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Otto H. Kegel's Beautiful Contributions to Sylow Theory in Locally Finite Groups

In remembrance of my adored masterly teacher

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ABSTRACT

Otto H. Kegel published two fundamental papers on Sylow Theory in locally finite groups: see [5] and [6]. The paper at hand summarises them, compares them and above all provides a list of their open issues which are still open until the present day. To study crucial configurations, Kegel developed in [6] the quite excogitated concept of the "*(smooth simple straight) split sequences of finite p-perfect subgroups with their associated ascending sequences of subgroups*" which is related to his equally very fine concept of the "*Sylow-separated (ascending) sequences of p-subgroups with associated sequences of Sylow p-subgroups*" he had developed already more then ten years earlier in [5]. When I met him personally in July 2022, I tried to find out how he detected these really ingenious ideas, but woefully was unsuccessful, although I was really very privileged to witness up very close the creation of the idea of [6] (see [3]). A scheduled new attempt in July 2025 (see [3]) could not be realised … ☹.

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Let p be a prime. A *p-group* is a group all of whose elements have order a power of p. A *Sylow p-subgroup* of a group is a maximal (w.r.t. inclusion) p-subgroup. A group satisfies *the Sylow Theorem for the prime p* (or *the Sylow p-Theorem*) if all its Sylow p-subgroups are conjugate, it satisfies *the strong Sylow Theorem for the prime p* if each of its subgroups satisfies the Sylow Theorem for p. Mahmut Kuzucuoğlu wrote a beautiful paper about the life and work of Otto H. Kegel (see [7]). On his Page 5 he mentions [6] and Philip Hall's universal locally finite group into which for every prime p every countably infinite locally finite p-group can be embedded as a Sylow p-subgroup hence it does not satisfy the Sylow p-Theorem. But he did not appreciate Kegel's work on Sylow Theory in locally finite groups. He only states quite a bit nebulous that "*locally finite groups satisfying the strong Sylow Theorem for p must have some restricted properties*" and that [6] contains "*some results in this direction*" but without stating the results he means.

The AGTA-Research Paper [1] and the JMCA-Research Article [2] are both based on Kegel's such beautiful paper [6] each one proving a conjecture of it: the AGTA-paper answers the question on Page 10 with YES and the JMCA-paper finds a proof for the "inspection" of (2.4) Theorem on Page 13 both being centred around the gay concept of a *p-uniqueness subgroup* which is a finite p-subgroup being contained in a unique Sylow p-subgroup. In [3] I first present the main result of Kegel's paper [6] and comment then thoroughly his final considerations regarding p-length and $p = 3$ and $p = 2$. Therefore now is the time for summarising the beautiful paper [6].

In the following I summarise the introduction and the four lectures of [6]. The **Introduction** mentions Brian Hartley's tremendous result on locally finite and locally p-soluble groups satisfying the strong Sylow Theorem for the prime p if $p \neq 2$ (see references [7] of [1], [4] of [5] and [10] of [6]) and then states an extension of a weak form of Hartley's result which is also a fairly weak form of Kegel's main result (see Page 6):

> *If for the prime $p \geq 5$ the locally finite group G satisfies the strong Sylow Theorem, then there is a finite series of normal subgroups N_i of G with $G = N_0 \supseteq \ldots \supseteq N_i \supseteq N_{i+1} \supseteq \ldots \supseteq N_k = \langle 1 \rangle$ such that the factors N_i / N_{i+1} of this series are either a direct product of finitely many linear simple groups or locally p-soluble.*

Hartley's result allows one to refine the series so that the locally p-soluble factors are either p-groups or p'-groups.

Lecture I "Variations on Sylow's Theorem" first constructs a rather special Sylow p-subgroup (besides others) of a countably infinite locally finite group and then characterises the conjugacy of these special Sylow p-subgroups in those groups. After giving several examples and counterexamples, it is proved that a locally finite group without the Sylow p-Theorem contains a countably infinite subgroup with $2^{\aleph_0}$ Sylow p-subgroups and therefore a locally finite group G satisfies the strong Sylow Theorem for the prime p if and only if in every countable subgroup of G there are fewer than $2^{\aleph_0}$ Sylow p-subgroups. As a consequence the strong Sylow p-Theorem is inherited by every section. Kegel then introduces the fundamental concept of singular p-subgroups. The finite p-subgroup P of the locally finite group G is called *singular in G* if for every finite subgroup F of G containing P there is a unique Sylow p-subgroup of F containing P. The finite p-subgroup P of a locally finite group G is singular in G if and only if P is a p-uniqueness subgroup of G (see Proposition 2.3 of [1]). Kegel then proves fundamentally that for a locally finite group G satisfying the strong Sylow Theorem for the prime p there exists a finite p-subgroup which is singular in G, and asks the question "*If every subgroup S of the locally finite group G contains a finite p-subgroup which is singular in S, does G then satisfy the strong Sylow Theorem for the prime p ?*". This question is answered in the affirmative by Theorem 3.10 of [1]. **Lecture I** is closed with proving that singularity is inherited by factor groups and citing that linear locally finite groups satisfy the strong Sylow Theorem for all primes p.

Lecture II "Singular p-subgroups and simple locally finite groups" first reports the breathtaking results by Andrew Rae and Brian Hartley on locally finite and locally p-soluble groups with the strong Sylow Theorem for the prime $p \neq 2$ and then studies locally finite simple groups. The key for the simple groups is (2.4) Theorem: "*Let the p-subgroup P be singular in the finite simple group S which belongs to one of the seven rank-unbounded families. Then the rank of S is bounded in terms of P.*" Kegel "proves" this theorem "by inspection". [2] provides a proper proof. Brian Hartley states in his MR-review of [6] the following: *"If the simple locally finite group G satisfies the strong Sylow Theorem for the (even one) prime p, then G is linear. This depends on the classification of finite simple groups and an* assertion *about singular p-subgroups of classical groups. Another proof of this result has since been given by the reviewer (not yet published)."* However, due to the very tragic death of Brian Hartley on October 8, 1994, aged 55 (see [2]), this certainly highly interesting proof was never prepared for publication.

With someone of Hartley's stature, there is no question that his word is good enough and that in any case he supplied a new proof with probably quite a number of very new insights. It might therefore be worthwhile to inspect Hartley's estate *In Search of* **not** *Lost Notes* … .

A well-known theorem of Philip Hall states that an infinite group is simple if and only if it possesses a local system consisting of countably infinite simple subgroups (see 4.4 Theorem of [4] and (2.5) Theorem), and Kegel has detected on March 12, 1966 (see https://d-nb.info/1123426309/34) what is known as a *Kegel cover* (R_i, M_i) with *Kegel kernels* M_i and *Kegel simple sections* R_i / M_i: "If G is a countably infinite locally finite simple group then there exists a strictly ascending sequence $\{ R_i \}_{i \in \mathbb{N}}$ of finite subgroups of G with $G = \bigcup \{ R_i \mid i \in \mathbb{N} \}$ such that for each natural number i there exists a maximal normal subgroup M_{i+1} of R_{i+1} satisfying $M_{i+1} \cap R_i = \langle 1 \rangle$." According to their classification, the finite simple groups are distributed into finitely many families whence by the pigeon hole principle one may assume that the simple sections R_i / M_i $(i \in \mathbb{N})$ all belong to the same family. By the key (2.4) Theorem the rank of them is bounded. As a consequence, a locally finite simple group satisfies the strong Sylow Theorem for the prime p if and only if every countable subgroup contains a singular p-subgroup if and only if it is linear. **Lecture II** closes with remarks on the structure of locally finite simple groups: • Locally finite simple linear groups are countable. • Locally finite simple linear groups have a local system consisting of finite simple subgroups. • If the infinite locally finite simple group has a non-trivial linear representation of finite degree, then it is a (possibly twisted) Chevalley group over some infinite locally finite field.

