ISSN: 3033-3652

Journal of Physical Mathematics & its Applications

Review Article

Open Access

On the Convergence of Multi-Stage Differential Transform Method

Khalil Al Ahmad*, Farah Aini Abdullah and Zarita Zainuddin

School of Mathematical Sciences University Sains Malaysia 11800 USM Penang, Malaysia.

ABSTRACT

In many paper, Multi-stage Differential Transform Method (MsDTM) has been presented as a method to solve linear and nonlinear equations of various kinds. Multi-stage Differential Transform method has been used to find the exact or analytical approximate solution of the different problems. But only a few works have been considered of convergence of this method. In this paper, Convergence analysis of multi-stage differential transform method is presented. Some numerical examples are proposed to verify the convergence of the proposed method.

***Corresponding author**

Khalil Al Ahmad, School of Mathematical Sciences University Sains Malaysia 11800 USM Penang, Malaysia.

Received: December 19, 2023; **Accepted:** December 21, 2023; **Published:** January 11, 2024

Keywords: Convergence Analysis, Linear Systems, Nonlinaer Systems, Ordinary Differential Equations, Differential Transform Method, Multi-Stage Differential Transform Method.

Introduction

Ordinary differential systems have attracted much attention. It is mostly due to many applications in physics, engineering, biology and other fields represented as a system of ordinary differential equations [1-4]. Most of ordinary differential systems do not have analytical exact solution, this yield to use approximations and numerical techniques. Some of methods are series solution method which involve Adomian Decomposition Method (ADM),Homotopy perturbation Method (HPM), Differential Transform Method (DTM) and multi-stage Differential Transform Method(MsDTM) [5-10]. Differential Transform Method (DTM) was first presented by J Zhou during his researches on electrical circuits. This method is utilized to solve linear and nonlinear systems of ordinary differential equations [1,9,11-12]. The Differential Transform Method (DTM) has been successfully implemented to extend to partial, fractional, and algebraic differential equations [13-15]. The extended application of the DTM is due to it is applied directly without linearzation, discretization or perturbation transform [16]. On the other hand, DTM is still suffering of some drawbacks. One of them, It converges slightly over small time intervals [12-13,17]. To Avoid this issue, the Multi-stage Differential Transform Method (MsDTM) is utilized to enlarge the convergence range [10, 18]. DTM and MsDTM are implemented to linear and nonlinear systems of one or two Ordinary Differential Equations (ODEs) [17,2,19]. In this study, the new technique is continuity of the researchers' works to find general technique to enlarge the time interval of convergence of approximate solution series as a result to solve linear and nonlinear systems of several ordinary differential equations or more based on DTM and MsDTM. The new technique is applied to nonlinear systems of several ordinary differential equations. The aim of this study is proofing the series of the approximate solution is convergent. results show that the convergence of the series of approximate solution is proved by analytical and numerical methods. the numerical examples are used to verify the series solution is convergent. The paper is organized as follows. In section 2, some basic definitions and properties of DTM are given. In section 3, definition of MsDTM is given. the proposed new technique to solve system of several nonlinear ODEs is given in section 4. In section 5, convergence analysis of MsDTM is given, in addition to numerical examples in section 6. Conclusion remarks are presented in section 7.

Differential Transform Method (DTM)

Definition 1: If a function $u(t)$ is analytical with respect to t in the domain of interest, then [9, 20]

$$U(k)=\frac{1}{k!}\left[\frac{d^k u(t)}{dt^k}\right]_{t=t_0}, \tag{1}$$

is the transformed function of $u(t)$.

Definition 2: The differential inverse transforms of the set $\{U(k)\}_{k=0}^{n}$ is defined by [9, 20]

$$u(t)=\sum_{k=0}^{\infty} U(k)(t-t_0)^k . \tag{2}$$

Substituting (1) into (2), then the following is obtained

$$u(t)=\sum_{k=0}^{\infty}\frac{1}{k!}\left[\frac{d^k u(t)}{dt^k}\right]_{t=t_0}(t-t_0)^k. \tag{3}$$

From the above definitions (1) and (2), it is easy to see that the concept of the DTM is obtained from the power series expansion. To illustrate the application of the proposed DTM to solve systems of ordinary differential equations, we consider the nonlinear system

$$\frac{du(t)}{dt}=f(u(t),t),\ t\geq t_0, \quad (4)$$

where $f(u(t),t)$ is a nonlinear smooth function.
System (4) is supplied with some initial conditions

$$u(t_0)=u_0. \quad (5)$$

The DTM establishes the solution of (4) which can be written as

$$u(t)=\sum_{k=0}^{\infty}U(k)(t-t_0)^k, \quad (6)$$

where $U(0),U(1),U(2),...$ are unknowns which to be determined by the DTM. Applying the DTM to the initial conditions (5) and (4) respectively, the transformed initial conditions are obtained

$$U(0)=u_0, \quad (7)$$

and the recursion system

$$(1+k)U(k+1)=F(U(0),\ldots,U(k),k),\quad k=0,1,2,\ldots, \quad (8)$$

where $F(U(0),...,U(k),k)$ is the differential of $f(u(t),t)$.
Using (7) and (8), the unknown $U(k)$, $k=0,1,2,...$ can be determined. Then, the differential inverse transformation of the set of values $\{U(k)\}_{k=0}^{m}$ gives the approximate solution

$$u(t)=\sum_{k=0}^{m}U(k)(t-t_0)^k, \quad (9)$$

where m is the approximation order of the solution. Equation (6) gives the exact solution of problem (4)-(5).

If $U(k)$ and $V(k)$ are the differential transforms of $u(t)$ and $v(t)$ respectively, then the main operations of the DTM are shown in the table 1.

Table 1: Main Operation of DTM

Function	Differential transform
$\alpha u(t)\pm\beta v(t)$	$\alpha U(k)\pm\beta V(k)$
$u(t)v(t)$	$\sum_{r=0}^{k}U(r)V(k-r)$
$u(t)v(t)w(t)$	$\sum_{r=0}^{k}\sum_{l=0}^{r}U(l)V(r-l)W(k-r)$
$\frac{d^n}{dt^n}[u(t)]$	$(k+1)\ldots(k+n)U(k+n)$
$e^{\lambda t}$	$\frac{\lambda e^{\lambda t_0}}{k!}$
$\sin(\omega t)$	$\frac{w^k}{k!}\sin\left(\omega t_0+\frac{\pi k}{2}\right)$
$\cos(\omega t)$	$\frac{w^k}{k!}\cos\left(\omega t_0+\frac{\pi k}{2}\right)$

Applying the differential transform to the initial conditions (5) and (4) to obtain a recursion system for unknowns $U(0),U(1),U(2)$,.... The solution series obtained from DTM may have limited regions of convergence. In addition, the multi-stage version of this method is proposed to improve DTM.

Multi-Stage Differential Transform Method (MsDTM)

The multi-stage technique was used the first time by [10]. The multi-stage technique is utilized to enlarge the time intervals of the convergence. Due to the DTM method is failed to give convergent analytical approximate solutions over the large time intervals. The multi-stage DTM that has been improved in [10, 18, 21, 22]. The multistage technique depends on dividing the main interval into equally sub-intervals. Suppose that the main interval is $[0,T]$. This interval is divided into equally sub-intervals $[t_{i-1}-t_i]$, $i=1,2,...,N$.

The step size is $h=\frac{T}{N}$ and $t_i=ih$. The essential idea of the multistage DTM is, as shown in the first step, applying the DTM to Equation (2) over the sub-interval $[0,t_1]$, the approximate solutions are obtained as follow $u_i(t)=\sum_{l=0}^{K}U_l(t-t_i)^l, i=2,3...,N-1$. where K is the order of the approximation for the power series.

