ISSN: 2754-6659

Journal of Artificial Intelligence & Cloud Computing

Research Article

Open Access

Quantum Liquids He^3 and He^4 and Quantum Computing

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ABSTRACT

L. D. Landau described the quantum liquids He^3 and He^4 at low temperature (below 3K) by applying the quasi particle approach called the quasi particles (elementary excitations) as "rotons". We discuss in this paper how these physical materials may be candidates for building the processor of a quantum computer of type based on the quantum set theory of G. Takeuti.

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Received: November 06, 2024; **Accepted:** November 11, 2024; **Published:** November 22, 2024

Introduction

Gaisi Takeuti showed that set theory based on von Neumann´s quantum logic (on the lattice of all closed linear subspaces of a Hilbert space) satisfies the generalization of the ZFC axioms (Zermelo, Frenkel plus Axiom of Choice) of set theory [1,2]. Therefore, a reasonable mathematics can be derived from this set theory but a much richer mathematics, a "gigantic" mathematics by the words of Takeuti. He named *quantum set theory* this type of set theory.

We discussed computing in the framework of this quantum set theory in the paper [3]. We concluded that a) Computing based on quantum set theory offers a more general framework than the one based on the notion of the quantum bit, and as a corollary, b) it could and should offer a computing machinery exceeding the capacity of the computers we are using in these decades. The main points in this study are as follow:

- We investigated the elementary propositional systems of local field theories in the paper [4] and found that these propositions can not only take the values 0 and 1 but they have (infinitely many) third values, too, the so called **true-false** values [4-6]. Thus, in the case of systems with infinitely many degrees of freedom, von Neumann´s line of thoughts steps beyond the mathematics based on the two valued logic.

- The representations of the elementary propositional systems were looked for and the solutions of the commutation relations were studied on these representations. For this reason we had to turn to the *extension* of the basic tools of the theory of Hilbert spaces, to the theory of Hilbert A-modules H_A, where A is the C^*-algebra of operators in the Hilbert-space $L^2(R^3)$. Then it was found that the extended form of the von Neumann´s theorem holds true on these representations [7,8]. We call the representation space H_A as *the local state space of the quantized system*. We note that a local field theory consists generally of an infinite collection of (identical) systems of finitely many degrees of freedom connected in space [7,8].

- 1) This alternative solution of quantized fields with infinitely many degrees of freedom reproduces the physical implications of the conventional theory, legitimating in this way the alternative approach, 2) It uses (based on) the "gigantic"mathematics derivable from the quantum set theory of Takeuti [1,7-10].

- One can find the illustration of the geometrical structure of the system´s local state space both in references [7, p. 1059] and [8, p. 198]. It shows that one may think of this structure as a „non commutative" Hilbert bundle. Therefore the conclusion is that the local states of the system *[consisting of an infinite collection of (identical) quantum systems of finitely many degrees of freedom connected in space]* are sections of the bundle. The time evolution of these local states is governed, instead of the global/total Hamiltonian, by the **local Hamiltonian** of the system according to the eq. (30) in ref. [7] or to the eq.s (5.8a) and (5.8b) in ref. [8]. This geometrical structure and time evaluation equations implies that the different alternatives *[for the individual members of the infinite collection of (identical) quantum systems of finitely many degrees of freedom connected in space]* given by an initial value of the evolution equation described by a section of the "non commutative" Hilbert bundle can be computed in parallel [3].

In the paper we discussed further on this theoretical possibility by using an explicit example of a rigid body of cuboid form [11]. The universe $V^{(L)}$ of Takeuti was determined. A set of real numbers in this universe was explicitly described including a set of binary numbers. Thus we arrived at the foundations of von Neumann's theory of computing in terms of ordinary binary numbers. Then we concluded that this extension of computing to the universe $V^{(L)}$ provides a sound, mathematically well defined theory of quantum computing.

Now we discuss this theoretical possibility further on by studying its mathematical apparatus in a specific case of local field theory in the second section, the solution of the eigenvalue problem for the free field approximation in the **third** section and finally the possible application of the results in computing in the fourth section.

The Mathematical Apparatuses

Let we study the theoretical method of solving a specific field theoretical example, in the framework of the alternative quantization, for the illustrative case of N real classically relativistic scalar fields of Lagrangian density,

$$L(t,\boldsymbol{x}) = [1/2\sum_{\alpha=1}^{N}(\partial_\mu\phi_\alpha\partial^\mu\phi_\alpha - m_\alpha^2\phi_\alpha^2) - V(\phi_1,$$

$$\dots, \phi_N)](t,\boldsymbol{x}), (t,\boldsymbol{x})\in \boldsymbol{M}^4, (2.1)$$

Where $\boldsymbol{M}^4$ is the Minkowski space. This system consists of an infinite collection of identical classical anharmonic oscillators of N degrees of freedom connected in space. Then the corresponding quantum field theory (QFT) should consist of an infinite collection of identical quantum anharmonic oscillators of N degrees of freedom connected in space.