Lecture III "The study of crucial configurations" introduces the quite excogitated concept mentioned on Page 1 and uses it to show that it generates a subgroup with $2^{\aleph_0}$ Sylow p-subgroups and hence without the (strong) Sylow Theorem for the prime p. The sequence $\{ F_i \}_{i \in \mathbb{N}}$ of subgroups $F_i \neq \langle 1 \rangle$ of the group G will be called a *split sequence with associated ascending sequence* $\{ H_i \}_{i \in \mathbb{N}}$ if for every natural number i the subgroup $H_i := \langle F_j \mid 1 \leq j \leq i \rangle$ normalises the subgroup F_{i+1} and satisfies $H_i \cap F_{i+1} = \langle 1 \rangle$. A finite group is called *p-perfect* for the prime p if it is non-trivial perfect and generated by its p-elements. The split sequence $\{ F_i \}_{i \in \mathbb{N}}$ of finite p-perfect subgroups will be called *straight* with associated ascending sequence $\{ H_i \}_{i \in \mathbb{N}}$ if for every number i the group F_{i+1} does not contain any proper H_i-invariant normal p-perfect subgroup. In a straight split sequence of finite p-perfect groups the subgroups F_{i+1} have a unique p-soluble maximal H_i-invariant normal subgroup $M_{i+1} \neq F_{i+1}$. The straight split sequence $\{ F_i \}_{i \in \mathbb{N}}$ of finite p-perfect subgroups is called *simple* if all the factor groups F_{i+1} / M_{i+1} are simple. If $\{ F_i \}_{i \in \mathbb{N}}$ is a simple straight split sequence of finite p-perfect subgroups we shall consider the centralisers $C_{i,j} := \underline{\underline{\mathbf{C}}}_{H_i}(F_j / M_j)$ (i.e., all elements of H_i that commute with every element of F_j / M_j, equivalently, the set of elements x of F_j / M_j such that conjugation by x leaves each element of H_i fixed) for $j > i$; if for all triples i, j, k of natural numbers with $i < j < k$ one has $C_{i,j} = C_{i,k}$, we shall call the sequence *smooth* and denote the $C_{i,j}$ simply by C_i. If a locally finite group satisfies the strong Sylow Theorem for the prime p, a straight split sequence $\{ F_i \}_{i \in \mathbb{N}}$ of finite p-perfect subgroups cannot be simple (see the fourth property below) whence all but finitely many of the factor groups F_i / M_i are proper direct products. The group F_i induces a permutation representation on the simple direct factors of F_j / M_j for $j > i$. Let $P_{i,j}$ be the kernel of this permutation representation. We then may find with a rather considerable effort a straight split sequence $\{ F_i' \}_{i \in \mathbb{N}}$ which satisfies $P_{i,k}' \subseteq P_{i,k}$ for all indices i, j, k with $i < j < k$.

Kegel then proves basic properties of these split sequences: • From any split sequence $\{ F_i \}_{i \in \mathbb{N}}$ of finite p-perfect subgroups one can obtain a straight split sequence $\{ F_i' \}_{i \in \mathbb{N}}$ of finite p-perfect subgroups with $F_i' \subseteq F_i$ for all $i \in \mathbb{N}$. • If in the straight split sequence of finite p-perfect subgroups for infinitely many indices i the factor groups F_i / M_i are simple, then there is a simple straight split sequence of finite p-perfect groups. • If there exists a simple straight split sequence of finite p-perfect subgroups, then there will exist a smooth simple straight split sequence of finite p-perfect subgroups as well. • If $\{ F_i \}_{i \in \mathbb{N}}$ is a smooth simple straight split sequence of finite p-perfect subgroups, then the countably infinite group $< F_i \mid i \in \mathbb{N} >$ has $2^{\aleph_0}$ Sylow p-subgroups. • Let the straight split sequence $\{ F_i \}_{i \in \mathbb{N}}$ of finite p-perfect subgroups satisfy $P_{i,j} = P_{i,k} =: P_i$ for every triple i, j, k with $i < j < k$. Then the countable subgroup $< F_i \mid i \in \mathbb{N} >$ has $2^{\aleph_0}$ Sylow p-subgroups, if $p \geq 5$.

Summarising all these properties of split sequences, Otto H. Kegel obtains the following great theorem: *If for the prime $p \geq 5$ there is a straight split sequence $\{ F_i \}_{i \in \mathbb{N}}$ of finite p-perfect subgroups, then the countable subgroup $< F_i \mid i \in \mathbb{N} >$ contains $2^{\aleph_0}$ many Sylow p-subgroups. Hence it does not satisfy the (strong) Sylow Theorem for the prime p.*

Lecture IV "The strong finiteness results" first introduces the largest normal locally p-soluble subgroup $\mathbf{S}_p(G)$ of a locally finite group G and then restates Brian Hartley's marvellous result on locally finite and locally p-soluble groups satisfying the strong Sylow Theorem for the prime $p \neq 2$. The lecture then uses **Lecture III** to deduce the following theorem: *If a locally finite group contains an infinitely descending sequence of normal p-perfect subgroups, it cannot satisfy the strong Sylow Theorem for the prime $p \geq 5$.*

Otto H. Kegel then analyses locally finite groups satisfying the strong Sylow Theorem for the prime p but being not p-soluble and gets the following result: *If the locally finite group G is not p-soluble and satisfies the strong Sylow Theorem for the prime $p \geq 5$, then the socle* **soc**(X) *of the factor group $X = G / \mathbf{S}_p(G)$ (i.e., the subgroup generated by all minimal normal subgroups of X) is the direct product of finitely many linear simple p-perfect subgroups. Also the centraliser $\underline{\underline{\mathbf{C}}}_X(\mathbf{soc}(X))$ is trivial.*

Together with Brian Hartley's result on locally finite and **locally p-soluble** groups with strong Sylow Theorem for the prime p ($\neq 2$) we now have the following really grandiose result by Otto H. Kegel on locally finite groups with strong Sylow Theorem for the prime p (≥ 5) **in general**:

(4.4) Theorem. *If the locally finite group G satisfies the strong Sylow Theorem for the prime $p \geq 5$, then there are characteristic subgroups*

$$\langle 1 \rangle \subseteq \mathbf{O}_p(G) \subseteq \mathbf{O}_{p,p'}(G) \subseteq \mathbf{O}_{p,p',p}(G) \subseteq \mathbf{S}_p(G) \subseteq S \subseteq A \subseteq P \subseteq G$$

such that $S/\mathbf{S}_p(G) = \mathrm{soc}(G/\mathbf{S}_p(G)$ is a direct product of finitely many locally finite simple linear groups, A/S is an abelian group of rank bounded by the number of simple direct factors of $S/\mathbf{S}_p(G)$, the factor group P/A is a finite soluble group of order bounded by the number and a function of the types of the simple direct factors of $S/\mathbf{S}_p(G)$, and the factor group G/P permutes these direct factors faithfully. If none of the characteristics of the underlying locally finite fields of the infinite simple direct factors of $S/\mathbf{S}_p(G)$ is p, then the factor group $G/\mathbf{O}_p(G)$ satisfies the minimum condition for p-subgroups. In any case, the factor group $G/\mathbf{O}_{p,p',p}(G)$ is countable.

Kegel's book "Locally Finite Groups" (see [4]), co-authored with Bertram A.F. Wehrfritz, is a well-known classical book on locally finite groups. As Brian Hartley mentions in the Zbl-review of the book, the purpose of writing the book was "*directed mainly towards the kind of material needed for the proof that locally finite groups satisfying the minimal condition are abelian-by-finite, but a brief and up-to-date account is given of recent work in many areas other than those treated in detail. In this way, and by means of its excellent bibliography, having* 16 *pages with* 323 *titles, the book gives a good picture of the present state of locally finite group theory.*" He also mentions that the book contains in its Chapter 1 "*a good survey of known results on maximal p-subgroups of locally finite groups*". Page 10 of [3] shows the dust jacket of the book containing a summary of its contents.

