ISSN: 2754-6705

Journal of Mathematical & Computer Applications

SCIENTIFIC Research and Community

Review Article

Open Access

Three Beautiful Books about Sylow Theory in Locally Finite Groups

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ABSTRACT

This comprehensive review concerns the Trilogy about Sylow Theory in Locally Finite Groups which has been published in four books by Books on Demand (BoD), Norderstedt, Germany, namely: Part 1 - ISBN 978-3-7543-6087-3 (March 2023); Part 1 – Second edition - ISBN 978-3-7568-0801-4 (August 2023); Part 2 - ISBN 978-3-7543-3642-7 (August 2023); Part 3 - ISBN 978-3-7578-6001-1 (August 2023).

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Received: August 23, 2023; **Accepted:** August 31, 2023, **Published:** September 08, 2023

Keywords: locally finite group satisfying the Strong Sylow Theorem for the prime p (or the Strong Sylow p-Theorem); singular (Sylow) p-subgroup; (very) good Sylow p-subgroup; p-uniqueness subgroup; minimal (w.r.t. order) p-unique subgroup; (numerical) Sylow p-invariant a_p ; simple group; nested local system; family T of types of known finite simple groups; simple locally finite group of type $\Xi \in T$; rank of a locally finite simple group; P-invariant Sylow p-subgroup; Kegel cover; ★-sequence; Kegel sequence; conjugacy class; P-isomorphic P-orbit; p-length of a p-soluble finite group; Hall-Higman Theory; locally finite field F ; algebraic closure of the prime field in characteristic p ; General Linear Group; Special Linear Group; Projective Special Linear (PSL) Group; G-module over a (locally finite) field F ; irreducibility; complete reducibility; non-modular G-module; modular G-module; G-isomorphic G-modules; Jordan normal form; Classical Group; group of Lie type; twisted Chevalley group.

Introduction

The books are based on the author's Diplomarbeit [1] and continue his paper "Characterising Locally Finite Groups Satisfying the Strong Sylow Theorem for the Prime p" which appeared in Adv. Group Theory Appl. 13 (June 2022) on pp. 13-39 (see MR4441631 and Zbl 1496.20065). Since they contain a lot of material which cannot be published in a mathematical journal, they have been published as Books on Demand (BoD) (see https://www.bod.de/buchshop/ and https://www.bod.fr/librairie/). In the following we describe the mathematical subject matter. Regarding the non-mathematical contents, including dedications to famous or beloved persons, see the contained four recensions: Part 1, page v; Part 2, page i; Part 3, page i and page 42.

Part 1

Part 1, "Characterising Locally Finite Groups Satisfying the Strong Sylow Theorem for the Prime p" of the Trilogy [2] is based on the BoD-Book "Characterising locally finite groups satisfying the strong Sylow Theorem for the prime p – Revised edition" (see under ISBN 978-3-7562-3416-5) which in turn has been based on the author's quite pioneering research paper in the Volume 13 of Advances in Group Theory and Applications (AGTA) (see https://www.advgrouptheory.com/journal/Volumes/ 13/Flemisch.pdf and the Summary in Zbl 1496.20065). The First edition of Part 1 removes the highlights in light green of the Revised edition and adds the albeit very considerably improved Pages from i to vi and Pages from 27 to 34 to the AGTA-paper. In addition, Part 1 adds the ten new Pages from 35 to 44 to the Revised edition and therefore had to renumber the Pages from xv to xviii into the Pages from 45 to 48. It includes the second paper of Professor Otto H. Kegel about Sylow theory in locally finite groups (see MR981832 and Zbl 0659.20024) as Appendix 1 and his first paper on the beautiful subject (see MR414714 and Zbl 0328.20029) as Appendix 2. Finally, it calls to mind Kegel's fine contribution to the conference Ischia Group Theory 2016.

Most importantly, Part 1 introduces the basic concepts of the (First) Trilogy about Sylow Theory in Locally Finite Groups: locally finite group satisfying the Strong Sylow Theorem for the prime p (or the Strong Sylow p-Theorem); singular (Sylow) p-subgroup; (very) good Sylow p-subgroup; p-uniqueness subgroup; minimal (w.r.t. order) p-unique subgroup; (numerical) Sylow p-invariant ap ; P-invariant Sylow p-subgroup; P-isomorphic P-orbit.

The Second edition of Part 1 introduces happily a uniform page numbering, adds page numbers to the appendices, improves Pages iv and v, Page 22, Pages 25 to 34 and pages 39, 45, 49, 50, 75, 76, 105 and 106, adds Pages 109 to 112, and adds a two-page Table of Contents of the Trilogy. Part 1 has vi + 48 + 27 + 31 + 8 = 120 pages and includes 15 references according to the AGTA-paper followed by • a two-page Postscript about Kegel's Theorem/Conjecture on simple locally finite groups with the Strong Sylow Theorem for the prime p (see Part 2) and about the (First) Trilogy, by • the section "About the author", by • a Note on Kegel's Theorem/Conjecture, by • a fine section about "Sylow p-intersections and p-uniqueness subgroups", by • the rather illuminating section "Comparison of Kegel's two papers on Sylow theory in locally finite groups", by • the section "Five personal Notes about Mathematics", and finally followed by • the very insight gaining section "Augustin-Louis Cauchy's and Évariste Galois' contributions to Sylow theory in finite groups"

descri-bing and summarising a very pioneering follow-up paper. It also adds the section "Mathematical Reviews (MR) (MathSciNet) and zbMATH Open (Zentralblatt MATH)" for the AGTA-paper.