The next step is applying the DTM to Eq (2.2) over the sub-intervals $[t_{i-1},t_i]$, where $i=2,...,N-1$ by using the initial conditions $u_0^{(i)}(t)=\sum_{k=0}^{K}U_k^{(i)}(t-t_{i-1})^k$ the approximate solutions are obtained

$u_i(t)=\sum_{l=0}^{K}U_l(t-t_i)^l, i=2,3...,N-1$. the second step is repeated until $i=N$ then, the approximate solution over $[0,T]$ is obtained

$$u(t)=\begin{cases}u_1(t) & 0\leq t\leq t_1,\\ u_2(t) & t_1\leq t\leq t_2,\\ . \\ . \\ . \\ u_n(t) & t_{N-1}\leq t\leq T.\end{cases} \quad (10)$$

Solving Nonlinear Systems of Ordinary Differential Equations

Assume a system of nonlinear ODES that has the following form

$$\begin{cases}\frac{du_1(t)}{dt}=f_1(t,u_1(t),u_2(t),...,u_n(t)),\\ \frac{du_2(t)}{dt}=f_2(t,u_1(t),u_2(t),...,u_n(t)),\\ . \\ . \\ . \\ \frac{du_n(t)}{dt}=f_n(t,u_1(t),u_2(t),...,u_n(t)),\end{cases} \quad (11)$$

subject to the initial conditions

$$u_1(t_0)=u_1(0),u_2(t_0)=u_2(0),...,u_n(t_0)=u_n(0). \quad (12)$$

Based on the definitions of DTM which are presented previously, applying DTM to both sides of the system given in (11)and (12), the following is obtained

$$\begin{cases}(k+1)U_1(k+1)=F_1(k),\\ (k+1)U_2(k+1)=F_2(k),\\ . \\ . \\ . \\ (k+1)U_n(k+1)=F_n(k).\end{cases} \quad (13)$$

Therefore, regarding to DTM the nth term approximations for (13) can be presented as

$$\begin{cases} u_1(t) = \sum_{k=1}^{n} U_1(k)t^k \\ u_2(t) = \sum_{k=1}^{n} U_2(k)t^k \\ . \\ . \\ . \\ u_n(t) = \sum_{k=1}^{n} U_n(k)t^k. \end{cases} \tag{14}$$

As mentioned in the introduction, the DTM is not valid over large intervals so, the Multi-stage differential transform method (MsDTM)is applied to enlarge the interval of the convergence. The main range $[0,T]$ is divided into N equal sub-intervals then, applying DTM in every sub-intervals to obtain the approximate solutions over $[0,T]$ as

$$u(t) = \begin{cases} u_1(t), & 0 \le t \le t_1 \\ u_2(t), & t_1 \le t \le t_2 \\ . \\ . \\ . \\ u_n(t), & t_{N-1} \le t \le T. \end{cases} \tag{15}$$

Convergence Analysis of Multistage Differential Transform Method

In this section, a credible scheme for the convergence of multistage DTM when applied to non-autonomous nonlinear system of ODEs is discussed, also The enough condition of convergence is proposed, depending on the enough condition of convergence, and the estimation of the maximum absolute truncated error of the solutions is studied. Many researches addressed the convergence of semi analytical methods such as, HPM, Adomain decoposition method, the DTM method and multi-stage DTM method [23-31]. Depending on these researches, we present a convergence analysis of Multi-stage DTM method for non linear ODEs as non-autonomous system. The basic steps of MsDTM, as a device for solving nonautonomous system of nonlinear ODEs are the following. First, applying DTM over each sub-interval, then the approximate solution is obtained as a power series

$$y(t) = \sum_{k=0}^{\infty} Y(k)(t-t_0)^k \tag{16}$$

where $Y(t)$ is the differential transform of $y(t)$. The essential principle of multistage DTM consist of obtaining approximate solution as a power series.

$y(t) = \sum_{k=0}^{\infty} Y(k)(t-t_0)^k,\ t \in I,$ where $I = (t_0, t_0 + r)$, $r > 0$.

Theorem 1:

Let $y_k(t) = a_k(t - t_0)^k$, then the power series solution $\sum_{k=0}^{\infty} y(k)t^k$,

converges if $\exists\ 0 \le \alpha \le 1$ such that $\| y_{k+1} \| \le \alpha \| y_k \|, \forall k \ge k_0$, for some $k_0 \in N$.

Theorem1 is a special case of Banach's space.

Proof

Denote as $(C[I], \|.\|)$ the Banach's space of all continuous functions on I with norm $\|f(t)\| = max_{t\in I}|f(t)|$. Define in the sequence of partial sums $\{S_n\}_{n=0}^{\infty}$ as

$$S_n = y_0(t) + y_1(t) + \ldots + y_n(t), \tag{17}$$

where $y_k(t) = a_k(t - t_0)^k$, we are going to show that $\{S_n\}_{n=0}^{\infty}$ is a Cauchy sequence in this Banach's space. For this reason consider,

$$\|S_{n+1} - S_n\| = \|y_{n+1}\| \le \alpha\|y_n\| \le \alpha^2\|y_{n-1}\| \le \ldots \le \alpha^{n+1}\|y_0\|. \tag{18}$$

For every $n,m \in \mathbb{N}$, $n \ge m$, we have

$$\|S_n - S_m\| = \|(S_n - S_{n-1}) + (S_{n-1} - S_{n-2}) + \ldots + (S_{m+1} - S_m)\| \le \|S_n - S_{n-1}\| + \|S_{n-1} - S_{n-2}\| + \ldots + \|S_{m+1} - S_m\| \le \alpha^n\|y_0\| + \alpha^{n-1}\|y_0\| + \ldots + \alpha^{m+1}\|y_0\| \tag{19}$$

$$\le (\alpha^{m+1} + \alpha^{m+2} + \ldots)\|y_0\| = \frac{\alpha^{m+1}}{1-\alpha}\|y_0\|,\ \ n \ge m,\ \ |\alpha| < 1,$$

$$\lim_{n,m\to\infty}\|S_n - S_m\| = 0. \tag{20}$$

Therefore, $\{S_n\}_{n=0}^{\infty}$ is a Cauchy sequence in the Banach's space $(C[I], \|.\|)$, so the series solution $\sum_{k=0}^{\infty} y_k(t)$, defined in (4.1) converges, and the proof is complete.

Definition 1:

For every i $\in \mathbb{N} \cup \{0\}$ we define,

$$\alpha_i = \begin{cases} \frac{\|y_{i+1}\|}{\|y_i\|}, & \|y_i\| \ne 0 \\ 0, & \|y_i\| = 0 \end{cases}. \tag{21}$$

Corollary 1. In therorem 1. $\sum_{i=0}^{\infty} y(i)$ converges to exact solution y, when $0 \le \alpha_i \le 1$, $i = 1,2,3,\ldots$

Corollary 2: If y_i and $\tilde{y}$ are obtained by standard and multistage DTM methods, respectively, and both the αi's and α̃i' s are less than one then the rate of convergence of $\sum_{i=0}^{\infty} y(i)$ and $\sum_{i=0}^{\infty} \tilde{y}(i)$ to exact solution y is depend to values of α_i's, i.e., if $\tilde{\alpha}_i$¡ α_i for all i, it implies that the rate of convergence of $\sum_{i=0}^{\infty} \tilde{y}(i)$ to exact solution is higher than $\sum_{i=0}^{\infty} y(i)$.

Numerical Examples

Example 1:

Consider non-autonomous system of nonlinear ODEs [32].

$$\begin{cases} \frac{du_1}{dt} = 2u_2^2, \\ \frac{du_2}{dt} = e^{-t}u_1, \\ \frac{du_3}{dt} = u_2 + u_3. \end{cases} \tag{22}$$

with the exact solution

$$\begin{cases} u_1 = e^{2t}, \\ u_2 = e^{t}, \\ u_3 = te^{t}. \end{cases} \tag{23}$$

The initial conditions are $u_1(0) = 1$, $u_2(0) = 1$, $u_3(0) = 0$,

By applying the differential transform on both sides of system (22), the following recursion system is obtained

$$\begin{cases} (k+1)U_1(k+1) = DT[2u_2^2] = F_1(k), \qquad k \geq 0, \\ (k+1)U_2(k+1) = DT[e^{-t}u_1] = F_2(k), \\ (k+1)U_3(k+1) = DT[u_2+u_3] = F_3(k). \end{cases} \tag{24}$$

Then, the following is obtained

$$F(0) = \begin{bmatrix} 2U_{02}^2 \\ e^{-t_0}U_{01} \\ U_{02}+U_{03} \end{bmatrix}, \tag{25}$$

$$F(1) = \begin{bmatrix} 4U_{02}U_{12} \\ -e^{-t_0}U_{01}+e^{-t_0}U_{11} \\ U_{12}+U_{13} \end{bmatrix}, \tag{26}$$

$$F(2) = \begin{bmatrix} 4U_{02}U_{22}+2U_{12}^2 \\ \frac{1}{2}e^{-t_0}U_{01}-e^{-t_0}U_{11}+e^{-t_0}U_{21} \\ U_{22}+U_{23} \end{bmatrix}, \tag{27}$$

$$F(3) = \begin{bmatrix} 4U_{02}U_{32}+4U_{12}U_{22} \\ -\frac{1}{6}e^{-t_0}U_{01}+\frac{1}{2}e^{-t_0}U_{11}-e^{-t_0}U_{21}+e^{-t_0}U_{31} \\ U_{32}+U_{33} \end{bmatrix}, \tag{28}$$

$$= \begin{bmatrix} 4U_{02}U_{42}+4U_{12}U_{32}+2U_{22}^2 \\ \frac{1}{24}e^{-t_0}U_{01}-\frac{1}{6}e^{-t_0}U_{11}+\frac{1}{2}e^{-t_0}U_{21}-e^{-t_0}U_{31}+e^{-t_0}U_{41} \\ U_{42}+U_{43} \end{bmatrix}, \tag{29}$$

