ISSN: 3033-3652

Journal of Physical Mathematics & its Applications

SCIENTIFIC Research and Community

Review Article

Open Access

On Self-Adjoint Boundary Value Problems for a Functional Equation of Even Order

Elena Vladimirovna Molchanova Alves[1]*, Manuel Joaquim Alves[2] and Sergey Labovskiy[3]

[1]PhD in Physics and Mathematics, Associate Professor of the School of Economy and Business Administration. Higher Institute of Sciences and Technology, Mozambique

[2]PhD in Physics and Mathematics, Full Professor of the Mathematics and Informatics Department. Eduardo Mondlane University, Mozambique

[3]PhD in Physics and Mathematics, Associate Professor of Plehanov University of Economy, Russia

ABSTRACT
Investigating a functional differential equation of even order, we explore a class of self-adjoint boundary value problems with two-point conditions. We establish the basis property of the system of eigenfunctions and compare the eigenvalues.

***Corresponding author**
Elena Vladimirovna Molchanova Alves, PhD in Physics and Mathematics, Associate Professor of the School of Economy and Business Administration. Higher Institute of Sciences and Technology, Mozambique. ORCID: https//orcid.org/0009-0000-1452-2553.

Received: October 10, 2023; **Accepted:** October 18, 2023; **Published:** October 27, 2023

Problem, Notation
Introduction
As is known, the spectral problem Lu = λu for a differential operator of nth order has a real spectrum and a complete orthogonal system of eigenvectors in the case of a self-adjoint operator and corresponding self-adjoint conditions. In was considered self-adjoint functional differential Sturm-Liouville boundary value problem [1,2]

$$-(pu')' + \int_0^l u(s)d_s r(x,s) = \rho f,$$

$$(pu')\big|_{x=0} = 0,\ ku(l) + pu'\big|_{x=l} = 0$$

($\rho(x)$ is a positive weight function). It can be considered as a generalization of the Sturm-Liouville boundary value problem for a string, the system of its eigen functions forms an orthogonal basis in two spaces. In were considered generalizations for equations of even order [3,4]

$$(-1)^m u^{(2m)} - \int_0^l u(s)q(x,ds) = \rho f, \quad x \in [0,l] \quad (m \geq 1).$$

with boundary conditions

$$u^{(k)}(0) = 0,\ k = 0,\dots,m-1,$$
$$u^{(k)}(l) = 0,\ k = m,\dots,2m-1.$$

These conditions are a special case of the boundary conditions considered in [5] and [6]. Usually such a boundary value problem is called focal [7]. A boundary value problem with periodic boundary conditions is also self-adjoint [8]. These problems are regular, the spectrum of such operators is real and discrete, the eigenfunctions are orthogonal and form a basis in the corresponding spaces. Taking this opportunity, we note that in for the singular operator [9]

$$\frac{1}{\rho(x)}(-1)^m(p(x)u^{(m)})^{(m)},\ x \in [0,\infty)$$

necessary and sufficient conditions for the discreteness of the spectrum are obtained.

This paper describes a class of two-point boundary conditions under which the boundary value problem operator remains self-adjoint (self-adjoint extension). For any such problem, the above properties remain the same. We consider the operator (symbol ':=' is read as is equal by definition)

$$\mathcal{L}u(x) := \frac{1}{\rho(x)}\left((-1)^m u^{(2m)} + \int_0^l u(s)r(x,ds)\right),$$

$$x \in [0,l] \quad (m \geq 1). \qquad (1.1)$$

Assumptions regarding the function r(x, ds) are given below. We write the boundary conditions in the form

$$B_S u = 0, \qquad (1.2)$$

where the operator of boundary conditions B_S is defined in section of notations in (1.7).

Notations and assumptions

The integral in (1.1) is the Stieltjes integral, instead of r(x, ds) it can be written dsr(x, s). In the case of r(x, ds), the function r(x, ·) is considered as a charge (depending on the parameter x). If the charge is discrete

$$\int_0^l u(s)r(x,ds) = \sum_{i=1}^{\infty} r_i(x)u(h_i(x)),$$

we obtain an equation with a deviating argument. Let us introduce the following definitions and assumptions.

