ISSN: 3033-3652

Journal of Physical Mathematics & its Applications

SCIENTIFIC Research and Community

Research Article

Open Access

A Comprehensive Exploration of Quantum Calculus: Extending Classical Calculus to Discrete Frameworks

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ABSTRACT

This manuscript explores quantum calculus (q-calculus), a framework that extends classical calculus to discrete settings without the use of limits. Beginning with a historical overview, it outlines the motivations and benefits of q-calculus across various fields. Key concepts, such as q-analogs and q-integers, are discussed with a focus on their application to Mersenne numbers in number theory. The paper also examines the q-derivative, its relevance to financial models, and how q-analogs enhance predictive capabilities. Additionally, it delves into q-integrals and their connection to Riemann-Stieltjes functions, extending the integral framework to discrete cases. The study concludes by linking q-calculus to quantum computing, highlighting recent advancements and identifying gaps in current research, aiming to underscore its potential role in shaping future quantum technologies.

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Received: October 21, 2025; **Accepted:** October 27, 2025; **Published:** November 14, 2025

Keywords: Quantum Calculus, Q-Analogs, Mersenne Numbers, Riemann-Stieltjes Functions, and Quantum Computing

Introduction

Quantum calculus, or calculus without limits, is a framework for looking at alternative forms of calculus. It looks at the more general notions of calculus on discrete or finite sets, rather than smooth, continuous changes. Among the quantum calculus branches, q-calculus and h-calculus are presented to derive "analogues" of the mathematical objects that tend to their classical counterparts within specific limits. In the case of q-calculus, the limit is taken while the parameter q tends to 1, whereas for h-calculus, the limit as the parameter h tends to 0 is considered. Both parameters, q and h, are connected through the following expression: $q = e^h$.

q-Analogs appear quite naturally over a very wide range of mathematical discipline, from combinatorics to special functions, number theory, representation theory and all the way to algebraic geometry, so the q-calculus is a very versatile and powerful tool. In this aspect, while the h-calculus has its applications in finite difference calculus and numerical analysis, it happens to be much more specialized to the discretization for linear approximation, such as numerical methods for differential equations. It lacks the wide range of connections and broad utility that appears for different mathematical disciplines, as with q-calculus.

The history of q-calculus dates back to the work of Bernouilli, Leonhard Euler, and Carl Gustav Jacobi [1,2]. It is a broad area of research in mathematics with deep historical roots and renewed relevance in modern times. Due to its vast applications in several areas, it is gaining the attention of modern mathematicians over the past several years. Lately, q-calculus has rapidly gained attraction in fields like mathematics, mechanics, and physics. Its applications span diverse topics, from quantum mechanics and analytic number theory to special functions, hypergeometric functions, and finite differences. It also connects to gamma function theory, Bernoulli and Euler polynomials, combinatorics, Sobolev spaces, and operator theory. More recently, it has been applied to the geometric theory of analytic and harmonic univalent functions, orthogonal polynomials, statistical mechanics, coding theory, and solving difference equations, showcasing its growing theoretical and practical importance.

Q-Analog

A q-analog is a mathematical object dependent on a parameter q, intended as a q-analog of its classical continuous counterpart. The defining feature of a q-analog is when the parameter q approaches 1, the q-analog converges to the original object in classical calculus. This approach is particularly useful in discrete settings, like combinatorics and number theory; allowing for generalizations of functions, derivatives, and integrals that retain some structural properties of their continuous forms. The idea is to explore how properties of mathematical structures change under discrete transformations controlled by q.

The introduction of the q-parameter allows mathematicians to study deformations of traditional structures and symmetries, providing a pathway to understanding other extensions of classical spaces [3].

The theory of quantum calculus starts by defining the q-analog of nonnegative integer n (q-integer) and eventually the q-analog of factorial (q-factorial).

Q-integers and the Mersenne Numbers

In traditional calculus, the difference is given by $f(x+\Delta x) - f(x)$. By bringing a q-perturbation such as $f(qx) - f(x)$, we are able to calculate the q-difference in the q-calculus.