Then, the following is obtained

$$U(0) = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad F(0) = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \quad U(1) = \frac{F(0)}{0+1} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix},$$

$$F(1) = \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}, \quad U(2) = \frac{F(1)}{1+1} = \begin{bmatrix} 2 \\ \frac{1}{2} \\ 1 \end{bmatrix}, \quad F(2) = \begin{bmatrix} 4 \\ \frac{1}{2} \\ \frac{3}{2} \end{bmatrix},$$

$$U(3) = \frac{F(2)}{2+1} = \begin{bmatrix} \frac{4}{3} \\ \frac{1}{6} \\ \frac{1}{2} \end{bmatrix}, \quad F(3) = \begin{bmatrix} \frac{8}{3} \\ \frac{1}{6} \\ \frac{4}{6} \end{bmatrix},$$

$$U(4) = \frac{F(3)}{3+1} = \begin{bmatrix} \frac{2}{3} \\ \frac{1}{24} \\ \frac{1}{6} \end{bmatrix},$$

$$F(4) = \begin{bmatrix} \frac{4}{3} \\ \frac{1}{24} \\ \frac{5}{24} \end{bmatrix}, \quad U(5) = \frac{F(4)}{4+1} = \begin{bmatrix} \frac{4}{15} \\ \frac{1}{120} \\ \frac{1}{24} \end{bmatrix},$$

.

.

.

Then, the approximate soution is obtained

$$u(t) = \sum_{k=0}^{\infty} U(k)t^k = \begin{bmatrix} 1+2t+2t^2+\frac{4}{3}t^3+\frac{4}{15}t^5+\ldots \\ 1+t+\frac{1}{2}t^2+\frac{1}{6}t^3+\frac{1}{24}t^4+\frac{1}{120}t^5+\ldots \\ t+t^2+\frac{1}{2}t^3+\frac{1}{6}t^4+\frac{1}{24}t^5+\ldots \end{bmatrix} \tag{30}$$

By applying the multistage DTM to the system (22), the main interval [0,3] is divided into 300 equal sub-interval then, applying DTM over every sub-intervals, the following approximate solutions are obtained over every equal sub-intervals

$$u_0 = \begin{bmatrix} 1.0+0.6666666668t^4+1.333333333t^3+2.0t^2+2.0t \\ 1.0+0.04166666658t^4+0.1666666667t^3+0.5t^2+t \\ 0.1666666667t^4+0.5t^3+t^2+t \end{bmatrix},$$

$0 \leq t < 0.01$,

$$u_1 = \begin{bmatrix} 1.00000000000038+0.680134226568056t^4+1.33306308418431t^3+2.00000270683551t^2+1.99999998645127t \\ 1.00000000000077+0.0420854236188209t^4+0.166658277505956t^3+0.500000083949229t^2+0.999999999503586t \\ 7.955950081\times 10^{-14}+0.168762548729955t^4+0.499957998519914t^3+1.00000042039392t^2+0.999999997892371t \end{bmatrix},$$

$0.01 \le t < 0.02$,

$$u_2 = \begin{bmatrix} 1.00000000082832+0.693873849399614t^4+1.33223779091248t^3+2.00002198356393t^2+1.99999977971996t \\ 1.00000000002588+0.0425083891766670t^4+0.166632885488572t^3+0.500000676768148t^2+0.999999993164928t \\ 0.0000000001277760874+0.170883724440929t^4+0.499830643185735t^3+1.00000339386174t^2+0.999999966010916t \end{bmatrix}$$

$0.02 \le t < 0.03$,

$$u_3 = \begin{bmatrix} 1.00000000673065+0.707891030858526t^4+1.33083513818321t^3+2.00007531880929t^2+1.99999886684168t \\ 1.00000000020768+0.0429356056053601t^4+0.166590149546253t^3+0.500002301253600t^2+0.999999965297976t \\ 0.000000001030927876+0.173030490460716t^4+0.499615880768684t^3+1.00001155797121t^2+0.999999826306065t \end{bmatrix},$$

$0:03 \le t < 0.04$,

.

.

.

$$u_{299} = \begin{bmatrix} 12077.7740430533+263.626906118873t^4-2625.72398211809t^3+10202.5194245100t^2-17985.4173173803t \\ 26.9475786942210+0.828570083600608t^4-6.59541786244451t^3+24.6587429270969t^2-39.2762075682179t \\ 198.009121154190+5.79170497903606t^4-49.4162519705458t^3+182.208027686134t^2-304.171353265752t \end{bmatrix},$$

$2.99 \le t < 3$.

Figures 1, 4 and 7 show that the approximate solutions in DTM do not agree with the exact solutions over the range of the time [0,3]. On the other hand, we can clearly note that the approximate solutions and the exact solutions agree together over the interval [0,3], where $K = 4$ is the order of the approximation, $N = 300$ is the number of subintervals and $h = 0.01$ is the size of the step time in MsDTM. The absolute error in DTM for the three components u_1, u_2, u_3 consequently is very large unlike that absolute error in MsDTM for the same components as shown in Figures 2, 5 and 8 consecutively. The accuracy in the computing is very high as shown in Figures 3, 6 and 9 consequently. It is very clear that the absolute error in MsDTM is very small if compared to the absolute error in RK4.

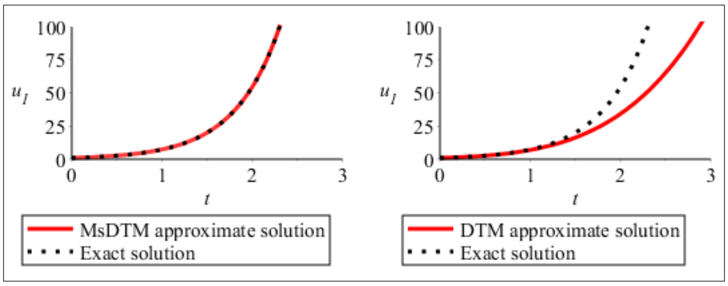

Figure 1: Comparison between MsDTM, DTM and Exact Solution of Component u1 for the System (22)

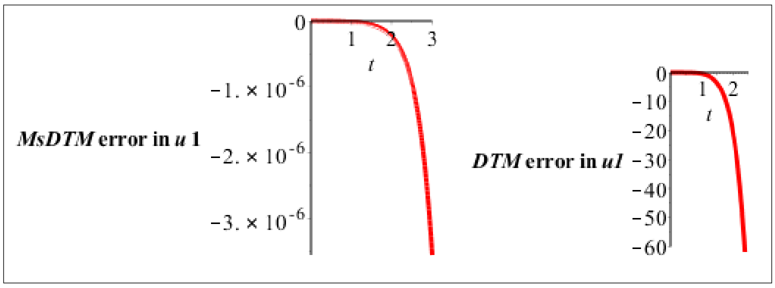

Figure 2: Comparison between MsDTM Error and DTM Error of component u1 for the System (22)

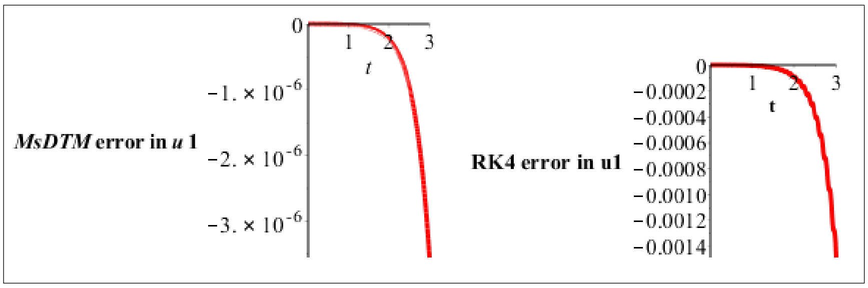

Figure 3: Comparison between MsDTM Error and RK4 Error of component u1 for the System (22)