Really, as it was shown the alternative quantization method substitutes the individual members of the system by their quantum mechanical counterparts [8]. The *local state pace* H_A is an A-valued Hilbert space (Hilbert A-module) of the form $L^2(\boldsymbol{R}^N)\otimes A$ [the tensor product of the complex separable Hilbert space $L^2(\boldsymbol{R}^N)$ and the C^*-algebra A of bounded operators of $L^2(\boldsymbol{R}^3)$]. In this approach the quantized system is described ***coherently*** because the algebra of bounded operators $B(H_A) = B(H)\otimes A$ of the local state space H_A is a ***factor*** [8-10].

Von Neumann's basic theorem of quantum mechanics (QM), namely that the canonical commutation relations (CCR) have a unique solution up to unitary equivalence (cf. yet the basic observations of John von Neumann in his first paper about QM), has an extended form in this framework: a B-irreducible set of unitary operators in the A-valued Hilbert space H_A satisfying the CCR's is uniquely determined up to A-unitary equivalence [8,12]. In this way this extension of von Neumann's theorem offers the possibility that one formulates QFT in terms of the A-valued Hilbert spaces in the same unique way, up to A-unitary equivalence as QM is formulated in terms of complex Hilbert spaces up to unitary equivalence [13].

The dynamics of the system is described by the unitary map

$$t \rightarrow \exp(-iHt)$$

of H_A onto itself, where H is the **local Hamiltonian** of the system obtained by replacing the *Hamiltonian density* of the classical system with its operator counterpart one gets by the quantization algorithm, i.e.

$$H = H(\phi, \pi, \partial\phi) =$$

$$= 1/2\sum_{\alpha=1}^{N} [\pi_\alpha^2 + (\partial\phi_\alpha)^2 + m_\alpha^2\phi_\alpha^2] + V(\phi_1, \dots, \phi_N),$$

Where the fields ϕ_α and their canonical momentum densities π_α as operators in H_A satisfy the CCR's [6, 7]. The classical equations of motion become well-defined operator equations in H_A and the local states (the ray's $\boldsymbol{\Phi}$ of H_A, i.e. for all $\Phi\in\boldsymbol{\Phi}$ we have $\langle\Phi|\Phi\rangle_A = 1$, where $\langle|\rangle_A$ denotes the A-valued inner product in H_A and 1 is the unity operator of A) are governed by the local Schrödinger equation [5, 6] :

$$i\hbar\partial\Phi(t)/\partial t = 1/2\sum_{\alpha=1}^{N}[\pi_\alpha^2 + (\partial\phi_\alpha)^2 + m_\alpha^2\phi_\alpha^2]\Phi(t) + V(\phi_1, \dots, \phi_N)$$

$$\Phi(t), \Phi\in H_A \quad (2.2)$$

Then one can apply the **extension** of the perturbation theory of QM to solve this equation by using the interaction picture [7]. The local Hamiltonian of the free fields is

$$H_0 = 1/2\sum_{\alpha=1}^{N}[\pi_\alpha^2 + (\partial\phi_\alpha)^2 + m_\alpha^2\phi_\alpha^2] =$$

$$= \sum_{\alpha=1}^{N} (N_\alpha + 1/2)p_0^\alpha, \quad (2.3)$$

where $p_0^\alpha = (\boldsymbol{p}^2 + m_\alpha^2)^{1/2}$, $\boldsymbol{p}^2 = (-i\hbar\partial)^2 = -\hbar^2\Delta$, Δ is the Laplace operator, and $N_\alpha = a_\alpha^+a_\alpha$, a_α^+ is the creation while a_α is the annihilation operator in the local Fock space F_A of the free fields [8]. As we see p_0^α is the energy component of a Klein-Gordon-like free particle of mass m_α, more precisely its energy operator, i.e. its Hamiltonian operator. In this framework the Haag-theorem does not block to solve the local Schrödinger equation (2.2) for non-trivial interactions in the local Fock space F_A of the free fields [8].