When the book was in press, Kegel developed the paper [5] which is based on the downright so magnificent concept mentioned on Page 1. Kegel not only had discovered during the eighties ("während der 80er Jahre"; see https://en.wikipedia.org/wiki/1980s) conditions for locally finite groups to satisfy the strong Sylow Theorem for the prime p (and presented them in June 9, 1987 in four fine lectures [6]) but already had presented such conditions, which are related to Sylow p-intersections, in lectures during December 11, 1973 [5]. Sequences of subgroups of a group G are either finite or countably infinite. The sequence ω_1 is *larger* than the sequence ω if and only if ω is an initial segment of ω_1. If p is a prime, the sequence $\{ S_i \}_{i \in \mathbb{N}}$ of p-subgroups of the group G is said to be *Sylow-separated* if there will exist a sequence $\{ P_i \}_{i \in \mathbb{N}}$ of Sylow p-subgroups of G (separating $\{ S_i \}_{i \in \mathbb{N}}$) so that one has $S_i \subseteq P_i \cap S_{i+1}$ and $S_{i+1} \not\subset P_i$ for all $i \in \mathbb{N}$. Clearly, there is an infinite Sylow-separated sequence of p-subgroups in the group G if and only if the set of Sylow-separated sequences of p-subgroups of G does not satisfy the maximum condition in the partial order defined above.

Kegel first proves that the existence of infinite Sylow-separated sequences is countably recognisable: *In the group G there exists an infinite Sylow-separated sequence of p-subgroups for the prime p if and only if there is a countable subgroup of G possessing an infinite Sylow-separated sequence of finitely generated p-subgroups.* He then relates the Sylow-separated sequences for the prime p to the Sylow p-Theorem: *If for the prime p and the subgroup S of the locally finite group G there are two Sylow p-subgroups of S which are not conjugate in S, then there exists an infinite Sylow-separated sequence of finite p-subgroups in G.* Using this result he proves: *If for the prime p every Sylow-separated sequence of p-subgroups of the locally finite group G is finite, then in every section of G any two Sylow p-subgroups of that section are conjugate.*

He then asks, if resp. when the converse of this result is valid: **Does the strong Sylow Theorem for the prime p force the maximum condition for Sylow-separated sequences of p-subgroups?**

Aforetime I describe his (partial) answer to this question, I present his famous concept of **Sylow p-intersections** and their relationship to p-uniqueness subgroups. If G is any locally finite group we define $\underline{\underline{\mathrm{Syl}}}_p G$:= set of all Sylow p-subgroups of G and $\underline{\underline{\mathrm{I}}}_p G := \{ S \cap T \mid S, T \in \underline{\underline{\mathrm{Syl}}}_p G \text{ with } S \neq T \}$. The elements of $\underline{\underline{\mathrm{Syl}}}_p G \cup \underline{\underline{\mathrm{I}}}_p G$ resp. of $\underline{\underline{\mathrm{I}}}_p G$ are called *Sylow p-intersections* resp. *proper Sylow p-intersections* of G. The set $\underline{\underline{\mathrm{I}}}_p G$ is partially ordered by inclusion and every ascending sequence of elements of $\underline{\underline{\mathrm{I}}}_p G$ will be a Sylow-separated sequence of p-subgroups of the group G. The maximum condition on Sylow-separated sequences of p-subgroups of G entails, of course, that $\underline{\underline{\mathrm{I}}}_p G$ satisfies the maximum condition. We now are considering two sets of subgroups of G: $\underline{\underline{\mathrm{I}}}_p^{\mathrm{e}} G := \{Q \subseteq G \mid Q$ is finite and is a with respect to inclusion maximal $\underline{\underline{\mathrm{I}}}_p G$-group$\}$ and $\underline{\underline{\mathrm{I}}}_p^{\mathrm{m}} \mathrm{G} := \{Q \subseteq G \mid Q$ is finite and is a with respect to order maximal $\underline{\underline{\mathrm{I}}}_p G$-group$\}$. Then we have $\underline{\underline{\mathrm{I}}}_p^{\mathrm{m}} G \subseteq \underline{\underline{\mathrm{I}}}_p^{\mathrm{e}} G$ and $\underline{\underline{\mathrm{I}}}_p G = \varnothing$ if and only if G is p-closed. We now define $\mathrm{i}_p = \mathrm{i}_p(G) \in \mathbb{N}_0$ as follows: if $\underline{\underline{\mathrm{I}}}_p^{\mathrm{m}} G = \varnothing$, let $\mathrm{i}_p := 0$, and if $\underline{\underline{\mathrm{I}}}_p^{\mathrm{m}} G \neq \varnothing$, let $p^{\mathrm{i}_p - 1} := |Q|$ for $Q \in \underline{\underline{\mathrm{I}}}_p^{\mathrm{m}} G$, that is, we let $\mathrm{i}_p - 1$ be the composition length of an arbitrary $\underline{\underline{\mathrm{I}}}_p^{\mathrm{m}} G$-group. i_p is a (numeric) Sylow p-invariant of G: if $Q_\mathrm{i} \in \underline{\underline{\mathrm{I}}}_p^{\mathrm{m}} G$ ($\mathrm{i} = 1, 2$) then there is a finite subgroup F of G containing Q_1 and Q_2; let $P_\mathrm{i} \in \underline{\underline{\mathrm{Syl}}}_p F$ with $Q_\mathrm{i} \subseteq P_\mathrm{i}$ ($\mathrm{i} = 1, 2$) and $x \in F$ with $P_2 = P_1^{\,x}$; then $Q_1^{\,x} \in \underline{\underline{\mathrm{I}}}_p G$ and thus $|Q_1| = |Q_1^{\,x}| \leq |Q_2|$; by symmetry $|Q_2| \leq |Q_1|$. We call i_p the *Sylow p-intersection* of G. Apparently $\mathrm{i}_p(G) = 1$ if and only if $S = T$ or $S \cap T = \langle 1 \rangle$ for all $S, T \in \underline{\underline{\mathrm{Syl}}}_p G$. In particular, if $\mathrm{i}_p(G) = 1$ then each Sylow p-subgroup of G is a *TI-set*, that is, either $S = S^x$ or $S \cap S^x = \langle 1 \rangle$ for all $x \in G$. The observation, that $\underline{\underline{\mathrm{I}}}_p^{\mathrm{e}} G \neq \varnothing$ implies the existence of so-called "localised" p-uniqueness subgroups of G, delivers the relationship between Sylow p-intersections and the p-uniqueness subgroups: let $Q \in \underline{\underline{\mathrm{I}}}_p^{\mathrm{e}} G$; then there are $S, T \in \underline{\underline{\mathrm{Syl}}}_p G$ such that $S \neq T$ and $Q = S \cap T$; if $x \in S \setminus Q$ then $\langle Q, x \rangle$ is a p-uniqueness subgroup of G *with respect to S*, and if $y \in T \setminus Q$ then $\langle Q, y \rangle$ is a p-uniqueness subgroup of G *with respect to T*. Therefore, if $\underline{\underline{\mathrm{I}}}_p^{\mathrm{e}} G \neq \varnothing$ then $\mathrm{a}_p(G)$ is defined (see Page 37 of [1]) and if $\underline{\underline{\mathrm{I}}}_p^{\mathrm{m}} G \neq \varnothing$ then $\mathrm{i}_p(G) \leq \mathrm{a}_p(G) + 1$.