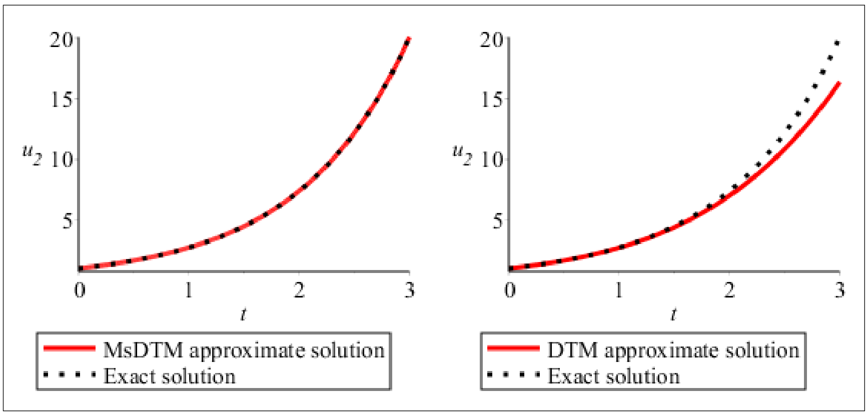

Figure 4: Comparison between MsDTM, DTM and Exact Solution of Component u2 for the System (22)

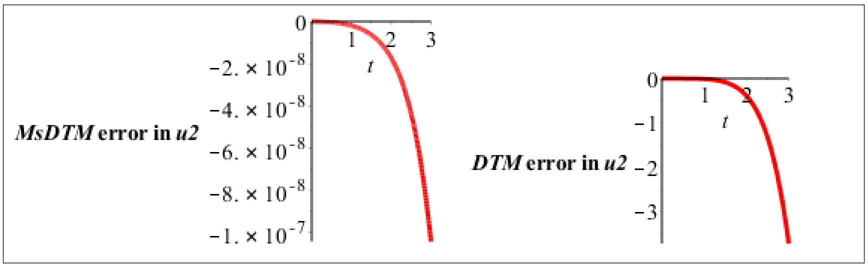

Figure 5: Comparison between MsDTM Error and DTM Error of component u2 for the System (22)

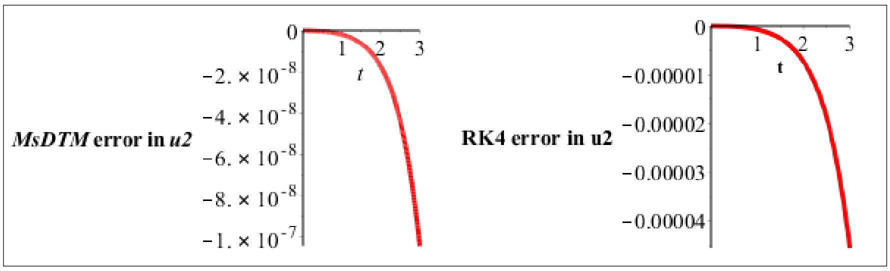

Figure 6: Comparison between MsDTM Error and RK4 Error of component u2 for the System (22)

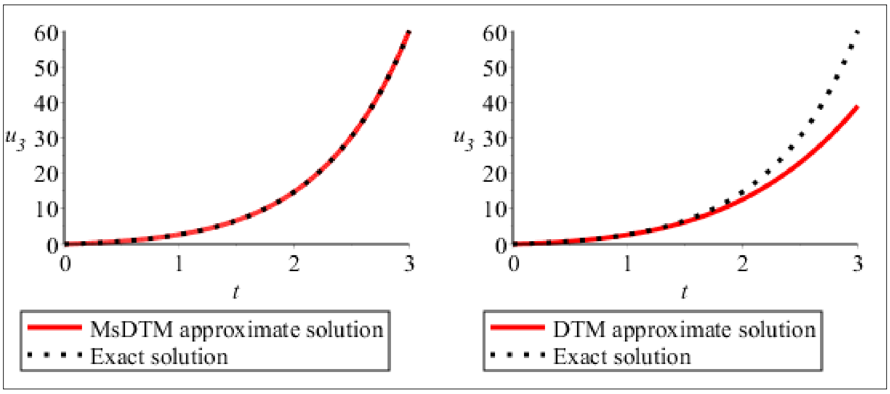

Figure 7: Comparison between MsDTM, DTM and Exact Solution of Component u3 for the System (22)

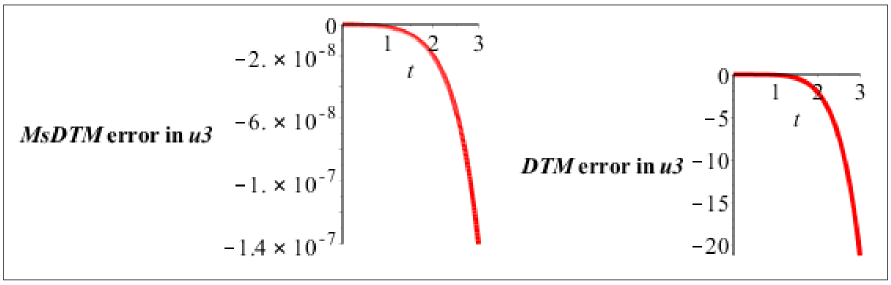

Figure 8: Comparison between MsDTM Error and DTM Error of component u3 for the System (22)

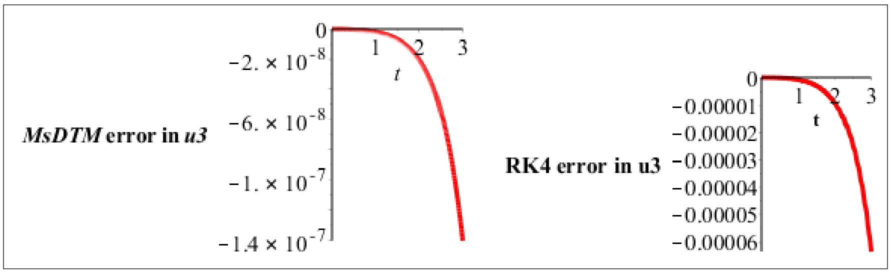

Figure 9: Comparison between MsDTM Error and RK4 Error of component u3 for the System (22)

System (22) was solved by Adomian decomposition method [32]. The new technique in this research is more effective and accuracy than the Adomian decomposition method. We obtain the approximate solution after applying DTM over interval $I = [0,3]$ as follows.

$$u(t) = \sum_{k=0}^{\infty} U(k)t^k = \begin{bmatrix} 1 + 2t + 2t^2 + \frac{4}{3}t^3 + \frac{4}{15}t^5 + \dots \\ 1 + t + \frac{1}{2}t^2 + \frac{1}{6}t^3 + \frac{1}{24}t^4 + \frac{1}{120}t^5 + \dots \\ t + t^2 + \frac{1}{2}t^3 + \frac{1}{6}t^4 + \frac{1}{24}t^5 + \dots \end{bmatrix} \tag{31}$$

α_i's are computed to show the approximate solution series is not convergent to exact solution over $I = [0,3]$. The power series approximate solution for component u_1 as follows.

$$\begin{cases} u_{10} = 1, \\ u_{11} = 1+2t, \\ u_{12} = 1+2t+2t^2, \\ u_{13} = 1+2t+2t^2+\frac{4}{3}t^3, \\ u_{14} = 1+2t+2t^2+\frac{4}{3}t^3+\frac{2}{3}t^4, \\ . \\ . \\ . \end{cases} \tag{32}$$

$$\begin{cases} y_{11} = u_{11}-u_{10} = 2t, \\ y_{12} = u_{12}-u_{11} = 2t^2, \\ y_{13} = u_{13}-u_{12} = \frac{4}{3}t^3, \\ y_{14} = u_{14}-u_{13} = \frac{2}{3}t^4, \\ . \\ . \\ . \end{cases} \tag{33}$$

$$\begin{cases} \alpha_{11} = \frac{\|y_{12}\|_\infty}{\|y_{11}\|_\infty} = 3 > 1, \\ \alpha_{12} = \frac{\|y_{13}\|_\infty}{\|y_{12}\|_\infty} = 1.999999999 > 1, \\ \alpha_{13} = \frac{\|y_{14}\|_\infty}{\|y_{13}\|_\infty} = 1.500000001 > 1, \\ . \\ . \\ . \end{cases} \tag{34}$$

In the same method, all αi's are computed for component u_2 and u_3 over interval $I = [0,3]$ as follows.

$$\begin{cases} \alpha_{21} = \frac{\|y_{22}\|_\infty}{\|y_{21}\|_\infty} = 1.5 > 1, \\ \alpha_{22} = \frac{\|y_{23}\|_\infty}{\|y_{22}\|_\infty} = 1, \\ \alpha_{23} = \frac{\|y_{24}\|_\infty}{\|y_{23}\|_\infty} = 0.7499999983 < 1, \\ . \\ . \\ . \end{cases} \tag{35}$$