With the help of this expression, we calculate q-derivatives in the following manner:

$$D_q f = \frac{f(qx) - f(x)}{qx - x}$$

When q → 1, we retain the classical calculus.

Now, let us consider a function $f(x) = x^n$ and differentiate it using the q-derivative formula.

$$D_q x^n = \frac{(qx)^n - x^n}{qx - x} = \frac{q^n - 1}{q - 1} x^{n-1}$$

From traditional calculus, we know that the derivative of $f(x) = x^n$ is nx^{n-1}

Comparing the two derivatives we define q-integers as: $[n]_q = \frac{q^n - 1}{q - 1}$

From here we tread into the world of *Quantum Calculus*. q-Analogs occur naturally in various contexts and q-integers and q-derivatives become the basis to study them. For example, q-analogs are most commonly explored in the mathematical areas of combinatorics and special functions.

As a consequence of q-integers, we can compute q-factorials.

$$[n]_q! = [n]_q \cdot [n]_{q-1} \cdot [n]_{q-1} \cdots [3]_q \cdot [2]_q \cdot [1]_q$$

q-Factorials are used to define q-binomial coefficients also called *Gaussian Polynomials.*

$$\frac{n}{k_q} = \frac{[n]_q!}{[n-k]_q! \, [k]_q!}$$

Gaussian polynomials count the number of k-dimensional subspaces of an n-dimensional vector space over a finite field with q elements. It is widely used in enumerative theory and coding theory [4]. Exploring these is beyond the scope of this exposition.

The Mersenne numbers

Another interesting derivation from q-integers is the case when $q = 2$.

$$q = 2: [n]_2 = \frac{2^n - 1}{2 - 1} = 2^n - 1$$

These are known as the Mersenne numbers. Among these occur Mersenne Primes. The numbers are named after Marin Mersenne (1588–1648), a French Minim friar who researched them in the early 17th century.

Mersenne primes are one of the most important constituents of number theory research. Their properties, concerning the structure of perfect numbers, brought a good deal of historical connections to prominent mathematical concepts. Marin Mersenne developed the science of measurement, named these primes Mersenne primes. In the 4th century BC, Euclid demonstrated that if 2^n-1 is prime, then 2^n-1(2^n-1) is a perfect number, a later extension in the 18th century by Leonhard Euler. The Euclid–Euler theorem proves that all of the even perfect numbers are given by Mersenne primes, but to this day it is not known if there are any odd perfect numbers. The study of Mersenne primes in number theory has also established that they are also very important in cryptography for secure encryption of data with the help of their mathematical properties [5]. For example,

Quick Primality Test

Mersenne primes have some historical interest in the development of algorithms for testing primality. And perhaps the most effective test for determining if a number, represented as 2^{p-1} is prime is the Lucas-Lehmer test, tailored to Mersenne primes. The algorithm is useful in cryptographic applications such as RSA key generation. In RSA key generation the efficiency of the testing algorithm for prime numbers plays a significant role in generating secure keys.

Potential Use in Post-Quantum Cryptography

Mersenne primes are currently used to optimize classical algorithms like (ECC) Elliptic Curve Cryptography. ECC over Mersenne primes is known for its computational efficiency. This efficiency allows for faster key exchanges, signature generation, and encryption/decryption processes. While they don't directly address quantum vulnerabilities, ongoing research is investigating their potential in post-quantum cryptography. The hardware and software optimizations derived from using Mersenne primes can serve as a foundation for developing more efficient post-quantum cryptographic systems.

q-Derivatives and Their Application in Finance Models

q-Derivative or Jackson derivative is the q-analog of ordinary derivative and is defined as:

$$\left(\frac{d}{dx}\right)_q f(x) = \frac{f(qx) - f(x)}{qx - x}$$

It is denoted by $D_q f(x)$. In 1908, Frank Hilton Jackson reintroduced this q-difference operator and hence it is also called *Jackson derivative.*

q-Derivatives are involved in q-difference equations. A q-difference equation is a discrete analog of a differential equation representing the case of discrete intervals. These equations form a basic part of q-calculus, generalize classical differential equations, and find applications in many areas of mathematics and physics, particularly where discrete or q-deformed structures are concerned. In settings where quantities are discrete or deformed in a non-linear way implementing the q-difference equation becomes convenient and more effective.