- ':=' *means equal by definition*, $\Delta := [0, l]$, ≡ means *is equivalent* for measurable functions.
- $L_2(\Delta, \rho)$ is the space of functions whose square is Lebesgue integrable with positive and measurable weight $\rho(x)$ on Δ and scalar product

$$(f,g) := \int_\Delta f(x)g(x)\rho(x)\,dx, \qquad (1.3)$$

$L_2(\Delta) := L_2(\Delta, 1)$. We assume that $\int_\Delta \rho(x)\,dx < \infty$ (regularity condition).

- For almost all $x \in \Delta$, the function $r(x, \cdot)$ has limited variation on Δ,

for any $s \in \Delta$ function $r(\cdot, s)$ measurable on Δ, $r(x, 0) = 0$; total variation $|r|(x) := \operatorname*{var}_{s\in\Delta} r(x,s)$ is Lebesgue integrable. It is known that $r = r_+ - r_-$, where r_+ and r_- are non-decreasable functions, and $|r|(x) = r_+(x, l) + r_-(x, l)$, $r_+(x, 0) = r_-(x, 0) = 0$.

Let $\xi_+(x,y) := \int_0^x r_+(s,y)\,ds,\ \xi_-(x,y) := \int_0^x r_-(s,y)\,ds,\ \xi = \xi_+ - \xi_-.$

Assume that ξ is symmetrical $\xi(y, x) = \xi(x, y)$. Then ξ_+ and ξ_- are symmetrical.

- AC^k $(k \geq 0)$ is the set of functions u, that have absolutely

continuous on Δ derivative $u^{(k)}$, $u^{(0)} := u$.

- $[u, v]$ is bilinear form defined on AC^{m-1} by equality

$$[u,v] := \int_\Delta u^{(m)}v^{(m)}\,dx + \int_\Delta v(x)\,dx \int_\Delta u(s)d_s r(x,s).$$

The symmetry of the function ξ implies the symmetry of $[u, v]$, since by Lemma 3.12

$$[u,v] = \int_\Delta u^{(m)}v^{(m)}\,dx + \int_{\Delta\times\Delta} u(s)v(x)d\xi. \qquad (1.4)$$

Assume that $[u, u] > 0$ if $u \not\equiv 0$. To do this we require that

$$\int_{\Delta\times\Delta} u(s)u(x)d\xi > 0 \qquad (1.5)$$

if $u \not\equiv 0$.

- W is Hilbert space (Lemma 3.1) of functions from AC^{m-1} satisfying $[u, u] < \infty$, with scalar product $[u, v]$.
- W_S: Let the index sets $I, J \subset \{0, 1, \ldots, m-1\}$ be arbitrary, $S = (I, J)$ and

$$W_S = \{u \in W : u^{(i)}(0) = 0,\ \forall i \in I,\ u^{(j)}(l) = 0,\ \forall j \in J\}. \qquad (1.6)$$

Obviously, W_S is a (closed) subspace of W. In particular, $W = W_S$ if $S = (\emptyset, \emptyset)$. On W_S we will consider the form [u, v] defined by the equality

- $T : W \to L_2(\Delta, \rho)$ is the (set-theoretic embedding) operator defined by the equality

$$Tu(x) := u(x),\ x \in \Delta.$$

The definition is correct and T is continuous (Lemma 3.2).

- T^* is the adjoint operator to T ($T^* : L_2(\Delta, \rho) \to W$), T^*s is the adjoint operator to T ($T^* : L_2(\Delta, \rho) \to W_S$).
- Operator $B_S : AC^{2m-1} \to R^{2m}$ is linear operator of boundary conditions. Define for $S = (I, J)$ the index sets

$$U := \{2m-1-i : i \notin I, i \in \{0,1,\ldots,m-1\}\},$$
$$V := \{2m-1-i : i \notin J, i \in \{0,1,\ldots,m-1\}\}.$$

We use all these indices to form the boundary conditions operator:

$$B_S u := (u^{(i)}(0) : i \in I \cup U, u^{(k)}(l) : k \in J \cup V). \qquad (1.7)$$

For example, if $m = 3$, $I = \{0, 2\}$, $J = \{2\}$, then $S = (0, 2; 2)$, $U = \{4\}$, $V = \{4, 5\}$,

$$B_S u = (u(0), u''(0), u^{(4)}(0), u''(l), u^{(4)}(l), u^{(5)}(l)).$$

G_S is the Green's operator of the boundary value problem $Lu = f$, $B_S u = 0$, that is, $u = G_S f$, if the problem is uniquely solvable.