Application

Let we apply the formalism of the foregoing section to a system of Lagrangian (2.1) localized in space to a box of cuboid form with side-edges a, b and c. In that case the basic Hilbert space in the Takeuti's approach reduces to the Hilbert space $L^2([0,a],[0,b],[0,c])$ of the square integrable functions over the domain of the cuboid form. The local Hamiltonian operator of the free fields has a diagonal form like equation (2.3) in the corresponding local state space $H_A = F_A$ where of course A is the C^*-algebra of operators in the Hilbert-space $L^2([0,a],[0,b], [0,c])$. It means that its eigenvalues are hermitian operators in $L^2([0,a],[0,b], [0,c])$. For example, in the lowest energy local state, in the local vacuum state Φ_0 when the local number operator N_α equals to zero for all α, the hermitian eigenvalue operator of the local free field Hamiltonian operator is

$$1/2(\sum_{\alpha=1}^{N} p_0^\alpha) \quad (3.1)$$

Let we diagonalizes p_0^α. It means the solutions of the eigenvalue equations

$$p_0^\alpha\phi_n = e^\alpha_n\phi_n, \quad \phi_n\in L^2 ([0,a],[0,b],[0,c]) \quad (3.2)$$

or equivalently

$$p_0^{\alpha 2}\phi_n = e^{\alpha}_n{}^2\phi_n, \quad \phi_n \in L^2([0,a],[0,b],[0,c]) \quad (3.2)$$

$$-\hbar^2\Delta\phi_n + m_\alpha^2 \phi_n = e^\alpha_n{}^2\phi_n, \quad (3.3)$$

Thus we get the partial differential equations for all α by rearrangement

$$\Delta\phi_n + \hbar^{-2}(e^\alpha_n{}^2 - m_\alpha^2)\phi_n = 0, \quad \phi_n\in L^2([0,a],[0,b], [0,c]) \quad (3.4)$$

For all α the equation (3.4) becomes three one dimensional equations having solutions of the form [11]:

$$\phi_1(x) = A \sin k_1 x + B \cos k_1 x, \quad 0<x<a,$$
$$\phi_2(y) = C \sin k_2 y + D \cos k_2 y, \quad 0<y<b,$$
$$\phi_3(z) = F \sin k_3 z + G \cos k_3 z, \quad 0<z<c,$$

where

$$(k_1)^2 + (k_2)^2 + (k_3)^2 = \hbar^{-2} (e_\alpha^2 - m_\alpha^2)$$

and

$$\phi_1(x)\phi_2(y)\phi_3(z) = \phi(x,y,z)$$

The wave functions of norm 1 have the form

$$\phi_{n1,n2,n3}(x,y,z) = (8/abc)^{1/2} \sin (n_1\pi/a)x \sin (n_2 \pi/b)y \sin (n_3 \pi/c)z \quad (3.5)$$

The square of the eigenvalues of the Klein-Gordon-like particle's energy are discrete in the form

$$(e^{\alpha}_{n1,n2,n3})^2 = \pi^2\hbar^2 (n_1{}^2/a^2 + n_2{}^2/b^2 + n_3{}^2/c^2) + m_\alpha{}^2, \qquad n_1, n_2, n_3 = 1, 2, 3, \ldots \quad (3.6)$$

Then, in the special case when $a = b = c$, i.e. when the cuboid form is a cube, we have

$$e^{\alpha}_{n1,n2,n3} = [\pi^2\hbar^2 a^{-2}(n_1{}^2 + n_2{}^2 + n_3{}^2) + m_\alpha{}^2]^{1/2}, \; n_1, n_2, n_3 = 1, 2, 3, \ldots \quad (3.6)$$

As we see the physical system described by the Lagrangian density (2.1) and localised it in a cuboid form in space, after quantization, has a discrete energy spectra in the first approximation (in the free fields approximation). We know such a phenomenon from condensed matter physics. Namely the collective behaviour of the atoms in the quantum liquids He^3 and He^4 at low temperature, below 3 K, can be described by the quasi-particle approach as it was showed by L. D. Landau and we have studied its mathematical model in the non-relativistic framework in the paper [11].

Application in Computing

Let we discuss the possibility of applying Takeuti's quantum set theory in computing theory. With the relations, formulas and mathematical objects of the foregoing sections we can again determine the components of Takeuti's approach. The basic Hilbert space is the state space $L^2([0,a],[0,b],[0,c])$ spanned by the orthonormal functions in (3.5) which set of functions constitutes a basis for this Hilbert space. L is the lattice of all closed linear subspaces of $L^2([0,a],[0,b],[0,c])$ (the quantum logic of von Neumann [2]). Then the totality of all L-valued functions provides the universe $V^{(L)}$ for us [remember: the totality, the set of all (0,1)-valued functions (the characteristic functions of the sets in classical set theory) gives the universe V in classical set theory] [1].