Suppose the locally finite group G contains a finite Sylow p-subgroup S and let P be any finite p-subgroup of G; since $\langle S, P \rangle$ is finite, S contains a conjugate of P; thus $|S|$ is an upper bound for the orders of the finite p-subgroups of G; hence $T \in \underline{\underline{\mathrm{Syl}}}_p G$ has a maximal finite subgroup M; if $x \in T$ then $\langle M, x\rangle$ is finite and so $\langle M, x\rangle = M$; hence $T = M$ is finite; thus S and T are Sylow p-subgroups of the finite group $\langle S, T\rangle$ and so are conjugate; so G satisfies the strong Sylow Theorem for the prime p and all Sylow p-subgroups of G have order $|S| =: p^{\mathrm{b}_p(G)}$ and we have $\mathrm{i}_p(G) - 1 \leq \mathrm{a}_p(G) \leq \mathrm{b}_p(G)$. Hence in a countable locally finite group all Sylow p-subgroups have the same order. We can now summarise these findings about Sylow p-intersections of G: *The locally finite group G contains a finite Sylow p-subgroup* $\Rightarrow \underline{\underline{\mathrm{I}}}_p^{\mathrm{m}} G \neq \varnothing$ *and G satisfies the strong Sylow Theorem for the prime p* $\Rightarrow \underline{\underline{\mathrm{I}}}_p^{\mathrm{e}} G \neq \varnothing$ *and* $\mathrm{i}_p(G) - 1 \leq \mathrm{a}_p(G) \leq \mathrm{b}_p(G) \Rightarrow$ *G contains through* $\underline{\underline{\mathrm{I}}}_p^{\mathrm{e}} G$ *localised p-uniqueness subgroups.*

Kegel shows two fine theorems about $\underline{\underline{\mathrm{I}}}_p G$: *If* $\underline{\underline{\mathrm{I}}}_p G$ *satisfies the maximum condition and* $\underline{\underline{\mathrm{N}}}_S (S \cap T) \neq S \cap T \neq \underline{\underline{\mathrm{N}}}_T (S \cap T)$ *for all* $S \cap T \in \underline{\underline{\mathrm{I}}}_p G$ *, then G satisfies the Sylow Theorem for the prime p.* and *If* $\underline{\underline{\mathrm{I}}}_p G$ *satisfies the maximum condition and all p-subgroups of G satisfy the normaliser condition (that is, every proper subgroup is properly contained in its own normaliser) and G has a maximal p-subgroup which is countable, then G satisfies the strong Sylow Theorem for the prime p.* However, ***his premise of the existence of a countable maximal p-subgroup is dispensable***: let C be a countable subgroup of G and P be a good Sylow p-subgroup of C (see Page 15 of [1]); if the Sylow p-subgroups of C are not conjugate, there is one Q which is not conjugate to P; choose Q so that $P \cap Q$ is as large in $\underline{\underline{\mathrm{I}}}_p C$ as possible; one then can find elements $a \in \underline{\underline{\mathrm{N}}}_P (P \cap Q) \setminus P \cap Q$ and $b \in \underline{\underline{\mathrm{N}}}_Q (P \cap Q) \setminus P \cap Q$ and can consider the group $\langle a, b\rangle$; since it is finite there will be an element $g \in \langle a, b\rangle$ so that the group $\langle a, b^g\rangle$ is a p-group; then there is a Sylow p-subgroup X of C containing the p-group $\langle a, b^g\rangle(P \cap Q)$; also one has $Q^g \cap X \supsetneq P \cap Q$, and so the Sylow p-subgroup Q^g is conjugate to X; but this means that the Sylow p-subgroups P and Q of C are conjugate, contrary to assumption; so there is no such $P \cap Q \in \underline{\underline{\mathrm{I}}}_p C$, and two Sylow p-subgroups of C are conjugate.

We return now to the question of Page 8 and describe Otto H. Kegel's (partial) answer. Kegel shows first, spending a considerable effort: *If in the locally finite group G every* $\underline{\underline{\mathrm{I}}}_p G$*-group satisfies the minimum condition for subgroups, then G satisfies the strong Sylow Theorem for the prime p if and only if G satisfies the maximum condition for Sylow-separated sequences of p-subgroups.* Kegel then deduces rather easily for the locally finite group G

satisfying the minimum condition for p-subgroups equivalent conditions for satisfying the strong Sylow Theorem for the prime p, namely *the maximum condition for Sylow-separated sequences of p-subgroups* and *the maximum condition for* $\underline{\underline{I}}_p G$. Using Brian Hartley's tremendous result on the locally p-soluble locally finite groups with the strong Sylow p-Theorem he then obtains a corollary: *Let p be an odd prime and G be a locally finite group which is locally p-soluble, then G satisfies the strong Sylow Theorem for the prime p if and only if G satisfies the maximum condition on Sylow-separated sequences of p-subgroups.* He closes by asking the final question: **Does the maximum condition for $\underline{\underline{I}}_p G$ imply the strong Sylow Theorem for the prime p?**

I now summarise the **Open Issues** of Kegel's beautiful papers [5] and [6]. They are open since 1973 resp. since 1987. First the **Open Issues of [5]**:

- **Open Issue 1:** Does the strong Sylow Theorem for the prime p force the maximum condition for Sylow-separated sequences of p-subgroups?
- **Open Issue 2:** Does the maximum condition for $\underline{\underline{I}}_p$G imply the strong Sylow Theorem for the prime p?

For the **Open Issues of [6]** see [3] about their origin and solutions (*Ich habe eine* "**List of Open Issues**" *veröffentlicht, von denen ich die meisten (OI 3, OI 4, OI 5, OI 6, OI 7, OI 10, OI 11) in unveröffentlichten (draft and partly German) Arbeiten bereits gelöst habe.*):

- **Open Issue 3:** Extend Theorems 2, 3 and 4 of the JMCA-paper [2] to the remaining four Locally Finite Classical Groups (i.e., the Symplectic Groups, the Unitary Groups, the Orthogonal Groups in char ≠ 2 and the Orthogonal Groups in char 2).
- **Open Issue 4:** Summarise the work by **B. Hartley** and **A. Rae** regarding λ_p and p^{a_p} and the foregoing work on the classical Hall-Higman theory regarding λ_p and p^{b_p}, c_p, d_p, p^{e_p} and r_p by **P. Hall**, **G. Higman**, **A.H.M. Hoare**, **T.R. Berger**, **F. Gross**, **E.G. Bryukhanova** and **A. Turull**.
- **Open Issue 5:** Let p be a prime. Let G be a p-soluble finite group, $\lambda_p(G)$ be its p-length, and $a_p(G)$ be its p-uniqueness (see Page 37 of [1]). Then (best possible) $\lambda_p(G) \leq a_p(G) + 1$.
- **Open Issue 6:** Deduce the three rectangles/tableaux shown below and use them to prove **Lagrange**'s theorem and **Cauchy**'s concealed second and third group theorems:

complete right transversal for G in H	the **first row** consists of ***all*** elements z_k of G $(1 \le k \le \mathrm{M})$ acting on H in the **following rows** via multiplication from the left by their inverses				correspondence	$\mathrm{set}_H \mathrm{Orbi}(G) := G \backslash H$ of ***all*** orbits of H under G acting by left translation
$t_1 := 1 =: z_1$	z_2	z_3	…	z_M	↔	$G = {}_1\mathrm{Orb}(G)$
t_2	$z_2\, t_2$	$z_3\, t_2$	…	$z_\mathrm{M}\, t_2$	↔	$Gt_2 = {}_{t_2}\mathrm{Orb}(G)$
t_3	$z_2\, t_3$	$z_3\, t_3$	…	$z_\mathrm{M}\, t_3$	↔	$Gt_3 = {}_{t_3}\mathrm{Orb}(G)$
…	…	…	…	…	…	…
t_R	$z_2\, t_\mathrm{R}$	$z_3\, t_\mathrm{R}$	…	$z_\mathrm{M}\, t_\mathrm{R}$	↔	$Gt_\mathrm{R} = {}_{t_\mathrm{R}}\mathrm{Orb}(G)$