The Green's operator G_S has an integral representation (see, for example, [4])

$$u(x) = \int_0^l G_S(x,s)f(s)\rho(s)\,ds,$$

The Green's function $G_S(x, s)$ is symmetric i.e. $G_S(x, s) = G_S(s, x)$. For each s, the section $G_S(\cdot, s)$ is a solution to the problem $Lu = 0$, $B_S u = 0$ (but a jump in the $(2m-1)$th derivative is allowed).

***Remark* 1.1.** A special case of the form $[u, v]$ is the obviously positive form defined by the bilinear form $(q \geq\not\equiv 0,\ \xi_1 \geq 0)$

$$[u,v] = \int_\Delta u^{(m)}v^{(m)}\,dx + \int_\Delta q(x)u(x)v(x)\,dx + \frac{1}{2}\int_{\Delta\times\Delta}(u(x) - u(s))(v(x)-v(s))d\xi_1. \qquad (1.8)$$

It can be represented in the form

$$[u,v] := \int_\Delta u^{(m)}v^{(m)}\,dx + \int_\Delta R(x)u(x)v(x)\,dx - \int_\Delta v(x)\,dx \int_\Delta u(s)d_s r_1(x,s).$$

$R(x) \geq r_1(x)$, $r_1(x) := r_1(x,l) - r_1(x,0) = r_1(x,\Delta)$, and

$$\int_\Delta (R(x) - r_1(x))\,dx > 0.$$

Investigation of the Spectral Problem

We follow the scheme in [3]. The difference is that we will consider the form $[u, v]$ in two spaces: in W and its closed subspace W_S, considered as an independent Hilbert space. Therefore, what is given below for W is also true for W_S.

Minimizing the functional $(1/2)[u,u] - (f, Tu)$, $u \in W$

leads to an equation in variational form

$$[u, v] = (f, Tv),\ \forall v \in W,\ f \in L_2(\Delta, \rho). \tag{2.1}$$

The operator T is bounded (Lemma 3.2), so the functional v 7↦ (f, Tv) is bounded, and the equation (2.1) has a unique solution $u = T^* f \in W$.

Since the image of $T(W)$ is dense in $L_2(\Delta, \rho)$ (Lemma 3.5), then [8, Lemma 5.1] the adjoint operator T^* is an injection. It follows that the inverse operator

$$\widetilde{\mathcal{L}} := (T^*)^{-1} \tag{2.2}$$

is defined on the image $R(T^*) := T^*(L_2(\Delta, \rho))$. Let us denote the domain of Let us denote the domain of definition of the operator $\widetilde{\mathcal{L}}$ by $D(\widetilde{\mathcal{L}}) := R(T^*)$. The spectrum of the operator $\widetilde{\mathcal{L}}$ is determined by the spectral problem $\widetilde{\mathcal{L}}u = \lambda T u$, i.e. by esolvent $(\widetilde{\mathcal{L}} - \lambda T)^{-1}$. This problem is equivalent to $u = \lambda T^* T u$. The operators T and T^* are compact (Lemma 3.6), therefore the spectrum of the operator $\widetilde{\mathcal{L}}$ is discrete, real and positive. The above is also true for the restriction of the form [u, v] to the subspace W_S. We denote the corresponding L operator by $\widetilde{\mathcal{L}}_S := (T_S^*)^{-1}$.

It should be noted that the operators T^* and T_s^* are different because they are associated with T defined in different domains W and W_S.

Theorem 2.1. *The spectral problem* $\widetilde{\mathcal{L}}_S u = \lambda T u$ *has a complete and orthogonal system of eigenfunctions in* W_S (as well as in $L_2(\Delta, \rho)$):

$\widetilde{\mathcal{L}}_S u_k = \lambda_k T u_k$, *k= 0, 1, 2, ... The eigenvalues are positive and have a single condensation point* $+\infty$, *i.e.* $0 < \lambda_0 \leq \lambda_1 \leq \cdots$, *and* $\lambda_k \to \infty$. *For* λ *different from the eigenvalues, the equation e* $Lu - \lambda T_u = f$ *has a unique solution in* W_S *for any* $f \in L_2(\Delta, \rho)$.

