ISSN: 2754-6659

Journal of Artificial Intelligence & Cloud Computing

Research Article

Open Access

Quantum Liquid He^3 of Fermi Statistics and Quantum Computing in Terms of the Alternative Field Quantization

Miklós Banai

Retired Theoretical Physicist, Budapest, Hungary

ABSTRACT

The quantum liquid He^3 of Fermi statistics described by L. D. Landau at low temperature (below 3 K) using the quasiparticle approach is studied in the framework of the alternative field quantization [1]. We discuss in this paper how this physical material may be a candidate for building the processor of a quantum computer of type based on the quantum set theory of G. Takeuti [2].

***Corresponding author**

Miklós Banai, Retired Theoretical Physicist, Budapest, Hungary. E-mail: banai.m@t-online.hu

Received: July 23, 2025; **Accepted:** July 25, 2025; **Published:** August 05, 2025

Introduction

The main Results this Paper are based on are as Follow:

Gaisi Takeuti in [2] showed that set theory based on von Neumann´s quantum logic [3] (on the lattice of all closed linear subspaces of a Hilbert space) satisfies the generalization of the ZFC axioms (Zermelo, Frenkel plus Axiom of Choice) of set theory. Therefore, a reasonable mathematics can be derived from this set theory but a much richer mathematics, a „gigantic" mathematics by the words of Takeuti. He named quantum set theory this type of set theory.

We discussed computing in the framework of this quantum set theory in the paper [4]. We concluded that a) computing based on quantum set theory offers a more general framework than the one based on the notion of the quantum bit, and as a corollary b) it could and should offer a computing machinery exceeding the capacity of the computers we are using in these decades. The main points in this study are as follow:

- We investigated the elementary propositional systems of local field theories in the paper [1.A)] and found that these propositions can not only take the values 0 and 1 but they have (infinitely many) third values, too, the so called **True-False** values. Thus, in the case of systems with infinitely many degrees of freedom, von Neumann´s line of thoughts steps beyond the mathematics based on the two valued logic.

- The representations of the elementary propositional systems were looked for and the solutions of the commutation relations were studied on these representations. For this reason, we had to turn to the ***extension*** of the basic tools of the theory of Hilbert spaces, to the theory of Hilbert A-modules H_A, where A is the C*-algebra of operators in the Hilbert-space $L^2(\boldsymbol{R}^3)$. Then it was found that the extended form of the von Neumann´s theorem holds true on these representations [1.D]. We call the representation space H_A as *the local state space of the quantized system*. We note that a local field theory consists generally of an infinite collection of (identical) systems of finitely many degrees of freedom connected in space [1.C, D].

- We showed in [1.E]. that 1) this alternative solution of quantized fields with infinitely many degrees of freedom reproduces the physical implications of the conventional theory, legitimating in this way the alternative approach (cf. [6]), and in [1.C, D] that 2) it uses (based on) the „gigantic" mathematics derivable from the quantum set theory of Takeuti [2].

- One can find the illustration of the geometrical structure of the system´s local state space both in references [1.C] and [1.D)]. It shows that one may think of this structure as a „non commutative" Hilbert bundle. Therefore, the conclusion is that the local states of the system [consisting of an infinite collection of (identical) quantum systems of finitely many degrees of freedom connected in space] are sections of the bundle. The time evolution of these local states is governed, instead of the global/total Hamiltonian, by the **local Hamiltonian** of the system according to the eq. (30) in [1.C] or to the eq.s (5.8a) and (5.8b) in ref. [1.D)] This geometrical structure and time evaluation equations implies that the different alternatives [for the individual members of the infinite collection of (identical) quantum systems of finitely many degrees of freedom connected in space] given by an initial value of the evolution equation described by a section of the "non commutative" Hilbert bundle can be computed in *parallel* [4].

In the paper [5.A] we discussed further on this theoretical possibility by using an explicit example of a rigid body of cuboid form. The universe $V^{(L)}$ of Takeuti was determined. A set of real numbers in this universe was explicitly described including a set of binary numbers. Thus we arrived at the foundations of von Neumann's theory of computing in terms of ordinary binary numbers. Then we concluded that this ***extension*** of computing to the universe $V^{(L)}$ provides a sound, mathematically well-defined theory of quantum computing. In the next paper [5.B]. this theoretical possibility was discussed further on by studying its mathematical apparatus in a specific case of local scalar fields including the solution of the eigenvalue problem for the free field approximation and the possible application of the results in computing [6].