rectangle $|G| \times [H:G]$ of elements

set of ***certain*** orbits of H under G acting by left translation	the **first row** consists of ***all*** right cosets $G\boldsymbol{x_1}^k$ of G in H $(0 \le k \le p-1)$ with the powers of some p-blank $\boldsymbol{x_1}$ of G in H; the **following rows** consist of right cosets of G in H with the powers of left conjugates of $\boldsymbol{x_1}$				correspondence	$\boldsymbol{X} := \langle \boldsymbol{x_1} \rangle$; set of ***all*** orbits of H under G ↺ ↻ $\boldsymbol{X}$, the **simultaneous** actions of G by left translation and of $\boldsymbol{X}$ by right translation
$G\boldsymbol{x_1}^0\, t_1 = G$	$G\boldsymbol{x_1}$	$G\boldsymbol{x_1}^2$	…	$G\boldsymbol{x_1}^{p-1}$	↔	cosets $G \langle \boldsymbol{x_1} \rangle = G\boldsymbol{X}$ = double coset $G\,1\,\boldsymbol{X}$
$G\boldsymbol{x_2}^0\, t_2 = Gt_2$	$G\boldsymbol{x_2}\, t_2$	$G\boldsymbol{x_2}^2\, t_2$	…	$G\boldsymbol{x_2}^{p-1}\, t_2$	↔	cosets $G \langle \boldsymbol{x_2} \rangle t_2$ = double coset $G\, t_2\, \boldsymbol{X}$
$G\boldsymbol{x_3}^0\, t_3 = Gt_3$	$G\boldsymbol{x_3}\, t_3$	$G\boldsymbol{x_3}^2\, t_3$	…	$G\boldsymbol{x_3}^{p-1}\, t_3$	↔	cosets $G \langle \boldsymbol{x_3} \rangle t_3$ = double coset $G\, t_3\, \boldsymbol{X}$
…	…	…	…	…	…	…
$G\boldsymbol{x_\mathrm{S}}^0\, t_\mathrm{S}$	$G\boldsymbol{x_\mathrm{S}}\, t_\mathrm{S}$	$G\boldsymbol{x_\mathrm{S}}^2\, t_\mathrm{S}$	…	$G\boldsymbol{x_\mathrm{S}}^{p-1}\, t_\mathrm{S}$	↔	cosets $G \langle \boldsymbol{x_\mathrm{S}} \rangle t_\mathrm{S}$ = double coset $G\, t_\mathrm{S}\, \boldsymbol{X}$

tableau $p \times [H:G]\,/\,p$ of cosets

set of ***certain*** orbits of H under G acting by left translation	the **first row** consists of ***all*** right cosets $G\,\boldsymbol{x_{1c}}$ of G in H $(0 \le c \le \lvert H \rvert_p - 1)$ with the elements of some Sylow p-subgroup $\boldsymbol{X}$ of H, all of whose **elements of order** p are p-blanks of G in H; the **following rows** consist of right cosets of G in H with the elements of left conjugates of $\boldsymbol{X}$				correspondence	$\lvert \boldsymbol{X} \rvert = \lvert H \rvert_p = p^b$; set of ***all*** orbits of H under G ↺ ↻ $\boldsymbol{X}$, the **simultaneous** actions of G by left translation and of $\boldsymbol{X}$ by right translation
$G\, x_{10}\, t_1 = G$	$G\, \boldsymbol{x_{11}}$	$G\, \boldsymbol{x_{12}}$	…	$G\, \boldsymbol{x_{1\,p^b-1}}$	↔	cosets $G\, \{\boldsymbol{x_{1c}} \mid 0 \le c \le p^b - 1\}$ $= G\boldsymbol{X}$ = double coset $G\, 1\, \boldsymbol{X}$
$G\, x_{20}\, t_2 = G\, t_2$	$G\, \boldsymbol{x_{21}}\, t_2$	$G\, \boldsymbol{x_{22}}\, t_2$	…	$G\, \boldsymbol{x_{2\,p^b-1}}\, t_2$	↔	cosets $G\, \{\boldsymbol{x_{2c}} \mid 0 \le c \le p^b - 1\}\, t_2 =$ double coset $G\, t_2\, \boldsymbol{X}$
$G\, x_{30}\, t_3 = G\, t_3$	$G\, \boldsymbol{x_{31}}\, t_3$	$G\, \boldsymbol{x_{32}}\, t_3$	…	$G\, \boldsymbol{x_{3\,p^b-1}}\, t_3$	↔	cosets $G\, \{\boldsymbol{x_{3c}} \mid 0 \le c \le p^b - 1\}\, t_3 =$ double coset $G\, t_3\, \boldsymbol{X}$
…	…	…	…	…	…	…
$G\, x_{\mathrm{T}0}\, t_\mathrm{T} = G\, t_\mathrm{T}$	$G\, \boldsymbol{x_{\mathrm{T}1}}\, t_\mathrm{T}$	$G\, \boldsymbol{x_{\mathrm{T}2}}\, t_\mathrm{T}$	…	$G\, \boldsymbol{x_{\mathrm{T}\,p^b-1}}\, t_\mathrm{T}$	↔	cosets $G\, \{\boldsymbol{x_{\mathrm{T}c}} \mid 0 \le c \le p^b - 1\}\, t_\mathrm{T} =$ double coset $G\, t_\mathrm{T}\, \boldsymbol{X}$

rectangle $\lvert H \rvert_p \times [H:G]\,/\,\lvert H \rvert_p$ of cosets

- **Open Issue 7:** Deduce **Cauchy**'s fundamental theorem of 1812/1815 from $[H{:}\langle x\rangle] \geq |G|$, if x is a *p-blank of G in H*, that is, an element of H of order p with $x \notin G$.

- **Open Issue 8:** Determine all the (minimal) p-uniqueness subgroups for the known finite simple groups and their natural overgroups, the symmetric and the linear groups, and for the (locally) p-soluble groups, distinguishing $p \geq 5$, $p = 3$ and $p = 2$.

- **Open Issue 9:** Specify for a finite group the relationships of **the properties** of its p-subgroups which are **minimal** (w.r.t. order or w.r.t. inclusion) w.r.t. being contained in a unique **Sylow** p-subgroup to **the properties** (e.g., conjugacy) of its p-subgroups which are **maximal** w.r.t. being contained in a (unique) **Sylow** p-subgroup (i.e., its **Sylow** p-subgroups).

- **Open Issue 10:** Introduce for every prime p a generalised p-length for locally finite groups which is finite if and only if they satisfy the strong **Sylow** Theorem for p.

- **Open Issue 11:** Making a revision of **Kegel**'s (3.5) Theorem [6] thereby relating it to rarely known articles and extending it to $p \geq 3$, extend **Kegel**'s (4.4) Theorem [6] (see Page 6) to the case $p = 3$.

- **Open Issue 12:** Summarise **Kegel**'s wonderful **Sylow** paper [6] thereby integrating **first of all** the AGTA-paper (see https://www.advgrouptheory.com/journal/Volumes/13/Flemisch.pdf) and then **secondly** the JMCA-paper (see https://www.onlinescientificresearch.com/articles/the-strong-sylow-theorem-for-the-prime-p-in-simple-locally-finite-groups--de-luxe-edition.pdf) and extending **thirdly** his main Theorem (4.4) (see Page 6) to $p = 3$ and $p = 2$ by using the Open Issues 11 and 5.

Having proved and stated his main result (4.4) Theorem [6] (see Page 6), Kegel states the following: *Making a suitable definition for the p-length of the group* $S/\mathbf{S}_p(G)$ *– possibly simply the number of simple direct p-perfect factors – one gets that with this extended notion of p-length the locally finite group G satisfying the strong Sylow Theorem for the prime* $p \geq 5$ *will have finite p-length.* I have defined in unpublished (partly German) work for every prime p such a generalised p-length for locally finite groups which is finite if and only if they satisfy the strong Sylow Theorem for p. This is Open Issue 10. Kegel continues as follows: *It seems desirable to remove the restrictions on the prime p in the above result. This would mean finding an argument in the finite soluble case to prove a qualitative result like* (2.1) [this is Open Issue 5] *and finding a rather different argument to prove* (3.5) [this is Open Issue 11] *which might use different properties of finite simple groups.* I solved Open Issues 5 and 11 in unpublished work without using properties of simple groups.