As follows from the statement below, the operator L defined by the equality (1.1) gives a representation of the operator $\widetilde{\mathcal{L}}_S$ defined on the set W_S.

Theorem 2.2. The operator defined by the equality (2.2) in the domain $D(\widetilde{\mathcal{L}}_S) := R(T_S^*) \subset W_S$ coincides with the operator $\mathcal{L}$ defined by the equality (1.1). Equation

$$(-1)^m u^{(2m)} + \int_0^l u(s) r(x, ds) = \rho f, \quad x \in [0, l] \tag{2.3}$$

is equivalent on $D(\widetilde{\mathcal{L}}_S)$ to equation (2.1).

Proof. It follows from Lemma 3.10.

Corollary 2.1. Boundary value problem (2.3), $B_S u = 0$ is uniquely solvable in $D(\widetilde{\mathcal{L}}_S)$ for any $f \in L_2(\Delta, \rho)$.

Remark 2.1. The form (1.8) corresponds to the functional differential operator

$$\mathcal{L}_1 u(x) := \frac{1}{\rho(x)} \left((-1)^m u^{(2m)} + R(x) u - \int_0^l u(s) r_1(x, ds) \right), \quad x \in [0, l]. \tag{2.4}$$

If we consider the equation

$$[u, v] = (f, Tv),\ u \in W,\ \forall v \in W_S, \tag{2.5}$$

then in a wider space W one can obtain nonzero solutions to the homogeneous equation $Lu = 0$. In WS this equation has a unique solution $u_0 = T_S^* f$. Let $e1, \ldots, en$ be a basis in the orthogonal complement of $W_S^\perp$. Here n is the total number of elements of the sets I and J, where $S = (I, J)$. Then the general solution to the equation (2.5) has the form

$$u = T_S^* f + \sum_{k=1}^n c_k e_k.$$

The orthogonal complement of $W_S^\perp$. is the null space of L, i.e. $Le_k = 0$, $k = 1, \ldots, n$.

Let us note an important estimate for the first eigenvalue (depending on S)

$$\lambda_0 = \inf_{u \in W_S, u \neq 0} \frac{[u,u]}{(Tu, Tu)} = \inf_{u \in W_S, u \neq 0} \frac{[u,u]}{\int_\Delta u^2 \rho\, dx}. \tag{2.6}$$

From this estimate we obtain a comparison of the first eigenvalue for different boundary conditions. Let $S_1 = (I_1, J_1)$, $S_2 = (I_2, J_2)$, and λ_{01}, λ_{02} be the corresponding first eigenvalues. If $I_1 \subset I_2$, $J_1 \subset J_2$, then

$$\lambda_{01} \leq \lambda_{02}.$$

Auxiliary Lemmas. Properties of Space and Operators

Space W. Boundedness and compactness of T

Lemma 3.1. *The space W with scalar product [u, v] is Hilbert.*

Proof: We need to show that every fundamental sequence un in W converges (in W). Let us represent $u_n(x)$ as the sum $u_{0n}(x) + p_n(x)$, where

$$u_{0n}(x) = \int_0^x \frac{(x-s)^{m-1}}{(m-1)!} u_n^{(m)}(s)\, ds,$$

and $pn(x)$ is polynom of degree at most $m-1$. From (1.4) it follows that $u_n^{(m)}$ is fundamental in $L_2(\Delta)$. Therefore $u_n^{(m)} \to \varphi \in L_2(\Delta)$.

This implies convergence

$$u_{0n}(x) \to \int_0^x \frac{(x-s)^{m-1}}{(m-1)!}\varphi(s)\,ds.$$

This convergence is not only uniform, but also in W. The sequence pn is fundamental in W. Due to the condition (1.5) pn converges on a set of positive measure. But pn is a polynomial of degree at most $m-1$, so pn converges in the sense of W.