q-Derivatives are deeply connected with special functions such as basic hypergeometric functions, q-orthogonal functions, q-Bessel Functions, q-exponential functions, and quantum harmonic oscillator, quantum groups and lattice systems. These functions, groups, and models have broad and diverse applications across many areas of mathematics, physics, engineering, and other fields. They occur naturally in the solutions of q-difference equations.

q-Derivatives, especially when combined with quantum algorithms, have been applied across various fields that use discrete models to improve computational efficiency and accuracy. For example, finance, optimization, and machine learning. Let's explore its application in finance.

Application of q-Derivatives in Finance

Direct applications include computing sensitivities for better risk management and more accurate pricing. The quantum algorithms speed up the simulations and gradient estimations which are necessary to price complex derivatives and optimize portfolios

[6,7]. These advantages find significant applications in high dimensional, discrete models where classical methods struggle because of non-scalability due to resource constraint.

From the mathematical perspective, quantum algorithms such as Quantum Amplitude Estimation (QAE) improve the productivity of pricing complex financial derivatives through the use of principles of quantum mechanics, especially in Monte Carlo simulations [8].

In classical Monte Carlo methods, to estimate the price P of a financial derivative (e.g., options, futures) with a given accuracy ϵ, the the number of paths required is $O(1/\epsilon^2)$. This is because the classical Monte Carlo simulation balances as: ERROR $\propto \frac{1}{\sqrt{N}}$, where N is the number of Monte Carlo paths or samples.

In contrast, QAE can reduce the number of required simulations by a quadratic factor, scaling as $O(1/\epsilon)$. Mathematically, this improvement is described as: ERROR $\propto \frac{1}{\sqrt{N}}$, meaning that for the same accuracy ϵ, using q-derivatives with QAE can achieve the result with a quadratic speedup compared to classical Monte Carlo [9].

This means that, for a high-accuracy derivative price, quantum algorithms may drastically reduce the number of needed simulations, thereby enhancing the speed of pricing complex derivatives.

A closely related and very important financial application is the calculation of sensitivity of derivative prices to model and market parameters, these sensitivities are called *greeks*. They play a crucial role in financial markets by managing risk.

For risk management, the greeks (e.g., Delta, Gamma, Vega) are sensitivities that measure the rate of change in a derivative's price with respect to various parameters such as the underlying asset price (S) or volatility (σ). Let P be the price then,

- Delta $\Delta = \frac{\delta P}{\delta S}$
- Gamma $\Gamma = \frac{\delta^2 P}{\delta S^2}$
- Vega $\nu = \frac{\delta P}{\delta \sigma}$

In classical methods, calculating these greeks often involves using finite differences, where you slightly perturb the input (e.g., the underlying price or volatility) and observe how the price changes. For instance, to compute Delta, you would price the derivative at two slightly different values of the underlying asset S and subtract the results:

$$\Delta \approx \frac{P(S+\Delta S)-P(S)}{\Delta S};\ P = \frac{1}{N}\sum_{i=1}^{N} f(S_i)$$

where N is the number of paths, and $f(S_i)$ represents the payoff function evaluated for each simulated price path S_i. This is a case of the classical Monte Carlo method.

However, this requires multiple derivative price evaluations, making it computationally expensive, especially for high-dimensional models or portfolios with many assets.

Quantum gradient algorithms, a broader part of quantum computing, offer faster and efficient computation of these greeks. While one has to vary the inputs and recalculate this derivative price many times, quantum algorithms would estimate the gradient directly. This would be especially useful for complex derivatives such as *exotic options*, where classical finite-difference methods may require many recalculation procedures.