The journal **Archiv der Mathematik** dedicates its Issue 1 of Volume 85 (July 2005) (see under https://link.springer.com/journal/13/volumes-and-issues/85-1 and https://link.springer.com/article/10.1007/s00013-005-0009-y) to Otto H. Kegel on the occasion of his **seventieth birthday on July 20, 2004**. It writes: *Dem Archiv der Mathematik ist Otto H. Kegel besonders verbunden. Zunächst als Autor, dann als ein langjähriger Herausgeber, dessen tatkräftiger Unterstützung sich die Redaktion stets sicher sein konnte, und anschließend als Beiratsmitglied bis 2002. … Als Mensch besitzt O.H. Kegel eine Eigenschaft, die nicht allzu viele Erfolgspersönlichkeiten auszeichnet: Es ist die Konstanz seines liebenswerten, freundlichen Charakters.*

Ten and a good half years later Kegel visited the Ischia Group Theory 2016 (IGT 2016) conference (see the beautiful portrait below) which took place from March 29, 2016 to April 2, 2016 in Italy at the beautiful Isola d'Ischia (see under https://en.wikipedia.org/wiki/Ischia) directly opposite of Napoli (see http://www.dipmat2.unisa.it/ischiagrouptheory/IGT2016/home_2016.html). He gave a talk entitled "REMARKS ON UNCOUNTABLE SIMPLE GROUPS", a topic which he discovered together with Philip Hall (see Page 4). It is presented in detail in my JMCA-paper [2] with references to the "known" finite simple groups according to the famous Classification of Finite Simple Groups (see https://en.wikipedia.org/wiki/Classification_of_finite_simple_groups).

Another eight years later the awaited Ischia Group Theory 2024 (IGT 2024) conference took place, again at the great Isola d'Ischia, from April 8, 2024 to April 13, 2024 with April 11, 2024 being the 120th birthday of Philip Hall, but Otto H. Kegel could not participate (see https://sites.google.com/unisa.it/igt/home and https://sites.google.com/unisa.it/igt/agenda). However, Otto H.

Kegel was very friendly honoured on the occasion of his **ninetieth birthday on July 20, 2024**. Mahmut Kuzucuoğlu showed **40 slides** entitled "**Selected Works of Otto H. Kegel**" (see https://drive.google.com/drive/folders/18NLK4hHZeU5km-hhImWNokBlFn6wImy8) and I gave a talk of **13 slides** entitled "**The Strong Sylow Theorem for the Prime *p* in Simple Locally Finite Groups**" (see https://drive.google.com/drive/folders/1rD92YXaJ5fs6VvK1SSNfXBpMK6uDjjtr and [2]).

Otto H. Kegel passed away on his birthday July 20, 2025 at the age of 91. Being in the deepest mourning, I miss him dreadfully and will always honour his memory .

Otto H. Kegel (Ischia, 2016) • courtesy of **Nikolay Aleksandrovich Vavilov** (see https://www.advgrouptheory.com/GTArchivum/Pictures/gtphotos/OttoKegel.jpg)

The journal **Archiv der Mathematik** issued a Call for Papers for a Special Issue "In memory of Otto Kegel" (see https://link.springer.com/journal/13/updates/: August 1st, 2025 – March 31st, 2026). It writes: *The* Archiv der Mathematik *will publish a beautiful Special Issue in memory of Otto Kegel (1934 – 2025). Professor Otto Kegel had strong ties to the journal. From 1975 – 1989 he was an editor of the* Archiv der Mathematik*, and over the years he contributed several articles to the journal with a continuous impact in group theory.*

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Reference [6]: Page iii and Page iv

Group Theory

Proceedings of the Singapore Group Theory Conference
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Reference [6]: MR-Review and Zbl-Review and Botticelli

90c:20037 **20 GROUP THEORY AND GENERALIZATIONS** **1452**

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Kegel, O. H. (D-FRBG)
Four lectures on Sylow theory in locally finite groups.
Group theory (*Singapore*, 1987), 3–27, *de Gruyter*, *Berlin-New York*, 1989.

It is an easy consequence of Zorn's lemma that if p is a prime and G an arbitrary group, then every p-subgroup of G is contained in a maximal one. However, unlike the situation for finite groups, there will usually be little relation between these maximal p-subgroups. This is true even if G is required to be locally finite, in which case, demanding the conjugacy of the maximal p-subgroups of G actually leads to strong restrictions on the structure of G. These lectures give an up-to-date survey of results along these lines, and in particular provide a proof of some important results of the author, announced some time ago.

For the rest of this review, all groups appearing are taken to be locally finite. The group G is said to satisfy the Sylow theorem for the prime p if the maximal p-subgroups of G are conjugate in G, and to satisfy the strong Sylow theorem for p if every subgroup of G satisfies the Sylow theorem for p. An important notion arising in this context is that of a singular p-subgroup; a finite p-subgroup P of G is called singular if for each finite $F \geq P$, P lies in a unique Sylow p-subgroup of F. If G satisfies the strong Sylow theorem for p, then G contains a finite singular p-subgroup (1.5). The existence of a singular p-subgroup of given order p^k in a finite group is already a restriction. For example if $p \neq 2$ and G is finite p-soluble, a result of A. Rae [J. London Math. Soc (2) **7** (1973), 117–123; MR **48** #420; corrigendum; MR **51** #13030] states that the p-length of G is at most $2^{k+2}-1$. This implies that if $p \neq 2$, then any locally p-soluble group satisfying the strong Sylow theorem for p has finite p-length.

Now it is known that any locally finite linear group satisfies the strong Sylow theorem for all p. One of the main theorems in these lectures is a partial converse: If the simple locally finite group satisfies the strong Sylow theorem for (even one) p, then G is linear. This depends on the classification of finite simple groups and an assertion about singular p-subgroups of classical groups. Another proof of this result has since been given by the reviewer (not yet published).

On the basis of the above results on simple groups and previous work on the locally p-soluble case, the author goes on to prove an impressive structure theorem for groups satisfying the strong Sylow theorem for a prime $p \geq 5$. Let $S_p(G)$ denote the largest normal locally p-soluble subgroup of G. From previously known results, $S_p(G)/O_p(G)$ satisfies the minimal condition on p-subgroups, and so $S_p(G)/O_{p,p',p}(G)$ is finite. Let $\overline{G} = G/S_p(G)$ and $\overline{S}$ be the socle of $\overline{G}$. The new information is that $\overline{S}$ is the direct product of a finite number of simple linear groups and $C_{\overline{G}}(\overline{S}) = 1$. Knowledge about the automorphism groups of simple linear groups then shows that $\overline{G}/\overline{S}$ is abelian-by-finite of rather restricted structure.

{For the entire collection see MR 89j:20001.}

B. Hartley (Manchester)

Kegel, O. H.
Four lectures on Sylow theory in locally finite groups. (English) Zbl 0659.20024
Group theory, Proc. Conf., Singapore 1987, 3-27 (1989).

[For the entire collection see Zbl 0652.00004.]

It is well known that the famous Sylow theorems for finite groups do not hold for locally finite groups in general. Also, if in a locally finite group G the maximal p-subgroups are conjugate for the prime p, this need not be the case for the maximal p-subgroups of a subgroup of G. A locally finite group G is said to satisfy the strong Sylow theorem for the prime p if in every subgroup of G the maximal p-subgroups are conjugate. It can be shown that this implies the validity of the strong Sylow theorem in every section of G.

The following theorem is proved. If the locally finite group G satisfies the strong Sylow theorem for the prime $p \geq 5$ then there exists a finite series of normal subgroups N_i of G leading from 1 to G such that the factors N_{i+1}/N_i are either direct products of finitely many linear simple groups or locally p-soluble. By a result of B. Hartley this series can even be refined in such a way that the locally p-soluble factors N_{i+1}/N_i are either p-groups or p'-groups. It seems to be unknown at present whether in this theorem the restriction on the prime p can be removed. - The proof of the above theorem requires a large number of facts about locally finite groups and especially about finite groups. In particular structure theorems for simple linear locally finite groups and finite simple groups are needed. These lectures explain the necessary arguments in detail and exhibit the flavour of the theory of locally finite groups very well.