Lemma 3.2. *The operator T acts from W to $L_2(\Delta, \rho)$ and is bounded.*

Proof: The statement follows from the estimate

$$(Tu, Tu) = \int_0^l \left(\int_0^x \frac{(x-s)^{m-1}}{(m-1)!}u^{(m)}(s)\,ds\right)^2 \rho(x)\,dx$$

$$\leq \int_0^l \rho(x)dx \int_0^x \left(\frac{(x-s)^{m-1}}{(m-1)!}\right)^2 ds \int_0^x (u^{(m)}(s))^2\,ds$$

$$\leq [u,u] \int_0^l \rho(x)\,dx \int_0^x \left(\frac{(x-s)^{m-1}}{(m-1)!}\right)^2 ds.$$

The next two lemmas have an auxiliary role.

Lemma 3.3. *Let* $\int_\Delta f(x)h(x)\,dx = 0$ *for any* $h \in L_2(\Delta)$ such that

$$\int_\Delta (c_1 + c_2x + \cdots + c_mx^{m-1})h(x)\,dx = 0 \ (\forall c_1, \ldots, c_m). \qquad (3.1)$$

Then $\exists c_1, \ldots, c_m$ such that

$$f(x) = c_1 + c_2x + \cdots + c_mx^{m-1}.$$

Lemma 3.4. $\int_\Delta f(x)v^{(m)}(x)\,dx = 0$ for any $v \in AC^{m-1}$ such that $v^{(m)} \in L_2(\Delta)$, and satisfying the equalities

$$v(0) = \ldots = v^{(m-1)}(0) = v(l) = \ldots = v^{(m-1)}(l) = 0. \qquad (3.2)$$

Then $\exists c_1, \ldots, c_m$ such that

$$f(x) = c_1 + c_2x + \cdots + c_mx^{m-1}.$$

Proof: We use Lemma 3.3. Any h satisfying the condition (3.1) can be represented as $h = v^{(m)}$, where v satisfies the conditions of our lemma. Indeed, as such a function we need to take

$$v(x) := \int_0^x \frac{(x-s)^{m-1}}{(m-1)!}h(s)\,ds.$$

From (3.1) it follows (3.2).

Since $\int_\Delta f(x)h(x)\,dx = \int_\Delta f(x)v^{(m)}(x)\,dx = 0$, the statement of our lemma follows from Lemma 3.3.

Lemma 3.5. The images $T(W_S)$ and $T(W)$ are dense in $L_2(\Delta, \rho)$.
Proof: It is enough to consider $T(W_S)$. Suppose that the closure $T(W_S)$ does not coincide with $L_2(\Delta, \rho)$. Then there exists $h \in L_2(\Delta, \rho)$ orthogonal to $T(W_S)$, i.e.

$$(\forall u \in W_S) \ \int_0^l h(x)u(x)\rho(x)\,dx = 0.$$

Using the integration by parts

$$\int_0^l hu\rho\,dx = \sum_{k=1}^m (-1)^{k-1}H^{(m-k)}u^{(k-1)}\Big|_0^l + (-1)^m\int_0^l H(x)u^{(m)}(x)\,dx,$$

where $H^{(m)} = h\rho$. We will assume that u satisfies (3.2). Then

$$0 = \int_0^l H(x)u^{(m)}(x)\,dx.$$

From Lemma 3.4 $H = c_1 + c_2x + \cdots + c_mx^{m-1}$. Therefore $h = H^{(m)}/\rho \equiv 0.$

Lemma 3.6. The operator T is compact.

Proof: We refer, for example, to the work [3], which establishes (very simply, by the way) compactness for a form containing only the term $\int_\Delta u^{(m)}v^{(m)}\,dx$. This will also imply compactness in the case of the form (1.4). There is a difference due to the boundary conditions, but they are finite-dimensional and do not affect compactness.

Euler's equation
The next two statements are obtained by integration by parts.
Lemma 3.7. Let $u^{(2m-1)}$ be absolutely continuous. Then

$$\int_0^l u^{(m)}v^{(m)}\,dx = \sum_{i=1}^m (-1)^{i-1}u^{(m+i-1)}v^{(m-i)}\Big|_0^l + (-1)^m\int_0^l u^{(2m)}v\,dx. \qquad (3.3)$$

Lemma 3.8. Let φ be Lebesgue integrable on $[0, l]$ and let the function v have an absolutely continuous derivative of order $m-1$. Then

$$\int_0^l \varphi v\,dx = \sum_{i=0}^{m-1}(-1)^iF^{(m-1-i)}v^{(i)}\Big|_0^l + (-1)^m\int_0^l F(x)v^{(m)}\,dx, \qquad (3.4)$$

where $F^{(m)} = \varphi$.