For component u_3, the following is obtained:

$$\begin{cases} \alpha_{31} = \frac{\|y_{32}\|_\infty}{\|y_{31}\|_\infty} = 3 > 1, \\ \alpha_{32} = \frac{\|y_{33}\|_\infty}{\|y_{32}\|_\infty} = 1.5 > 1, \\ \alpha_{33} = \frac{\|y_{34}\|_\infty}{\|y_{33}\|_\infty} = 1, \\ . \\ . \\ . \end{cases} \tag{36}$$

Most values of αi's greater than one consequently, according to (Corollary.1) the power series approximate solution diverges over the time interval $I = [0,3]$ so, the multistage DTM is applied over interval $I = [0,3]$ to enlarge the range of the convergence. the main interval I is divided into 300 sub-intervals, then the standard DTM is applied over every sub-interval to obtain the approximate power series solution as follows.

$$u_0 = \begin{bmatrix} 1.0+0.6666666668t^4+1.333333333t^3+2.0t^2+2.0t \\ 1.0+0.04166666658t^4+0.1666666667t^3+0.5t^2+t \\ 0.1666666667t^4+0.5t^3+t^2+t \end{bmatrix},$$

$0 \le t < 0.01$,

$$u_1 = \begin{bmatrix} 1.00000000000038+0.680134226568056t^4+1.33306308418431t^3+2.00000270683551t^2+1.99999998645127t \\ 1.00000000000077+0.0420854236188209t^4+0.166658277505956t^3+0.500000083949229t^2+0.999999999503586t \\ 7.955950081\times10^{-14}+0.168762548729955t^4+0.499957998519914t^3+1.00000042039392t^2+0.999999997892371t \end{bmatrix},$$

$0.01 \le t < 0.02$,

.

.

.

$$u_{299} = \begin{bmatrix} 12077.7740430533+263.626906118873t^4-2625.72398211809t^3+10202.5194245100t^2-17985.4173173803t \\ 26.9475786942210+0.828570083600608t^4-6.59541786244451t^3+24.6587429270969t^2-39.2762075682179t \\ 198.009121154190+5.79170497903606t^4-49.4162519705458t^3+182.208027686134t^2-304.171353265752t \end{bmatrix},$$

$2.99 \le t < 3$. In the same method, all αi's are computed for component $u_0, u_1, \ldots u_{299}$ over sub-intervals [0,0.01],[0.01,0.02],...[2.99,3] respectively as follows. The first, for u_0 over sub-interval [0,0.01]

$$\begin{cases} \alpha_{11} = \frac{\|y_{12}\|_\infty}{\|y_{11}\|_\infty} = 0.01 < 1, \\ \alpha_{12} = \frac{\|y_{13}\|_\infty}{\|y_{12}\|_\infty} = 0.006666666665 < 1, \\ \alpha_{13} = \frac{\|y_{14}\|_\infty}{\|y_{13}\|_\infty} = 0.005000000002 < 1, \\ . \\ . \\ . \end{cases} \tag{37}$$

$$\begin{cases} \alpha_{21} = \frac{\|y_{22}\|_\infty}{\|y_{21}\|_\infty} = 0.005 < 1, \\ \alpha_{22} = \frac{\|y_{23}\|_\infty}{\|y_{22}\|_\infty} = 0.003333333334 < 1, \\ \alpha_{23} = \frac{\|y_{24}\|_\infty}{\|y_{23}\|_\infty} = 0.002499999994 < 1, \\ . \\ . \\ . \end{cases} \tag{38}$$

$$\begin{cases} \alpha_{31} = \frac{\|y_{32}\|_\infty}{\|y_{31}\|_\infty} = 0.01 < 1, \\ \alpha_{32} = \frac{\|y_{33}\|_\infty}{\|y_{32}\|_\infty} = 0.005 < 1, \\ \alpha_{33} = \frac{\|y_{34}\|_\infty}{\|y_{33}\|_\infty} = 0.003333333334 < 1, \\ . \\ . \\ . \end{cases} \tag{39}$$

The second, for u_1 over sub-interval [0.01,0.02]

$$\begin{cases} \alpha_{11} = \frac{\|y_{12}\|_\infty}{\|y_{11}\|_\infty} = 0.01000001360 < 1, \\ \alpha_{12} = \frac{\|y_{13}\|_\infty}{\|y_{12}\|_\infty} = 0.006665306398 < 1, \\ \alpha_{13} = \frac{\|y_{14}\|_\infty}{\|y_{13}\|_\infty} = 0.005102040817 < 1, \\ . \\ . \\ . \end{cases} \tag{40}$$

$$\begin{cases} \alpha_{21} = \frac{\|y_{22}\|_\infty}{\|y_{21}\|_\infty} = 0.005000000842 < 1, \\ \alpha_{22} = \frac{\|y_{23}\|_\infty}{\|y_{22}\|_\infty} = 0.003333164991 < 1, \\ \alpha_{23} = \frac{\|y_{24}\|_\infty}{\|y_{23}\|_\infty} = 0.002525252526 < 1, \\ . \\ . \\ . \end{cases} \tag{41}$$

$$\begin{cases} \alpha_{31} = \frac{\|y_{32}\|_\infty}{\|y_{31}\|_\infty} = 0.01000000422 < 1, \\ \alpha_{32} = \frac{\|y_{33}\|_\infty}{\|y_{32}\|_\infty} = 0.004999577885 < 1, \\ \alpha_{33} = \frac{\|y_{34}\|_\infty}{\|y_{33}\|_\infty} = 0.003375534529 < 1, \\ . \\ . \\ . \end{cases} \tag{42}$$

Finally,for u_{299} over sub-interval [2.99,3]

$$\begin{cases} \alpha_{11} = \frac{\|y_{12}\|_\infty}{\|y_{11}\|_\infty} = 0.005672662045 < 1, \\ \alpha_{12} = \frac{\|y_{13}\|_\infty}{\|y_{12}\|_\infty} = 0.002573603513 < 1, \\ \alpha_{13} = \frac{\|y_{14}\|_\infty}{\|y_{13}\|_\infty} = 0.001004016065 < 1, \\ . \\ . \\ . \end{cases} \tag{43}$$

$$\begin{cases} \alpha_{21} = \frac{\|y_{22}\|_\infty}{\|y_{21}\|_\infty} = 0.006278290206 < 1, \\ \alpha_{22} = \frac{\|y_{23}\|_\infty}{\|y_{22}\|_\infty} = 0.002674677246 < 1, \\ \alpha_{23} = \frac{\|y_{24}\|_\infty}{\|y_{23}\|_\infty} = 0.001256281408 < 1, \\ . \\ . \\ . \end{cases} \tag{44}$$

$$\begin{cases} \alpha_{31} = \frac{\|y_{32}\|_\infty}{\|y_{31}\|_\infty} = 0.005990308611 < 1, \\ \alpha_{32} = \frac{\|y_{33}\|_\infty}{\|y_{32}\|_\infty} = 0.002712078748 < 1, \\ \alpha_{33} = \frac{\|y_{34}\|_\infty}{\|y_{33}\|_\infty} = 0.001172024334 < 1, \\ . \\ . \\ . \end{cases} \tag{45}$$

All values of α_i's less than one consequently, according to (Corollary 1) the power series approximate solution converges over the interval [0,3].