Reviewer: B. Amberg

Alessandro di Mariano di Vanni Filipepi (* 1 March 1445 in Florence – † 17 May 1510 ibidem), better known as **Sandro Botticelli** (/ˌbɒtɪˈtʃɛli/ BOT-ih-CHEL-ee; Italian: [ˈsandro bottiˈtʃɛlli]) or simply Botticelli, was an Italian painter of the Early Renaissance ☺. Botticelli's posthumous reputation suffered until the late 19th century, when he was rediscovered by the Pre-Raphaelites who stimulated a reappraisal of his work. Since then, his paintings have been seen to represent the linear grace of late Italian Gothic and some great Early Renaissance painting, even though they date from the latter half of the Italian Renaissance period (see under https://en.wikipedia.org/wiki/Sandro_Botticelli and KUNSTKOPIE.DE https://www.kunstkopie.de/a/botticelli.html).

Sandro Botticelli: **Venere e Marte** (1482-1483) (see https://en.wikipedia.org/wiki/Venus_and_Mars_(Botticelli)) and **La nascita di Venere** (mid-1480s) (see https://en.wikipedia.org/wiki/The_Birth_of_Venus)

Reference [5]: Page 5 and Page 6

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Reference [5]: Page 7 and Page 199

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O. H. KEGEL

CHAIN CONDITIONS AND SYLOW'S THEOREM IN LOCALLY FINITE GROUPS

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CHAIN CONDITIONS AND SYLOW'S THEOREM IN LOCALLY FINITE GROUPS (*)

O. H. KEGEL

Introduction.

In this short note we introduce the notion of a Sylow-separated sequence of p-subgroups, p a prime, for a group G. If in the locally finite group G the maximum condition holds for these sequences of p-subgroups then Sylow's theorem for the prime p holds in the following form in the group G: In every subgroup S of G any two maximal p-subgroups of S are conjugate. We conjecture that in turn the validity of Sylow's theorem in this strong form for the prime p implies the maximum condition for the set of Sylow-separated sequences of p-subgroups of the locally finite group G. Unfortunately, we can prove this conjecture only for rather special groups G, in particular for locally finite groups satisfying the minimum condition for their p-subgroups. Still, the fact that in some situations a conjugacy theorem may be expressed by a chain condition seems to merit some interest. In this context we also discuss the set $\boldsymbol{I}_pG$ of all intersections of pairs of maximal p-subgroups of G. Every ascending sequence of elements of $\boldsymbol{I}_pG$ is Sylow-separated. In some cases the maximum condition on $\boldsymbol{I}_pG$ is equivalent to Sylow's theorem for the prime p in its strong form.

1. Sylow-separated sequences.

Sequences of subgroups of the group G will either be finite or have order type ω. We shall say that sequence Σ_1 is *larger* than the sequence Σ if and only if Σ is an initial segment of Σ_1. Sets of sequences of subgroups will be considered with this partial order. If p is a prime, the

(*) I risultati conseguiti in questo lavoro sono stati esposti nella conferenza tenuta l'11 dicembre 1973.

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Reference [5]: Page 258 and Page 259 and MR-Review and Zbl-Review

we have shown the existence of elements g_n as required in lemma 6. So this lemma yields the existence of $2^{\aleph_0}$ maximal p-subgroups in C in contradiction to the assumption that in the countable group C any two maximal p-subgroups are conjugate. This contradiction derives from the assumption that in C there exists an infinite Sylow-separated sequence of p-subgroups; hence this assumption cannot be upheld.

Observing that locally finite p-group satisfying the minimum condition are both countable and N-groups (in fact, hypercentral), one may combine theorem 8 with proposition 5 to obtain a rather satisfactory result for locally finite groups satisfying the minimum condition for p-subgroups.

COROLLARY 9: *Let p be a prime and G a locally finite group satisfying the minimum condition for p-subgroups, then the following three conditions are equivalent:*

a) *In every subgroup S of G any two maximal p-subgroups of S are conjugate;*

b) *In the group G there do not exist infinite Sylow-separated sequences of p-subgroups;*

c) *The set* $\boldsymbol{I}_pG$ *of intersections of pairs of maximal p-subgroups of G satisfies the maximum condition.*

In view of proposition 3 and the very strong result of Hartley [4] that in a locally finite group G the Sylow theorem in its strong form for the odd prime p forces that $G/\boldsymbol{O}_pG$ satisfies the minimum condition for p-subgroups if G is locally p-soluble, one obtains a further corollary.

COROLLARY 10: *Let p be an odd prime and G a locally finite group which is locally p-soluble, then the following conditions are equivalent:*

a) *In every subgroup S of G any two maximal p-subgroups of S are conjugate;*

b) *In the group G there do not exist infinite Sylow-separated sequences of p-subgroups.*

If in this corollary one replaces the structural assumption that G is locally p-soluble by the assumption that $G/\boldsymbol{O}_pG$ satisfies the minimum condition for p-subgroups, then all of Corollary 9 holds, but I do not see how to obtain—in the situation of Corollary 10—that the maximum condition on $\boldsymbol{I}_pG$ implies the strong form of Sylow's theorem, in fact, I do not see how to get any information on the structure of a maximal p-subgroup of G in that case.

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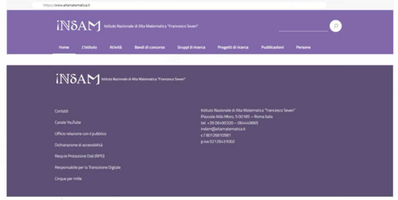
409 20 GROUP THEORY AND GENERALIZATIONS 54 ##2808–2814

Kegel, O. H. 2809
Chain conditions and Sylow's theorem in locally finite groups.
Symposia Mathematica, Vol. XVII (Convegno sui Gruppi Infiniti, INDAM, Roma, 1973), *pp.* 251–259. *Academic Press, London,* 1976.

If p is a prime, a sequence $\{S_i \mid i \in \mathbf{N}\}$ of p-groups of a group G is called Sylow-separated [Ss] if there is another sequence $\{P_i \mid i \in \mathbf{N}\}$ of maximal p-subgroups of G such that $S_i \subseteq P_i \cap S_{i+1}$ and $S_{i+1} \nsubseteq P_i$ for all $i \in \mathbf{N}$. In the first section of the paper conditions for the existence of infinite Ss-sequences of finitely generated or finite p-groups are given. These criteria are to be read in the light of the author's version of Sylow's theorem (8): Let G be a locally finite group such that for every pair P, Q of distinct maximal p-subgroups of G the intersection $P \cap Q$ satisfies the minimum condition for subgroups. In every subgroup S of G any two maximal p-subgroups are conjugate if and only if every Ss-sequence of p-subgroups of G has finite length.
{For the entire collection see MR **53** #5225.}
R. Göbel (Essen)



Kopie von subito e.V., geliefert für Flemisch (PVP08X01435)

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seems far from clear. The question of exactly which residually finite locally normal groups can be embedded in direct products of finite groups seems to be a very delicate one, and a satisfactory proof of some results such as those stated in this paper would be a very welcome contribution to it. B. Hartley.

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Kegel, O. H.:
Chain conditions and Sylow's theorem in locally finite groups.
Symp. math. 17, Anelli Funz. contin., Gruppi infin., Convegni 1973, 251-259 (1976).
[Dieser Artikel erschien in dem in diesem Zbl. 317.00007 angezeigten Sammelwerk.]
In dieser Note wird vorgeschlagen, die Gültigkeit des vollen Sylowsatzes für die Primzahl p als eine sehr starke Endlichkeitsbedingung in lokal endlichen Gruppen zu betrachten. Diese Betrachtungsweise wird untermauert durch einige Ergebnisse, die zeigen, daß die Gültigkeit des Sylowsatzes für die Primzahl p in lokal endlichen Gruppen aus aufsteigenden Kettenbedingungen für gewisse p-Untergruppen folgt. Insbesondere gilt der Sylowsatz für die Primzahl p in der lokal endlichen Gruppe G, in der die Minimalbedingung von p-Untergruppen erfüllt ist, genau dann, wenn die Menge der Durchschnitte von maximalen p-Untergruppen von G die Maximalbedingung erfüllt. Autorreferat.

