}^{+\infty} \frac{a^{(n)}}{n!b^{(n)}}A^n.$$

Theorem 3.5. *Let A be a matrix in $\mathcal{M}_m$ such that $\|A\| < 1$. Let a and b be two strictly positive real numbers. The matrix continued fraction*

$$\left[I; \frac{aA}{b}, \frac{-bA}{2\dfrac{b+1}{a+1}I + A}, \frac{\dfrac{-(b+n-2)(n-1)}{a+n-2}A}{\dfrac{b+n-1}{a+n-1}n\left(I + \dfrac{a+n-1}{b+n-1}\dfrac{A}{n}\right)}\right]_{n=3}^{+\infty}.$$

converges in $\mathcal{M}_m$. Furthermore, this continued fraction represents the function ${}_1F_1(a,b;A)$. So, we have

$${}_1F_1(a,b;A) = \left[I; \frac{aA}{b}, \frac{-bA}{2\dfrac{b+1}{a+1}I + A}, \frac{\dfrac{-(b+n-2)(n-1)}{a+n-2}A}{\dfrac{b+n-1}{a+n-1}n\left(I + \dfrac{a+n-1}{b+n-1}\dfrac{A}{n}\right)}\right]_{n=3}^{+\infty}. \tag{2}$$

Proof of Theorem 3.5. In order to prove the convergence of the continued fraction,

$$\left[I; \frac{aA}{b}, \frac{-bA}{2\frac{b+1}{a+1}I + A}, \frac{B_n}{A_n}\right]_{n=3}^{+\infty},$$

where $B_n = \frac{-(b+n-2)(n-1)}{a+n-2}A$, and $A_n = \frac{(b+n-1)n}{a+n-1}\left(I + \frac{a+n-1}{b+n-1}\frac{A}{n}\right)$ converges in $\mathcal{M}_m$, we should verify that the conditions of Theorem 2.8 are satisfied. So, we have

$$B_{2k-2}...B_2 = (-1)^{k-1}a\prod_{n=0}^{k-2}\frac{b+2n}{a+2n}(2n+1)A^{k-1}a,$$

and

$$B_{2k-1}...B_3 = (-1)^{k-1}\prod_{n=0}^{k-2}\frac{(b+2n+1)}{a+2n+1}(2n+2)A^{k-1}.$$

Then, we obtain

$$(B_{2k-2}...B_2)^{-1}A_{2k-1}^{-1}(B_{2k-1}...B_3) =$$
$$\frac{1}{a}\prod_{n=0}^{k-1}\frac{a+2n}{b+2n}\prod_{n=0}^{k-2}\frac{b+2n+1}{a+2n+1}\prod_{n=0}^{k-2}\frac{2n+2}{2n+1}\frac{1}{2k-1}\left(I + \frac{a+2k-2}{b+2k-2}\frac{A}{2k-1}\right)^{-1}.$$

According to Proposition 2.9 and the fact that

$$\lim_{k\to+\infty}\frac{a+2k-2}{(b+2k-2)(2k-1)} \mid\mid A \mid\mid = 0,$$

we obtain for all sufficiently large k

$$\mid\mid \left(I - \frac{a+2k-2}{(b+2k-2)(2k-1)}A\right)^{-1} \mid\mid \leq \left(1 - \frac{a+2k-2}{(b+2k-2)(2k-1)}||A||\right)^{-1} < 1.$$

So, one has

$$\mid\mid (B_{2k-2}...B_2)^{-1}A_{2k-1}^{-1}(B_{2k-1}...B_3) \mid\mid \leq$$
$$\frac{1}{a}\prod_{n=0}^{k-1}\frac{a+2n}{b+2n}\prod_{n=0}^{k-2}\frac{b+2n+1}{a+2n+1}\prod_{n=0}^{k-2}\frac{2n+2}{2n+1}\frac{1}{2k-1}\left(1 - \mid \frac{a+2k-2}{(b+2k-2)(2k-1)} \mid\mid\mid A \mid\mid\right)^{-1}.$$

Which becomes

$$\mid\mid (B_{2k-2}...B_2)^{-1}A_{2k-1}^{-1}(B_{2k-1}...B_3) \mid\mid \leq$$
$$\frac{1}{a}\prod_{n=0}^{k-1}\frac{a+2n}{b+2n}\prod_{n=0}^{k-2}\frac{b+2n+1}{a+2n+1}\prod_{n=0}^{k-2}\frac{2n+2}{2n+1}\frac{1}{2k-1} \tag{3}$$

We can see that the second side of the inequality (3) tends to 0, so let α be a real number such that $\alpha < \frac{1}{2}$, then for all sufficiently large k, we have

$$\mid\mid (B_{2k-2}...B_2)^{-1}A_{2k-1}^{-1}(B_{2k-1}...B_3) \mid\mid < \alpha.$$

To prove the second inequality of Theorem 2.8, we have

$$(B_{2k-1}...B_3)^{-1}A_{2k}^{-1}(B_{2k}...B_2) =$$

$$-a\prod_{n=0}^{k-1}\left(\frac{a+2n+1}{b+2n+1}\right)\left(\frac{b+2n}{a+2n}\right)\left(\frac{2n+1}{2n+2}\right)\left(I+\frac{a+2k-1}{b+2k-1}\frac{A}{2k}\right)^{-1}A$$

It becomes

$$|| \ (B_{2k-1}...B_3)^{-1}A_{2k}^{-1}(B_{2k}...B_2) \ ||\leq$$

$$|\ a\prod_{n=0}^{k-1}\left(\frac{a+2n+1}{b+2n+1}\right)\left(\frac{b+2n}{a+2n}\right)\left(\frac{2n+1}{2n+2}\right)\ |\ \left(1-\ |\ \frac{a+2k-1}{(b+2k-1)(2k)}\ |||\ A\ ||\right)^{-1}\ ||\ A\ ||\ . \qquad (4)$$

We also can see that the second side of the inequality (4) tends to 0, so let $\beta = \frac{1}{2}$, then for all sufficiently large k, we have $||\ (B_{2k-1}...B_3)^{-1}A_{2k}^{-1}(B_{2k}...B_2)\ ||< \beta$. Finally, the proof of Theorem 3.5 is complete.