Example 2:
Consider the Rabinovech-Fabrikant system [33].

$$\begin{cases} \frac{du_1}{dt} = u_2(u_3 - 1 + u_1^2) + \gamma u_1, \\ \frac{du_2}{dt} = u_1(3u_3 + 1 + u_1^2) + \gamma u_2, \\ \frac{du_3}{dt} = -2u_3(\alpha + u_1 u_2), \end{cases} \tag{46}$$

where, α, γ are constants that control the evolution of the system. $\alpha = 1.1$, $\gamma = 0.87$
The initial conditions are:

$$\begin{cases} u_1(0) = -1, \\ u_2(0) = 0, \\ u_3(0) = 0.5. \end{cases} \tag{47}$$

By applying the differential transform on the both sides of system (46) the following is obtained:

$$\begin{cases} (k+1)U_1(k+1) = DT[u_2(u_3 - 1 + u_1^2) + 0.87u_1] = F_1(k), \qquad k \geq 0, \\ (k+1)U_2(k+1) = DT[u_1(3u_3 + 1 + u_1^2) + 0.87u_2] = F_2(k), \\ (k+1)U_3(k+1) = DT[-2u_3(1.1 + u_1 u_2)] = F_3(k), \end{cases} \tag{48}$$

Then, the following is obtained:

$$F(0) = \begin{bmatrix} U_{02}{U_{01}}^2 + U_{02}U_{03} - U_{02} + 0.87U_{01} \\ -{U_{01}}^3 + 3U_{03}U_{01} + U_{01} + 0.87U_{02} \\ -2.2U_{03} - 2U_{02}U_{01} \end{bmatrix},$$

$$F(1) = \begin{bmatrix} U_{12}{U_{01}}^2 + 2U_{02}U_{01}U_{11} + U_{02}U_{13} + U_{12}U_{03} - U_{12} + 0.87U_{11} \\ -3U_{11}{U_{01}}^2 + 3U_{01}U_{13} + 3U_{03}U_{11} + U_{11} + 0.87U_{12} \\ -2.2U_{13} - 2.0U_{01}U_{13}U_{02} - 2U_{03}U_{01}U_{12} - 2U_{02}U_{03}U_{11} \end{bmatrix},$$

$$F(2) = \begin{bmatrix} U_{22}{U_{01}}^2 + U_{22}U_{03} - U_{22} + 2U_{12}U_{01}U_{11} + U_{12}U_{13} + 2U_{02}U_{01}U_{21} + U_{02}{U_{11}}^2 + U_{02}U_{23} + 0.87U_{21} \\ -3U_{21}{U_{01}}^2 + 3.0U_{21}U_{03} + U_{21} - 3U_{01}{U_{11}}^2 + 3.0U_{11}U_{13} + 3U_{01}U_{23} + 0.87U_{22} \\ -2.2U_{23} - 2.0U_{01}U_{23}U_{02} - 2U_{12}U_{01}U_{13} - 2U_{02}U_{11}U_{13} - 2U_{03}U_{01}U_{22} - 2U_{21}U_{03}U_{02} - 2.0U_{03}U_{11}U_{12} \end{bmatrix},$$

$$F(3) = \begin{bmatrix} \begin{bmatrix} U_{32}{U_{01}}^2 + 2U_{02}U_{01}U_{31} + 2U_{22}U_{01}U_{11} + 2U_{12}U_{01}U_{21} \\ -3U_{31}{U_{01}}^2 + 3U_{03}U_{31} + U_{31} - 6U_{21}U_{01}U_{11} \\ -2.2U_{33} - 2U_{01}U_{33}U_{02} - 2U_{12}U_{01}U_{23} - 2U_{02}U_{11}U_{23} - 2U_{13}U_{01}U_{22} - 2U_{13}U_{21}U_{02} \end{bmatrix} \\ \begin{bmatrix} +2U_{02}U_{11}U_{21} + U_{12}{U_{11}}^2 + U_{02}U_{33} + U_{32}U_{03} + U_{12}U_{23} + U_{22}U_{13} - U_{32} + 0.87U_{31} \\ +3U_{13}U_{21} - {U_{11}}^3 + 3U_{23}U_{11} + 3U_{01}U_{33} + 0.87U_{32} \\ -2U_{13}U_{11}U_{12} - 2U_{03}U_{01}U_{32} - 2U_{03}U_{31}U_{02} - 2U_{03}U_{11}U_{22} - 2U_{03}U_{21}U_{12} \end{bmatrix} \end{bmatrix},$$

$$F(4) = \begin{bmatrix} \begin{bmatrix} U_{42}{U_{01}}^2 + U_{42}U_{03} - U_{42} + 2U_{32}U_{01}U_{11} + U_{32}U_{13} + 2U_{22}U_{01}U_{21} + U_{22}{U_{11}}^2 + U_{22}U_{23} \\ -3U_{41}{U_{01}}^2 + 3U_{41}U_{03} + U_{41} - 6U_{31}U_{01}U_{11} + 3U_{31}U_{13} \\ -2.2U_{43} - 2U_{01}U_{43}U_{02} - 2U_{12}U_{01}U_{33} - 2U_{02}U_{11}U_{33} - 2U_{23}U_{01}U_{22} - 2U_{21}U_{23}U_{02} - 2U_{23}U_{11}U_{12} - 2U_{13}U_{01}U_{32} - 2U_{31}U_{13}U_{02} \end{bmatrix} \\ \begin{bmatrix} +2U_{12}U_{01}U_{31} + 2U_{12}U_{11}U_{21} + U_{12}U_{33} + 2U_{02}U_{01}U_{41} + 2U_{02}U_{11}U_{31} + U_{02}{U_{21}}^2 + U_{02}U_{43} + 0.87U_{41} \\ -3U_{01}{U_{21}}^2 - 3U_{21}{U_{11}}^2 + 3U_{21}U_{23} + 3U_{11}U_{33} + 3U_{01}U_{43} + 0.87U_{42} \\ -2U_{13}U_{11}U_{22} - 2U_{13}U_{21}U_{12} - 2U_{03}U_{01}U_{42} - 2U_{41}U_{03}U_{02} - 2U_{03}U_{11}U_{32} - 2U_{03}U_{31}U_{12} - 2U_{03}U_{21}U_{22} \end{bmatrix} \end{bmatrix},$$

Therefore, in a sequential pattern the following is obtained:

$$U(0) = \begin{bmatrix} -1 \\ 0 \\ 0.5 \end{bmatrix}, \quad F(0) = \begin{bmatrix} -0.87 \\ -1.5 \\ -1.1 \end{bmatrix}, \quad U(1) = \frac{F(0)}{0+1} = \begin{bmatrix} -0.87 \\ -1.5 \\ -1.1 \end{bmatrix},$$

$$F(1) = \begin{bmatrix} -1.5069 \\ 2.43 \\ 0.92 \end{bmatrix}, \quad U(2) = \frac{F(1)}{1+1} = \begin{bmatrix} -0.75345 \\ 1.215 \\ 0.46 \end{bmatrix},$$

$$F(2) = \begin{bmatrix} -1.0080015 \\ 5.195475 \\ 2.1980000001 \end{bmatrix}, \qquad U(3) = \frac{F(2)}{2+1} = \begin{bmatrix} -0.3360005 \\ 1.731825 \\ 0.7326666667 \end{bmatrix},$$

$$F(3) = \begin{bmatrix} -2.7345079352 \\ 5.353985 \\ -1.1351666672 \end{bmatrix}, \qquad U(4) = \frac{F(3)}{3+1} = \begin{bmatrix} -2.73450793519999990 \\ 5.35398499999999977 \\ -1.13516666720000004 \end{bmatrix},$$

$$F(4) = \begin{bmatrix} 0.4187716491 \\ 5.68230311 \\ -2.048973333 \end{bmatrix}, \qquad U(5) = \frac{F(4)}{4+1} = \begin{bmatrix} 0.08375432982 \\ 1.136460622 \\ -0.4097946666 \end{bmatrix}.$$

Then,we offer the approximate solution

$$u(t) = \sum_{k=0}^{\infty} U(k)t^k = \begin{bmatrix} -1.0 - 0.6836269838t^4 - 0.3360005t^3 - 0.75345t^2 - 0.87t \\ 1.33849625t^4 + 1.731825t^3 + 1.215t^2 - 1.5t \\ 0.5 - 0.2837916668t^4 + 0.7326666667t^3 + 0.46t^2 - 1.1t \end{bmatrix} \tag{49}$$

By applying the multi-stage differential transform on the system (46), the main interval [0,5] is divided into 300 equal sub-interval then, applying DTM over every sub-intervals, the following approximate solutions are obtained over every equal subinterval

$$u_0(t) = \begin{bmatrix} -1.0-0.6836269838t^4-0.3360005t^3-0.75345t^2-0.87t \\ 1.33849625t^4+1.731825t^3+1.215t^2-1.5t \\ 0.5-0.2837916668t^4+0.7326666667t^3+0.46t^2-1.1t \end{bmatrix},$$

$0 \leq t < 0{:}01666666667$,

$$u_1(t) = \begin{bmatrix} -0.999999999888680-0.672082536674801t^4-0.336437670328817t^3-0.753442271431072t^2-0.870000067500001t \\ 0.0000000002016869523+1.440583378584844t^4+1.72833914856530t^3+1.21505879618813t^2-1.50000049636959t \\ 0.500000000034265-0.318521999940339t^4+0.733830058006550t^3+0.459980567874375t^2-1.09999984056220t \end{bmatrix},$$

$0.01666666667 \leq t < 0{:}03333333334$,

$$u_2(t) = \begin{bmatrix} -0.999999988586273-0.650161446965580t^4-0.338692960117268t^3-0.753352277182874t^2-0.870001712405994t \\ 0.00000005624919752+1.55807094405523t^4+1.71650234630098t^3+1.21552230366789t^2-1.50000882741310t \\ 0.499999983743978-0.353880313450347t^4+0.737367168635968t^3+0.459842973044317t^2-1.09999738590747t \end{bmatrix},$$

$0.03333333334 \leq t < 0{:}05000000001$,

$$u_3(t) = \begin{bmatrix} -0.999999871272892-0.615746558008919t^4-0.344504692974333t^3-0.752979511079879t^2-0.870012469516512t \\ 0.0000004966778097+1.69179045279726t^4+1.69412362321935t^3+1.21694531307070t^2-1.50004955457117t \\ 0.499999870082238-0.389076188461875t^4+0.743230090637776t^3+0.459471849156237t^2-1.09998681429288t \end{bmatrix},$$

$0.05000000001 \leq t < 0{:}06666666668$,

.