Zbl 0328.20029 (https://zbmath.org/0328.20029; https://zbmath.org/pdf/03514049.pdf)

Note – I originally wanted to publish this article in the journal of **Advanced Group Theory and Applications** (AGTA) (see https:// www.advgrouptheory.com/journal/). However, **immediately (!) after submitting it**, the Editor-in-Chief sent an e-mail to me stating "Dear Author, following the editor's suggestion, I regret to inform you that your paper *Otto H. Kegel's Contributions* is not suitable for publication." and justifying this only (!) as follows: "We regret to inform you that your paper cannot be accepted for publication on AGTA as for we already have an official obituary describing the work of Otto Kegel." Moreover the Editor-in Chief replaced the Editor I had suggested **by himself (!)**. But I chose the Editor because he knows Sylow Theory which cannot be said about the Editor-in-Chief. I wrote a Reply to the Report: "The Editor-in-Chief claims that my paper is not suitable for publication following the Editor's suggestion. But the Editor did not even have time to read my paper let alone to reject it resp. to suggest a rejection. I chose the Editor since he knows Sylow Theory and has published about it. Also, replacement of the Editor by the Editor-in-Chief in my history is not correct. So I must ask the Editor-in-Chief for more honesty. The Editor did not suggest to reject my paper. There is the rather great misunderstanding that my paper is by no means meant to be a contribution to the official obituary describing the work of Otto H. Kegel. I doubt in any case that Kegel's work on Sylow Theory will be presented in the obituary such detailed as I wrote it but I am curious about it. Rather my paper is thought to be a supplement to Prof. Kuzucuoğlu's beautiful paper "Otto H. Kegel (1934 – 2025)" about life and work of Prof. Kegel being published in Volume 21 Issue 1 of the AGTA journal (see [7]). This is really needed since Prof. Kuzucuoğlu does not at all appreciate Prof. Kegel's contributions to Sylow Theory appropriately but mentions on his Page 5 only one of them and completely unsatisfactory. Thus my paper should be seen and published in conjunction with Prof. Kuzucuoğlu's paper. So please reset my paper from the state "rejected" to the previous state "Under Review" and send it to peer review." But there was not any answer at all by the Editor-in-Chief (nor by the Editor) … . I then tried to get this article published in the journal **Archiv der Mathematik** (see https://link.springer.com/journal/13). It was rejected without any reasonable rationale (!) … . They answered: "We very much appreciate your having allowed us to see your paper. We regret that it is not possible to accept it for publication. Due to the large number of manuscripts the Journal receives and the requisite high rejection rate (about 80%), all decisions regarding publications are final. As I am sure you are aware, the limited space in the journal prevents our including many excellent papers. Our decision, therefore, is not necessarily a reflection on your work." I replied as follows: "Ihre Hoffnung auf Verständnis kann ich nicht erfüllen. Zwar habe ich Verständnis für die Engpässe des Editorial Boards bei der Behandlung der großen Anzahl von Manuskripten, aber ich kann kein Verständnis haben für die Behandlung von Autoren wie sie mir widerfährt und wohl auch den meisten der ungefähr 80% abgelehnten Autoren widerfahren ist. Mein Papier, auf das ich stolz bin, war gerade einmal einen Tag beim*bei der Editor*in, konnte also nicht gelesen geschweige denn beurteilt werden. Ich habe mir (und durfte mir) ein ordentliches Review erwartet(*n), das ja mit einem Reject enden kann, aber wertvolle Erkenntnisse liefern würde. Ich habe mit viel Arbeit, aber auch mit viel Freude, das Papier entwickelt und ins Format des Arch.Math. gebracht. Diese viele Arbeit wird nicht gewürdigt. So geht es wohl auch den meisten anderen abgelehnten Autoren. Halten Sie es für sinnvoll, dass ich das Papier für das Special Issue "In memory of Otto Kegel" einreiche? Die Frage lässt sich allerdings wohl erst beantworten, wenn das Papier wenigstens diagonal gelesen wurde. Ich füge es daher noch einmal als Anhang bei." and got the following **sheerly incredible** answer: "Sie haben die Standard-Antwort erhalten für Artikel, die **ganz klar nicht geeignet sind für unser Journal**. Gerade für den Kegel Gedenkband rechne ich mit 50 exzellenten Artikeln mit neuen interessanten Resultaten aus denen die Editoren die besten (ca. 10) auswählen müssen." Therefore I had to ask "Bitte erklären Sie mir nachvollziehbar, warum mein Artikel "Otto H. Kegel's Beautiful Contributions to Sylow Theory in Locally Finite Groups" angeblich "ganz klar nicht geeignet ist für unser Journal", obwohl er doch ganz wichtige Errungenschaften von Prof. Otto Kegel, einem früheren Mitherausgeber des Arch.Math., zusammenfasst und v.a. wesentlich erweitert. Beim Special Issue "In memory of Otto Kegel" erwarten Sie offenbar dieselbe Ablehnungsrate von 80% wie sonst auch. Woher wissen Sie das heute schon? Nur 10 Beiträge für Kegel's Special Issue ist definitiv viel zu wenig. Es sollten doppelt so viele sein! Ich darf bescheiden darauf hinweisen, dass mein Artikel neue und m.E. sehr interessante Resultate enthält und die Sylow-Theorie in lokal endlichen Gruppen, eine bekanntlich große Herzensangelegenheit von Otto Kegel, ganz besonders würdigt. Ich habe den Artikel an das neueste Layout von Arch.Math.-Artikeln angepasst: siehe die Anlage. Die Items, die in Gold schattiert sind, müssen angepasst werden sobald der Artikel hoffentlich (nach sorgfältiger Revision) akzeptiert wurde." and added "Ich habe den Artikel korrigiert und verschönert: siehe die Anlage. Ich hoffe, dass er in der Special Issue "In memory of Otto Kegel" veröffentlicht werden kann, natürlich erst nach einer sehr sorgfältigen Revision, hier bin ich recht offen für gute Korrekturvorschläge. Das Editorial Office möge möglichst bald die Informationen mitteilen, die zum Update des Headers von Page 1 benötigt werden. Ich korrigiere dann Page 1 und die weiteren Seiten mit Goldschattierungen und mit Revisionsvorschlägen." The answer was "Wir haben Ihre beiden erneuten Einreichungen aus unserem System entfernt und bitten Sie, keine weiteren Modifizierungen Ihrer bereits abgelehnten Arbeit mit dem Titel ''Otto H. Kegel's Beautiful Contributions to Sylow Theory in Locally Finite Groups'' zu schicken. Eine weitere Bearbeitung würde kein anderes Ergebnis liefern, sondern lediglich Mehrarbeit verursachen. Bitte akzeptieren Sie nun unsere Entscheidung, dass **Ihre Arbeit zur Publikation im Archiv der Mathematik nicht geeignet ist**." Aber warum meine Arbeit **angeblich** "nicht zur Publikation geeignet ist" bzw. "ganz klar nicht geeignet ist für unser Journal" blieb unbeantwortet (!) … ☹. Eine beispiellos unwürdige Behandlung von Autoren!

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Turull, Alexandre [July 23, 1954 until today]
Wehrfritz, Bertram Arthur Frederick []

Tutzing am Starnberger See

Freitreppe mit flankierenden Löwen zum Seeufer in einer Parkanlage von Karl Effner aus der Zeit um 1870 mit **herrlichem Alpenblick**

Die Bayerischen Löwen am Midgardhaus in Tutzing

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