Part 2

Part 2, "About the Strong Sylow Theorem for the Prime *p* in Simple Locally Finite Groups" of the Trilogy [3] is based on Part 1. It first gives a profound overview of the structure of simple groups, and in particular of the simple locally finite groups, and reduces their Sylow theory for the prime *p* to an old (1987) conjecture of Professor Otto H. Kegel, contained in his second paper: "Let the *p*-subgroup *P* be a *p*-uniqueness subgroup in the finite simple group S which belongs to one of the seven rank-unbounded families. Then the rank of *S* is bounded in terms of *P*.", about the rank-unbounded ones of the known 19 families of finite simple groups.

Part 2 then introduces a new scheme to describe these 19 families, the family $T := \{$ abelian p , A^n , $A = PSL_{n,}$ $B = P\Omega_{\text{odd n,}}$ $C = PSp_n$, $D = P\Omega^{+}_{\text{even n,}}$ ${}^2A = PSU_n$ ${}^2D = P\Omega^{-}_{\text{even n,}}$ E_6 , E_7 , E_8 , F_4, G_2 , ${}^2B_2$, ${}^3D_4$, ${}^2E_6$, ${}^2F_4$, ${}^2G_2$, sporadic★ $\}$ of types, by using well- known symbols, defines the rank of each type, and emphasises the rôle of Kegel covers. This introductory part presents a unified picture of well-known results with proofs only by reference which is the reason why the title of Part 2 starts with "About".

Subsequently Part 2 applies new ideas to prove the conjecture for the alternating groups. A considerably thorough abstract of this quite pioneering approach is presented on page ii.

Thereupon Part 2 remembers the central rôle played by Kegel covers and ★-sequences. Finally, it presents a plan how to prove and even how to optimise the conjecture step-by-step or peu à peu by suggesting a way 1) and a way 2) which leads to further tough conjectures thereby happily unifying Sylow theory in locally finite simple groups with Sylow theory in locally finite and *p*-soluble groups being the work by adored professores Otto H. Kegel and Brian Hartley (see page 27 of Part 1). Part 2 has iv + 20 = 24 pages and includes 24 references and an overview by 12 references as well as two figures of the very famous classification of the finite simple groups. For any unexplained terminology it refers to Part 1.

Part 3

The Part 3, "The Strong Sylow Theorem for the Prime p in Projective Special Linear Locally Finite Groups" of the Trilogy [4] continues the program begun in Part 2 to optimise along the way 1), the Theorem there about the first type Ξ = "A_n" of infinite families of finite simple groups step-by-step to further types by proving it for the second type Ξ = "$A = PSL_n$". It first proves the Conjecture 2 from Part 2 about the General Linear Groups over (commutative) locally finite fields, stating that their rank is bounded in terms of their *p*-uniqueness, and then breaks down this insight to the Special Linear Groups and to the Projective Special Linear (PSL) Groups over locally finite fields. A thorough abstract of this three-stage-approach with a description of the new ideas regarding the proof for the General Linear Groups and with a figure showing the transitions to the Special Linear Groups and the PSL Groups is presented on page ii.

Part 3 closes with suggestions for future research • regarding the five remaining rank-unbounded types (the "Classical Groups") and the way 2), • regarding the (locally) finite and p-soluble groups by announcing a first proof of Conjecture 3 from Part 2, thereby somehow rounding off the venerable classical Hall-Higman theory, and thirdly • regarding Augustin-Louis Cauchy's and Évariste Galois' pioneering contributions to Sylow theory in finite groups showing three very beautiful rectangles / tableaux. These three suggestions culminate in the announcement of the Second Trilogy about Sylow Theory in Locally Finite Groups. Part 3 has iv + 44 = 48 pages and includes 32 references followed by • an insightful Postscript about • the Alternating Groups, ► the PSL Groups, ► the First Trilogy and ► the 19 families of finite simple groups, and finally followed by • the beautiful four-page section "A Contribution to the Mathematical Theory of Big Game Hunting". Its last page 42 shows the First Trilogy as a beautiful triptych (see https://en.wikipedia.org/wiki/Triptych and page 49 of Part 1 with an overview of three other triptychs).

Acknowledgements

The author is sincerely very grateful to the regrettably unknown peer referees for their corrections, suggestions and such friendly adjuvant advice which improved the manuscript considerably.

He wishes to thank so very heartfeltly his truly most fabulous wife / spouse Helga 💐. Without her tenderest and unconditional support and her love and greatest patience over many many years the reviewed publications would never have been born.

Most importantly, he is forever and ever grateful to Prof. Brian Hartley and to his teacher Prof. Otto H. Kegel (see page 27, page 30 and page 34 of Part 1) for their papers which provide incredible insights and give marvellous pleasure in reading and understanding the great Sylow theory of both (locally) *p*-soluble and simple (locally) finite groups.

Conflict of Interest

The author declares gently that there are no competing personal or organisational or financial conflicts of interest with this very original work or other conflicts of interest regarding the publication of this meticulous review paper.

which shepherds the Trilogy as a flock along all abysses (see the covers of [2], [3] and [4]).

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