We know that in the "quantum mathematics" based on $V^{(L)}$, the real numbers defined by Dedekind's cuts are self-adjoint operators of the basic Hilbert space $L^2([0,a],[0,b],[0,c])$ as it was shown by Takeuti in [1]. Therefore the "quantum real numbers" are self-adjoint operators and the algebra of them is the algebra of these operators. The binary numbers are replaced by the "quantum binary numbers", namely in symbols $(0, 1) \to (0, p(X), 1)$ [$p^2(X) = p(X)$, the orthogonal projector of the closed linear subspace X of $L^2([0,a],[0,b],[0,c])$, i.e. X is an element of L]. In this way we have in symbols:

the machine-made code of a classical program has the form of (1, 0, 0, 1, 1, ….),
then
the machine-made code of a "quantum program" should have the form of $(p(X), 1, 0, p(Y), p(Z), \ldots, 0, \ldots)$.

In this approach the unity operator 1 of the Hilbert space $L^2([0,a],[0,b],[0,c])$ belongs to the **true** logical value, the zero operator 0 belongs to the **false** logical value, while the projection operators $p(X), p(Y), p(Z), \ldots$ to the **true-false** values, e.g. $p(X)$ is **true** on the subspace X while it is false outside X (on the subspace $L\backslash X$). Clearly the number of the true-false values is **infinite.**

The local state space $H_A = L^2(\mathbf{R}^N)\otimes A$ is isomorphic to the countably infinite direct sum $H_A = \sum_1^\infty \oplus A$ of the Hilbert A-module A [8]. This means that we can represent H_A with infinite column vectors with operator entries from A. The local states are represented by the rays of norm 1 (the unity operator of A) in H_A. The expectation value of a local bounded observable F in the local state Φ in H_A is given by the formula

$$Exp\, F = \langle\Phi|F|\Phi\rangle_A \in A \quad (4.1)$$

using the A-valued inner product of H_A.

Therefore in this setting the local Hamiltonian of the quantized system of Lagrangian (2.1) is a real number valued function in this Takeuti's universe and we can write it in the form

$$H = \sum_{[n]} E_n P(\phi_n) \quad (4.2)$$

where $P(\phi_n)$ is the orthogonal projector of the one dimensional subspace of $L^2([0,a],[0,b],[0,c])$ spanned by the ray belonging to the eigenstate ϕ_n in (3.5), while E_n is also a hermitian element of A (or the unbounded extension of A) from the spectrum of the local Hamiltonian (which is a hermitian operator in the A-valued Hilbert space H_A,) [8]. Therefore it is also of the form

$$E_n = \sum_{[m]} e_m P(\phi_m) \quad (4.3)$$

where the (ordinary non-negative) real number e_m is from the spectrum of E_n (which of course may has not only discrete but continuous spectrum, too). Substituting (4.3) in the relation (4.2) and taking into account the relations

$P(\phi_m)P(\phi_n) = P^2(\phi_m)\delta_{m,n} = P(\phi_m)\,\delta_{m,n},\ \delta_{m,n} = 1$, if $m = n$ and 0 otherwise,

we get that the local Hamiltonian of the quantized system is a hermitian valued function in $V^{(L)}$ with values of form

$$H(\Phi) = Exp\, H = \sum_{[n]} e_n P(\phi_n) \in A \quad (4.4)$$

Then one can express the expectation value of the local Hamiltonian H as a linear combination of binary number valued functions in $V^{(L)}$ having the form

$$b = \sum_{[n]} b(n)P(\phi_n), \; b(n) = 0 \text{ or } 1, \quad (4.5)$$

where $P(\phi_n)$ is the orthogonal projector in $L^2([0,a],[0,b],[0,c])$ according to the relation of (4.2). The set of these binary numbers is a subset of the set of all binary numbers in $V^{(L)}$.

Thus we have seen that one can evaluate, in finite linear combinations, the evolution of the quantized system of Lagrangian (2.1), localised in a cuboid form, in the local Fock space F_A of the free fields by applying the eigenstates (3.5) of the energy operators of the Klein-Gordon-like particles of mass m_α, and thus in a finite steps of recursions. Therefore one can approach (or at least estimate) the real numbers in the universe $V^{(L)}$ by linear combinations of "quantum binary numbers" in this Takeuti's universe. Then we can conclude that a physical system having eigenstates of form (3.5) can help us to solve the the system's evolution equation of form (2.2) by exciting it and measuring its eigenstates and the corresponding eigenvalues while inserting the results in the appropriate mathematical relations.

Therefore the cuboid form of a "well and appropriately tuned up" rigid or condensed body, of our example in this paper, may be an essential part of the physical implementation of the processor for a "quantum computer" of this type [11].

As a closing note we remember again that, as it is well known, L. D. Landau described the quantum liquids He^3 and He^4 at low temperature (below 3K) by applying the quasi particle approach, outlined in a non-relativistic form and in a relativistic form in this paper in the foregoing sections [11]. He called the quasi particles (elementary excitations) as "rotons". Thus these physical materials and rotons may be the candidates for building the processor of a "quantum computer" of this type.

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