In this paper we will discuss the specific case of the fermions of spin ½ by using the Dirac equation and the standard notations [7] and for the sake of simplicity, we use the natural units $c = h/2\pi = 1$.

The Mathematical Formulation of the Alternative Quantization for Fermions

Let we study the theoretical method of solving a specific field theoretical example, in the framework of the alternative quantization [1] for Lagrangian density of the Dirac equation [7]

$$L(t,\boldsymbol{x}) = \{i/2[\psi^+\gamma^0\gamma^\mu\,\partial_\mu\psi - (\partial_\mu\psi^+)\gamma^0\gamma^\mu\psi] - m\psi^+\gamma^0\psi\}(t,\boldsymbol{x}),\ \ (t,\boldsymbol{x})\in\boldsymbol{M^4}, \quad (2.1)$$

where γ denotes the Dirac matrixes [7] and $\boldsymbol{M^4}$ is the Minkowski space. With a simple calculation one can check that the classical equation of motion $\partial L/\partial\psi - \partial_\mu(\partial L/\partial\partial_\mu\psi) = 0$ gives really the Dirac equation:

$$(i\gamma^\mu\,\partial_\mu + m)\psi^* = 0 \qquad (2.2)$$

The local state pace H_A is an A-valued Hilbert space (Hilbert A-module) of the form $[\sum_1^4 \oplus L^2{}_i(\boldsymbol{R})]\otimes A$ [the tensor product of the complex separable Hilbert space $H = \sum_1^4 \oplus L^2{}_i(\boldsymbol{R})$ and the C^*-algebra A of bounded operators of $L^2(\mathbf{R}^3)$]. In this approach the quantized system is described ***coherently*** because the algebra of bounded operators $B(H_A) = B(H)\otimes A$ of the local state space H_A is a ***factor*** [1.C, D)].

The spectral decompositions of the Dirac spinor operators are as follow:

$$\psi = \int_{[p\sigma]} \{(2\varepsilon)^{-1/2}[a_{p\sigma}u_{p\sigma}\exp(-ipx) + b_{p\sigma}{}^+u_{-p\text{-}\sigma}\exp(ipx)\}\ dE_{p\sigma}, \qquad (2.3a)$$

$$\psi^*\gamma^0 = \int_{[p\sigma]} \{(2\varepsilon)^{-1/2}[a_{p\sigma}{}^+u^*{}_{p\sigma}\gamma^0\exp(-ipx) + b_{p\sigma}u^*{}_{-p\text{-}\sigma}\gamma^0\exp(ipx)\}\ dE_{p\sigma}, \qquad (2.3b)$$

where $\varepsilon = (\boldsymbol{p}^2 + m^2)^{1/2} = p_0$ and $E_{p\sigma}$ denotes a partition of the unity opertor of A. In accordance with the rule of the Fermi statistics, the creation and annihilation operators, $a_{p\sigma}{}^+$, $b_{p\sigma}{}^+$ and $a_{p\sigma}$, $b_{p\sigma}$, respectively, satisfy the anti-commutation relations

$$\{a_{p\sigma}, a_{p\sigma}{}^+\}_+ = 1, \qquad \{b_{p\sigma}, b_{p\sigma}{}^+\}_+ = 1 \qquad (2.4)$$

These operators are anti-commute in all other combinations. The dynamics of the system is described by the unitary map

$$t \rightarrow \exp(-iHt)$$

of H_A onto itself, where H is the **local Hamiltonian** of the system obtained by replacing the Hamiltonian density of the classical system with its operator counterpart one gets by the quantization algorithm. The Hamiltonian density of the classical system is

$$H = H(\psi, \pi, \partial\psi) = \pi\partial_0\psi - L \qquad (2.5)$$

From (2.1) one gets that $\pi = \partial L/\partial\partial_0\psi = (i/2)\psi^*$ and with a straightforward algebra using the Dirac equation (2.2), too, we obtain for the Hamiltonian density (2.5) that