In classical method we have,

$$\Delta \approx \frac{P(S + \Delta S) - P(S)}{\Delta S}$$

This involves calculating the price at both S and $S + \Delta S$, with each scenario needing a separate Monte Carlo simulation.

In quantum method we get, $\Delta \propto$ Quantum Gradient Estimation on $f(S_i)$.

The mathematical advantage of quantum gradient algorithms is that they can calculate these sensitivities in fewer steps by calculating the gradient directly, than that is required for conventional algorithms to reduce the computational effort needed for measuring the risk, or greeks. Quantum algorithms achieve this by exploiting the principles of quantum superposition and entanglement, which allows multiple price evaluations to occur simultaneously, thereby greatly speeding up the process.

This integration of q-derivatives into QAE enables faster, more efficient pricing and risk analysis for complex financial instruments, providing a clear quantum advantage over classical methods.

q-Integrals and its Connection with the Riemann–Stieltjes integral

q-Integrals, also known as *Jackson Integral* or q-antiderivatives are used to extend classical integral definitions by incorporating a parameter q, which helps in defining integrals over discrete sets.

The basic q-integral on a real interval is defined as follows for a continuous function $f(x)$:

$$\int_x^q f(x)\, d_q x = (1-q)\sum_{j=0}^{\infty} xq^j f(xq^j)$$

The Riemann-Stieltjes integral is a generalization of the Riemann integral, where the integrator is a function rather than the traditional dx measure. For two functions $f(x)$ and $g(x)$, the Riemann-Stieltjes integral over an interval [a, b] is:

$$\int_a^b f(x)\, dg(x).$$

If the function $g(x)$ has jumps or discontinuities, the classical Reimann integral fails to operate. Hence, Riemann-Stieltjes integrals are powerful while we are dealing with jumps and discontinuities. To explore this property, we can illustrate it with the integrator being a step-function.

Step Function: A step function on an interval [a,b] is a function β: $[a,b] \to$R that satisfies the following conditions:

- It has finitely many points of discontinuities:

$$a \le s_1 < s_2 < \cdots < s_n \le b$$

- It is constant on subintervals: $[a,s_1),(s_1,s_2),(s_2,s_3),\cdot\cdot,(s_{n-1}),s_n)$ and $(sn,b]$

Now, if β defined above has discontinuities at $s_1,s_2,\cdots,s_n$, where $a \le s_1$ and $s_n \le b$ and there is a function f: $[a,b] \to R$ continuous at each s_j; $1 \le j \le n$, then it can be easily proved that f belongs to Reimann-Steiltjes integrals on [a,b] and,

$$\int_a^b f \, d\beta = \sum_{j=1}^{n} f(s_j)[\beta(b) - \beta(a)]$$

We will now show that, q-integral is a Riemann–Stieltjes integral with respect to a step function having infinitely many points of increase at the points q^j.

Proof: Let's define a step function $g_q(t)$ in a way that it has jumps at the points q^j, where (j=1,2,3,⋯). $g_q(t)$ can be expressed as:

$$g_q(t) = \begin{cases} 0; t < 1 \\ q^j; t \in [q^j, q^{j+1}) \end{cases}$$

This means that at each point q^j, there is a jump of si z.e. q^j.

Notice that, q-integral is given by: $\int_0^x f(t)\, d_q t = (1-q) \sum_{j=0}^{\infty} q^j f(q^j)$, and Riemann–Stieltjes integral with respect to a step function defined by $g_q(t)$ is expressed as:

$$\int_0^x f(t)\, dg_q(t), \text{ where } \Delta g(t) = g(q^j) - g(q^{j-1}) = q^j.$$

This implies, $dg_q(t) = d_q t$ at each point $t = q^j$, representing a jump of size q^j.

We can now express the q-integral using the Riemann–Stieltjes framework:

$$\int_0^x f(t)\, d_q x = \int_0^x f(t)\, dg_q(t)$$

This completes the proof that the q-integral is a Riemann–Stieltjes integral with respect to the step function $g_q(t)$. This relationship highlights how discrete calculus concepts extend traditional integration frameworks into discrete domains.