Lemma 3.9. Let $f \in L_2(\Delta, \rho)$, $u, v \in W$,

$$\int_\Delta u^{(m)}v^{(m)}\,dx + \int_\Delta v(x)\,dx\int_\Delta u(s)d_sr(x,s) = \int_\Delta fv\rho\,dx. \qquad (3.5)$$

Then

$$\int_\Delta \left(u^{(m)} - (-1)^mF\right)v^{(m)}\,dx = \sum_{i=0}^{m-1}(-1)^iF^{(m-1-i)}v^{(i)}\Big|_0^l, \qquad (3.6)$$

where $\varphi := -\int_\Delta u(s)d_sr(\cdot,s) + f\rho.$

Proof: The product $f\rho$ is integrable on Δ, since

$$\left(\int_0^l f\rho\,dx\right)^2 \leq \int_0^l f^2\rho\,dx\int_0^l \rho\,dx.$$

The equation (3.5) will take the form

$$\int_\Delta u^{(m)}v^{(m)}\,dx = \int_\Delta \varphi v\,dx. \qquad (3.7)$$

Substituting (3.4) into (3.7) we get (3.6).

The equality (3.6) can be simplified if we choose a function F that satisfies, for example, the conditions $F(0) = \ldots = F^{(m-1)}(0) = 0$. However, this is not necessary for our purpose.

Lemma 3.10 (Euler's equation). Let $f \in L_2(\Delta, \rho)$, and the function $u \in W_S$ be a solution to (3.5) for any $v \in W_S$.

Then $u \in AC^{2m-1}$ is a solution to the equation $Lu = f$, where L is defined by the equality (1.1), and satisfies the boundary conditions (1.2).

Proof: Function u satisfies (3.6), WS contains the set specified in lemma 3.4, so from (3.6), (3.2)

$$u^{(m)} - (-1)^m F = c_1 + c_2 x + \ldots + c_m x^{m-1}.$$

This implies the existence of $u^{(2m)}$ and the equality

$(-1)^m u^{(2m)} = F^{(m)} = \varphi.$ And this means equality

$$\mathcal{L}u = f,$$

where L is defined by (1.1). From (3.3) and (3.5)

$$(\forall v \in W_S)\ \sum_{i=1}^{m} (-1)^{i-1} u^{(m+i-1)} v^{(m-i)} \Big|_0^l = 0. \qquad (3.8)$$

From here it follows (1.2).

One Generalization of Fubini's Theorem

For completeness of presentation and ease of reading, we present the transformation of the integral we need (see also [3]).

Lemma 3.11 ([11]). Let (X,A) and (Y, B) be measurable spaces, μ be a measure in (X,A), K colon$X \times B \to [0,\infty]$ is core (i.e. for μ–almost all $x \in X$ $K(x, \cdot)$ it is a measure in (Y, B), $\forall B \in B$ $K(\cdot,B)$ is *μ–measurable* in X) .

Then

1. *the function v defined in $A \times B$ by equality*

$$\nu(E) = \int_X K(x, E_x)\mu(dx), \quad E_x = \{y\colon (x, y) \in E\},$$

is a measure,

2. if f : $X \times Y \to [-\infty,\infty]$ ν–measurable in $X \times Y$, then

$$\int\limits_{X\times Y} f(x,y)d\nu = \int\limits_X \left(\int\limits_Y f(x,y)K(x,dy) \right) \mu(dx).$$

Remark 3.1. The function v is a Lebesgue extension of the set of rectangles

$$\nu(A \times B) = \int_A K(x, B)\mu(dx), \quad A \in \mathcal{A}, B \in \mathcal{B},$$

Let us now apply this statement to our problem (in the notation of section 1.2). In our case $X = Y = \Delta$, the measure μ coincides with the Lebesgue measure.

Lemma 3.12. Let $f(x, y)$ be a ξ-measurable function. Under the assumptions of section 1.2 the following equality is valid:

$$\int_{\Delta\times\Delta} f(x, s)\, d\xi = \int_\Delta dx \int_\Delta f(x, s) r(x, ds). \qquad (3.9)$$

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