$$H = \psi^*\boldsymbol{\alpha}(i\partial\psi) + [\beta m(i\text{-}1)/2]\psi^*\psi, \qquad (2.6)$$

where $\boldsymbol{\alpha} = \gamma^0\boldsymbol{\gamma}$ and $\beta = \gamma^0$. Then by using the relation $u^*{}_{\pm p\sigma}\gamma^0 u_{\pm p\sigma} = 2\varepsilon = 2(\boldsymbol{p}^2 + m^2)^{1/2} = 2p_0$ and the rules (2.4) we get from (2.6) for the spectral decomposition of the local Hamiltonian

$$H = \int_{[p\sigma]}(a_{p\sigma}{}^+a_{p\sigma} + b_{p\sigma}{}^+b_{p\sigma} - 1)p_0\ dE_{p\sigma} = \int_{[p\sigma]}(N_{a,p\sigma} + N_{b,p\sigma} - 1)(\boldsymbol{p}^2 + m^2)^{1/2}\ dE_{p\sigma}. \qquad (2.7)$$

The classical equation of motion becomes well-defined operator equation in the local state space H_A and the local states (the rays $\boldsymbol{\Phi}$ of H_A, i.e. for all $\Phi \in \boldsymbol{\Phi}$ we have $\langle\Phi|\Phi\rangle_A = 1$, where $\langle|\rangle_A$ denotes the A-valued inner product in H_A and 1 is the unity operator of A) are governed by the local Schrödinger equation [1.C, D].

$$i\partial\Phi(t)/\partial t = [\int_{[p\sigma]}(N_{a,p\sigma} + N_{b,p\sigma} - 1)p_0\ dE_{p\sigma}]\Phi(t),\ \ \Phi \in H_A \qquad (2.8)$$

The energy component p_0 as an operator takes the form $p_0 = (\boldsymbol{p}^2 + m^2)^{1/2}$, $\boldsymbol{p}^2 = (-i\partial)^2 = -\Delta$, Δ is the Laplace operator, and the number operators $N_{a,p\sigma}$ and $N_{b,p\sigma}$ belong to the fermion of mass m and its anti-particle, respectively, in the local Fock space F_A [1.D]. We note that in this framework the Haag-theorem does not block to solve the local Schrödinger equation (2.8) for non-trivial interactions in the local Fock space F_A of the free fields [1.D].

Application in Computing and Conclusions

If we apply the formalism of the foregoing section to a system of Lagrangian (2.1) localized in space to a box of cuboid form with side-edges a, b and c then the basic Hilbert space in the Takeuti's approach reduces to the Hilbert space $L^2([0,a], [0,b], [0,c])$ of the square integrable functions over the domain of the cuboid form. The local Hamiltonian operator of the free field has a diagonal form like equation (2.7) in the corresponding local state space $H_A = F_A$ where of course A is the C*-algebra of operators in the Hilbert-space $L^2([0,a],[0,b], [0,c])$. It means that its eigenvalues are hermitian operators in $L^2([0,a],[0,b], [0,c])$. For example, in the lowest energy local state, in the local vacuum state $\boldsymbol{\Phi_0}$ when the local number operators equal to zero, the hermitian eigenvalue operator of the local free field Hamiltonian operator is

$$-p_0 \qquad (3.1)$$

It means that in the local vacuum state of the fermion this non vanishing term decreases the energy of the vacuum while we saw in [5.B]. that in the case of the scalar fields the lowest energy local state increases the energy of the local vacuum.

It is clear that the diagonalization of p_0 for this system under consideration is a special case of the one discussed in [5.B]. Therefore, we can write from [5.B] for the eigenstates of p_0 that

$$\phi_{n1,n2,n3}(x,y,z) = (8/abc)^{1/2}\sin(n_1\pi/a)x\,\sin(n_2\pi/b)y\,\sin(n_3\pi/c)z \qquad (3.2)$$

and for the eigenvalues that

$$e_{n1,n2,n3} = [\pi^2(n_1{}^2/a^2 + n_2{}^2/b^2 + n_3{}^2/c^2) + m^2]^{1/2}, \qquad n_1, n_2, n_3 = 1, 2, 3, \ldots.. \qquad (3.3)$$

So again, the physical system described by the Lagrangian density (2.1) and localised it in a cuboid form in space, after quantization, has a discrete energy spectra in the first approximation (in the free fields approximation). We know such a phenomenon from condensed matter physics. Namely the collective behavior of the atoms in the quantum liquid He^3 at low temperature, below 3 K, can be described by the quasi-particle approach as it was showed by L. D. Landau and He^3 satisfies Fermi statistics, i.e. it behaves like fermions.