.

.

$$u_{299}(t) = \begin{bmatrix} -151.830222441504-0.386759749143536t^4+7.10738398049597t^3-48.0377790404979t^2+140.930626429251t \\ 641.445864916163+0.753390434554313t^4-16.5948773527355t^3+135.250524355680t^2-483.628716658466t \\ -158.769843978622-0.279519324075883t^4+5.55428291742705t^3-40.8851984910278t^2+132.330230898842t \end{bmatrix},$$

$4.983333334 \le t \le 5.$

In system (46), the Figures 10, 12 and 14 show that the approximate and RK4 solution agree with together for the three components of system (39) u_1,u_2 and u_3 in MsDTM over the time interval [0,5], where $K = 4$ is the order of the approximation, $N = 500$ is the number of the subintervals and $h = 0.01$ is the size of the time step. In contrary, in DTM the same figures above illustrate a clear divergence between the approximate and RK4 solution over the interval [0,5] for the three components of system (46) consequently u_1,u_2 and u_3. The absolute error between the approximate and RK4 solution is very small in MsDTM if compared with the absolute error in DTM as shown the following figures respectively 11, 13 and 15. The Figures 16-19 explain the portrait phase of system (46) in 2−D views and 3−D views consequently in MsDTM comparing with the portrait phase in DTM in three coordinate planes.

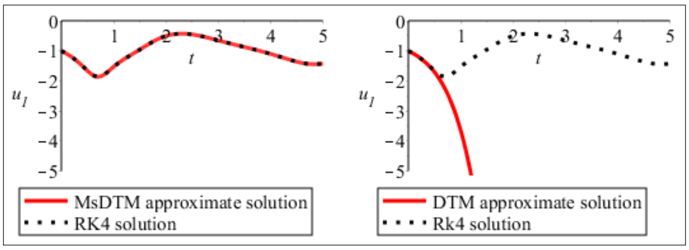

Figure 10: Comparison between MsDTM and DTM Solution of Component u_1 for the System (46)

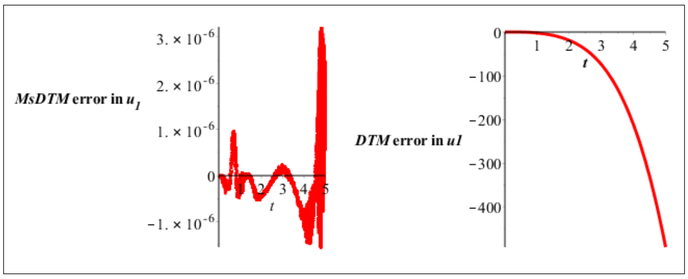

Figure 11: Comparison between MsDTM Error and DTM Error of Component u_1 for the System (46)

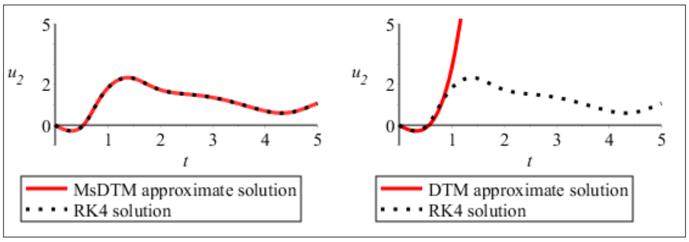

Figure 12: Comparison between MsDTM and DTM Solution of Component u_2 for the System (46)

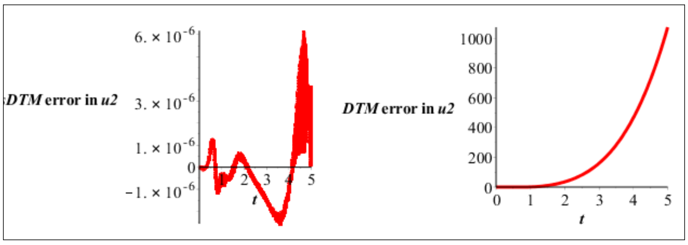

Figure 13: Comparison between MsDTM Error and DTM Error of Component u2 for the System (46)

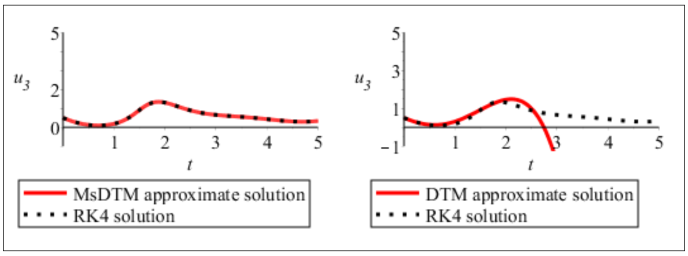

Figure 14: Comparison between MsDTM and DTM Solution of Component u3 for the System (46)

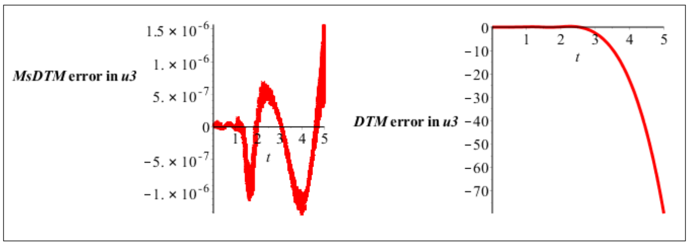

Figure 15: Comparison between MsDTM Error and DTM Error of Component u2 for the System (46)

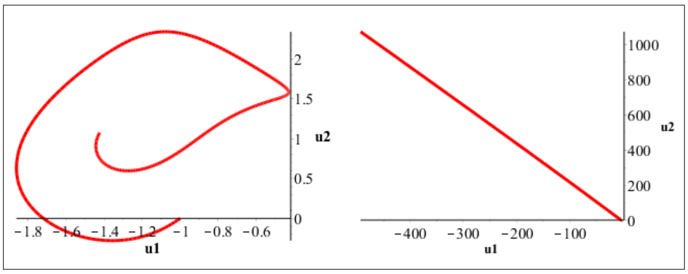

Figure 16: Comparison between u1-u2 Phase Portrait using MsDTM and DTM for the System (46)

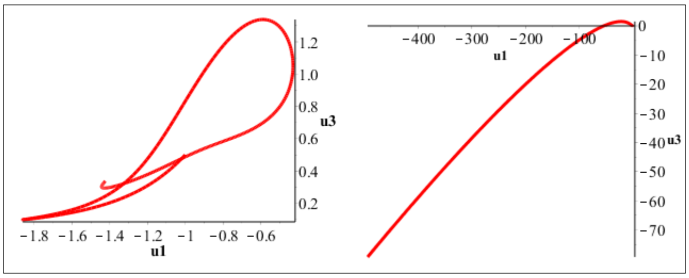

Figure 17: Comparison between u1-u3 Phase Portrait using MsDTM and DTM for the System (46)

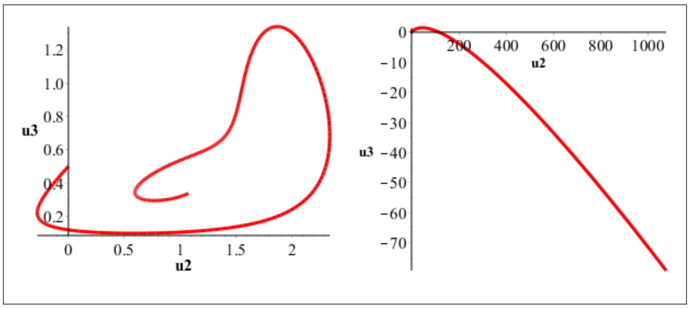

Figure 18: Comparison between u2-u3 Phase Portrait using MsDTM and DTM for the System (46)