We have observed that by introducing a certain type of q-perturbation into the classic Riemann–Stieltjes integral results into an equivalent q-integral [10]. From here we can establish the connection between the two concluding that q-integrals are indeed the q-analogs of Riemann–Stieltjes integrals.

q-Riemann-Stieltjes integrals is a versatile tool especially important in fields like stochastic processes and financial mathematics [11]. We have already explored how integration of q-derivatives into QAE enables faster, more efficient pricing and risk analysis for complex financial instruments. Similarly, q-integrals are the mathematical framework that can be applied in quantum computation techniques adapted by QAE, where it could be more accurate and faster for financial computations in scenarios where the other numerical methods fail to handle discrete jumps and non-smooth functions–common characteristics in financial models where asset prices, market conditions, or risk factors can shift abruptly.

Other applications are in q-series, Lie algebras and groups, number theory, orthogonal polynomials, quantum physics, etc.

Quantum Computing

Quantum computing is a field focused on developing computer technology based on quantum theory, which explains the behavior of matter and energy at atomic and subatomic levels. To fully grasp quantum computation, one must understand key mathematical concepts like probability theory, quantum calculus, and linear algebra.

Quantum calculus is crucial because it provides a comprehensive framework that unifies the fundamental operations used in quantum computing, such as unitary transformations and projective measurements. These operations form the backbone of quantum information processing in various quantum computing models, such as the quantum gate array and the one-way quantum computer. Quantum calculus allows for a more flexible computational approach by supporting not only unitary transformations but also multi-qubit projective measurements, and it integrates classical control mechanisms like loops and conditional structures. This versatility is vital for accommodating different quantum computing paradigms, including measurement-only models where projective measurements guide the entire computational process [12]. By enabling a unified and generalized model, quantum calculus simplifies the design and analysis of quantum algorithms, making it a fundamental tool for tackling complex quantum operations efficiently.

Conclusion

Recent advancements in quantum computing have led to significant breakthroughs, especially in the manipulation of qubits, the development of quantum algorithms, and quantum cryptography. These innovations have shown great potential in solving complex problems, particularly in fields such as climate modeling, drug discovery, finance, etc. For example, quantum optimization algorithms have opened new avenues in solving intricate computational problems more efficiently than classical systems.

However, there remain large gaps in the current research despite all these improvements. Quantum error correction is an important problem and remains so; indeed, the most sophisticated quantum systems are highly prone to errors through the environment [13]. Decoherence is what degrades the coherence of the quantum states-that is to say, what diminishes the loss of information-and therefore limits the stability and practicality of quantum systems. In consequence, the full power of quantum computing, especially that of attaining quantum supremacy-quantum systems that perform calculations beyond the capabilities of their classical counterparts-continues to fall short.

Meeting these challenges will be key to enabling the translation of quantum technologies into real-world applications. Research has shifted from relying mainly on the development of reliable qubits, better quantum gates, and more efficient error correction methodologies. In addition, next-generation more scalable and robust architectures of qubits need to be developed for the progress of this field.

China has already surpassed other countries in quantum supremacy by developing a quantum computer that is 10 billion times more effective than Google's quantum computer [14]. It would not be surprising if China became the first country to achieve quantum supremacy with billions of dollars spent on quantum technology. While Google's quantum computer is built using super cold superconducting metal, China has used a technology manipulating photons [15]. This implies that there can be different ways in which quantum computers can be built with improvisations.

The improvisations could indeed be enhanced by quantum calculus. The comprehensive framework of quantum calculus that combines unitary transformations and projective measurements are essential for error correction, tackling decoherence, and scalable quantum systems ensuring that the system can handle both classical and quantum components simultaneously. This mathematical consistency provides researchers with the opportunity to explore numerous hardware implementations and then adapt their algorithms appropriately, further driving innovation in the field.

Quantum computing holds immense potential to revolutionize computation, and quantum calculus is a very important tool in making that possible.

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