As to the application in computing one can simply repeat the 4. section of the reference [5.B]. Thus we can again conclude that one can evaluate, in finite linear combinations, the evolution of the quantized system of Lagrangian (2.1), localised in a cuboid form, in the local Fock space F_A of the free field of the fermion by applying the eigenstates (3.2) of the energy operator of the fermion of mass m, and thus in a finite step of recursions. Therefore, one can approach (or at least estimate) the real numbers in the universe $V^{(L)}$ of Takeuti by linear combinations of "quantum binary numbers" in this Takeuti's universe. Then we can conclude that a physical system having eigenstates of form (3.2) can help us to solve the the system's evolution equation of form (2.8) by exciting it and measuring its eigenstates and the corresponding eigenvalues while inserting the results in the appropriate mathematical relations.

Therefore, the cuboid form of a "well and appropriately tuned up" rigid or condensed body, of our example in this paper or the one in reference [5.B)] may be an essential part of the physical implementation of the processor for a "quantum computer" of this type.

It is an important observation that the initial values of the evolution equation (2.8) or (2.2) in [5. B)] are generated by the lattice of the projectors of the basic Hilbert space $L^2([0,a],[0,b], [0,c])$ which is an infinity set. This fact has a serious consequence in cyber security and cryptography based on the quantum computers discussed in the reference [5.B)] and in this paper.

As a closing note we remember again that, as it is well known, L. D. Landau described the quantum liquids He^3 and He^4 at low temperature (below 3 K) by applying the quasi particle approach. He called the quasi particles (elementary excitations) as „rotons". Thus, these physical materials and rotons may be the candidates for building the processors of "quantum computers" of this type.

References

1. 1. A. Banai M (1981) Propositional Systems in Field Theories and Lattice-valued Quantum Logics, Current Issues in Quantum Logic, eds. E. Beltrametti and B. C. van Frassen, Plenum Press, New York and London 425-435.
   1.B. Banai M (1981) Propositional Systems in Local Field Theories Int. J. Teor. Phys 20: 147-169.
   1.C. Banai M (1984) Quantum Relativity Theory and Quantum Space- Time, Int. J. Theor. Phys 23: 1043-1063.
   1.D. Banai M (1987) An unconventional canonical quantization of local scalar fields over quantum spacetime, J. Math. Phys 28: 193-214
   1.E. Banai M, Lukács B (1988) Unconventional Quantization of Field Theory Does Contain the Physics of the Conventional Theory, Modern Physics Letters A3: 1205-1212.
2. Takeuti G (1981) Quantum Set Theory, in Current Issues of Quantum Logic, eds. E. Beltrametti and B.C. van Frassen, Plenum Press, New York and London 303-322.
3. Birkhoff G, von Neumann J (1936) The logic of quantum mechanics, Ann. Math 37: 823-843.
4. Banai M (2023) Computing based on Quantum Set Theory, Int. Journal of Computer Science and Information Technologies 14: 35-36.

5 A. Banai M (2024) Computing in terms of Takeuti's Quantum Set Theory, Journal of Artificial Intelligence & Cloud Computing 3: 1-2.

5.B. Banai M (2024) Quantum Liquids He^3 and He^4 and Quantum Computing, Journal of Artificial Intelligence & Cloud Computing 3: 1-4.

6. Banai M (2025) Quantum Computing: The alternative way, ICAICC-25, Orlando, Florida, keynote speak, (to appear in the proceeding of the conference).
7. Landau LD, Lifsic EM (1971) Theoretical Physics IV., Nauka Publishing House, Moskva. https://cimec.org.ar/foswiki/pub/Main/Cimec/MecanicaRacional/84178116-Vol-1-Landau-Lifshitz-Mechanics-3Rd-Edition-197P.pdf.