4. Numerical applications

In this section, we will provide some numerical data to illustrate the preceding results, we begin with a real case.

Example 1. The following table clarifies the differences between the tabular value of the Kummer function ${}_1F_1(1, a+1; x)$ and its convergents f_k when applying the Theorem 3.1, in the case $a = 3/4$.

x	${}_1F_1(1,a+1;x)-$ $f_1(1,a+1,x)$	${}_1F_1(1,a+1;x)-$ $f_3(1,a+1,x)$	${}_1F_1(1,a+1;x)-$ $f_6(1,a+1,x)$	${}_1F_1(1,a+1;x)-$ $f_9(1,a+1,x)$	${}_1F_1(1,a+1;x)-$ $f_{12}(1,a+1,x)$	${}_1F_1(1,a+1;x)-$ $f_{15}(1,a+1,x)$
0.05	5.26e-4	7.35e-8	3.04e-14	4.14e-21	2.51e-28	8.07e-36
0.01	2.08e-5	1.16e-10	3.88e-19	4.23e-28	2.05e-37	5.28e-47
0.1	2.13e-3	1.18e-6	3.92e-12	4.26e-18	2.06e-24	5.3e-31
0.2	8.77e-3	1.93e-5	5.07e-10	4.4e-15	1.7e-20	3.49e-26
0.3	2.02e-2	9.96e-5	8.78e-9	2.56e-13	3.34e-18	2.3e-23
0.4	3.71e-2	3.2e-4	6.65e-8	4.58e-12	1.41e-16	2.31e-21
0.5	5.96e-2	7.97e-4	3.21e-7	4.31e-11	2.59e-15	8.28e-20

We note that the algorithm of the continued fraction converges very quickly. For example, for $x = 0.05$, we have 21 digits that coincide between the 9th convergent and the tabular value of the Kummer function ${}_1F_1(1, a+1; x)$. Now we pass to the matrix case, we illustrate the theoretical results obtained in Theorem 3.5.

Example 2. Let A be a 3×3 matrix, such that

$$A = \begin{pmatrix} 0.01 & 0.02 & 0.03 \\ 0.02 & 0.01 & 0.02 \\ 0.04 & 0.02 & 0.01 \end{pmatrix}$$

The norm of A is $||A|| = 0.07$, it is of course strictly less than 1. By applying the theoretical results obtained in Theorem 3.5, we obtain the difference between the tabular value ${}_1F_1(a, b; A)$ of the Kummer function and its k^{th} convergents f_k where $a = 1$ and $b = 2$.

$${}_1F_1(a,b;A) - f_1 = \begin{pmatrix} 2.86e-4 & 1.69e-4 & 1.7e-4 \\ 2.02e-4 & 1.52e-4 & 1.69e-4 \\ 2.04e-4 & 2.02e-4 & 2.86e-4 \end{pmatrix},$$

$${}_1F_1(a,b;A) - f_2 = \begin{pmatrix} 3.25e-6 & 2.69e-6 & 3.41e-6 \\ 2.95e-6 & 2.23e-6 & 2.69e-6 \\ 4.37e-6 & 2.95e-6 & 3.25e-6 \end{pmatrix},$$

$${}_1F_1(a,b;A) - f_5 = \begin{pmatrix} 3.76e-12 & 2.79e-12 & 3.31e-12 \\ 3.14e-12 & 2.35e-12 & 2.79e-12 \\ 4.18e-12 & 3.14e-12 & 3.76e-12 \end{pmatrix},$$

$${}_1F_1(a,b;A) - f_8 = \begin{pmatrix} 1.15e-18 & 8.6e-19 & 1.02e-18 \\ 9.67e-19 & 7.23e-19 & 8.6e-19 \\ 1.29e-18 & 9.67e-19 & 1.15e-18 \end{pmatrix},$$

$$ {}_1F_1(a,b;A) - f_{10} = \begin{pmatrix} 3.19e-23 & 2.38e-23 & 2.84e-23 \\ 2.68e-23 & 2.007e-23 & 2.38e-23 \\ 3.59e-23 & 2.68e-23 & 3.19e-23 \end{pmatrix}. $$

To better evaluate the precision, we calculate the norm of the difference between ${}_1F_1(a,b;A)$ and its k^{th} convergents f_k. We obtain

$$ ||_1\, F_1(a,b;A) - f_1\,|| = 6.93e-4, \qquad ||_1\, F_1(a,b;A) - f_2\,|| = 1.05e-5, $$

$$ ||_1\, F_1(a,b;A) - f_3\,|| = 1.27e-7, \qquad ||_1\, F_1(a,b;A) - f_5\,|| = 1.10e-11, $$

$$ ||_1\, F_1(a,b;A) - f_8\,|| = 3.41e-18, \qquad ||_1\, F_1(a,b;A) - f_{10}\,|| = 9.46e-23. $$

We can see that for n large, we have $\lim_{n\to+\infty} ||_1\, F_1(a,b;A) - f_n\,|| = 0$. This example proves that our approach is interesting.

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