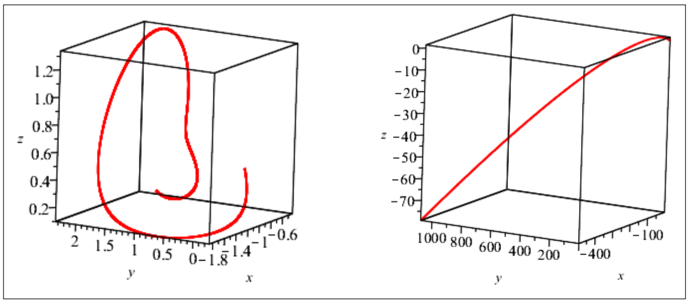

Figure 19: Comparison between u1-u2-u3Phase Portrait using MsDTM and DTM for the system (46)

System (46) was solved by MsDTM [33]. The same result is obtained using a new technique. This leads to the new technique becomes more general than other methods. Furthermore, the comparison between phase portrait DTM and phase portrait MsDTM is presented for the waves to control Chaotic system. The results show the movement of the waves is very clear in MsDTM, but it is difficult to note the movement of waves of Chaotic system using DTM. In system (46) the approximate solution is obtained as follows

$$u(t)=\sum_{k=0}^{\infty}U(k)t^k=\begin{bmatrix}-1.0-0.6836269838t^4-0.3360005t^3-0.75345t^2-0.87t\\ 1.33849625t^4+1.731825t^3+1.215t^2-1.5t\\ 0.5-0.2837916668t^4+0.7326666667t^3+0.46t^2-1.1t\end{bmatrix} \tag{50}$$

The approximate power series solution diverges over the time interval $I = [0,5]$ by computing the values of α_i's for the three components of u_1,u_2, and u_3 respectively as the method that is applied in example 1. The following is obtained

$$\begin{cases}\alpha_{11}=\frac{\|y_{12}\|_\infty}{\|y_{11}\|_\infty}=4.330172414>1,\\ \alpha_{12}=\frac{\|y_{13}\|_\infty}{\|y_{12}\|_\infty}=2.229746499>1,\\ \alpha_{13}=\frac{\|y_{14}\|_\infty}{\|y_{13}\|_\infty}=10.17300545>1,\\ .\\ .\\ .\end{cases} \tag{51}$$

$$\begin{cases}\alpha_{21}=\frac{\|y_{22}\|_\infty}{\|y_{21}\|_\infty}=4.050000000>1,\\ \alpha_{22}=\frac{\|y_{23}\|_\infty}{\|y_{22}\|_\infty}=7.126851852>1,\\ \alpha_{23}=\frac{\|y_{24}\|_\infty}{\|y_{23}\|_\infty}=3.864409654>1,\\ .\\ .\\ .\end{cases} \tag{52}$$

$$\begin{cases}\alpha_{31}=\frac{\|y_{32}\|_\infty}{\|y_{31}\|_\infty}=2.090909091>1,\\ \alpha_{32}=\frac{\|y_{33}\|_\infty}{\|y_{32}\|_\infty}=7.963768117>1,\\ \alpha_{33}=\frac{\|y_{34}\|_\infty}{\|y_{33}\|_\infty}=1.936703823>1,\\ .\\ .\\ .\end{cases} \tag{53}$$

According to (Corollary 1) the approximate power series solution diverges over [0,5] where all values of α_i's greater than one.

By applying the multi-stage differential transform on the system (46), the main interval [0,5] is divided into 300 equal sub-interval then, applying DTM over every sub-intervals, the following approximate solutions are obtained over every equal subinterval:

$$u_0(t)=\begin{bmatrix}-1.0-0.6836269838t^4-0.3360005t^3-0.75345t^2-0.87t\\ 1.33849625t^4+1.731825t^3+1.215t^2-1.5t\\ 0.5-0.2837916668t^4+0.7326666667t^3+0.46t^2-1.1t\end{bmatrix}, \tag{54}$$

$0 \le t < 0.01666666667$,

$$u_1(t)=\begin{bmatrix}-0.999999999888680-0.672082536674801t^4-0.336437670328817t^3-0.753442271431072t^2-0.870000067500001t\\ 0.000000002016869523+1.44058337858484t^4+1.72833914856530t^3+1.21505879618813t^2-1.50000049636959t\\ 0.500000000034265-0.318521999940339t^4+0.733830058006550t^3+0.459980567874375t^2-1.09999984056220t\end{bmatrix},$$

$0.01666666667 \le t < 0{:}03333333334$,

.

.

.

$$u_{299}(t)=\begin{bmatrix}-151.830222441504-0.386759749143536t^4+7.10738398049597t^3-48.0377790404979t^2+140.930626429251t\\ 641.445864916163+0.753390434554313t^4-16.5948773527355t^3+135.250524355680t^2-483.628716658466t\\ -158.769843978622-0.279519324075883t^4+5.55428291742705t^3-40.8851984910278t^2+132.330230898842t\end{bmatrix},$$

$4.983333334 \le t \le 5$. In the similar method, all α_i's are computed for component u_0,u_1,...u_{299} over sub-intervals [0,0.01666666667],[0.01666666667,0.03333333334], ...[4.983333334,5] respectively as follows

The first, for u_0 over sub-interval [0,0.01666666667]

$$\begin{cases} \alpha_{11} = \frac{\|y_{12}\|_\infty}{\|y_{11}\|_\infty} = 0.01443390806 < 1, \\ \alpha_{12} = \frac{\|y_{13}\|_\infty}{\|y_{12}\|_\infty} = 0.007432488330 < 1, \\ \alpha_{13} = \frac{\|y_{14}\|_\infty}{\|y_{13}\|_\infty} = 0.03391001820 < 1, \\ . \\ . \\ . \end{cases} \tag{55}$$

$$\begin{cases} \alpha_{21} = \frac{\|y_{22}\|_\infty}{\|y_{21}\|_\infty} = 0.01350000000 < 1, \\ \alpha_{22} = \frac{\|y_{23}\|_\infty}{\|y_{22}\|_\infty} = 0.02375617284 < 1, \\ \alpha_{23} = \frac{\|y_{24}\|_\infty}{\|y_{23}\|_\infty} = 0.01288136552 < 1, \\ . \\ . \\ . \end{cases} \tag{56}$$

$$\begin{cases} \alpha_{31} = \frac{\|y_{32}\|_\infty}{\|y_{31}\|_\infty} = 0.006969696968 < 1, \\ \alpha_{32} = \frac{\|y_{33}\|_\infty}{\|y_{32}\|_\infty} = 0.02654589373 < 1, \\ \alpha_{33} = \frac{\|y_{34}\|_\infty}{\|y_{33}\|_\infty} = 0.006455679409 < 1, \\ . \\ . \\ . \end{cases} \tag{57}$$

The second, for u_1 over sub-interval [0.01666666667; 0.03333333334]

$$\begin{cases} \alpha_{11} = \frac{\|y_{12}\|_\infty}{\|y_{11}\|_\infty} = 0.01443375887 < 1, \\ \alpha_{12} = \frac{\|y_{13}\|_\infty}{\|y_{12}\|_\infty} = 0.007442235085 < 1, \\ \alpha_{13} = \frac{\|y_{14}\|_\infty}{\|y_{13}\|_\infty} = 0.03329405891 < 1, \\ . \\ . \\ . \end{cases} \tag{58}$$

$$\begin{cases} \alpha_{21} = \frac{\|y_{22}\|_\infty}{\|y_{21}\|_\infty} = 0.01350064883 < 1, \\ \alpha_{22} = \frac{\|y_{23}\|_\infty}{\|y_{22}\|_\infty} = 0.02370720872 < 1, \\ \alpha_{23} = \frac{\|y_{24}\|_\infty}{\|y_{23}\|_\infty} = 0.01389178912 < 1, \\ . \\ . \\ . \end{cases} \tag{59}$$

$$\begin{cases} \alpha_{31} = \frac{\|y_{32}\|_\infty}{\|y_{31}\|_\infty} = 0.006969403550 < 1, \\ \alpha_{32} = \frac{\|y_{33}\|_\infty}{\|y_{32}\|_\infty} = 0.02658916881 < 1, \\ \alpha_{33} = \frac{\|y_{34}\|_\infty}{\|y_{33}\|_\infty} = 0.007234236242 < 1, \\ . \\ . \\ . \end{cases} \tag{60}$$

Finally,for u_{299} over sub-interval [4 .983333334 , 5]

$$\begin{cases} \alpha_{11} = \frac{\|y_{12}\|_\infty}{\|y_{11}\|_\infty} = 0.005681019601 < 1, \\ \alpha_{12} = \frac{\|y_{13}\|_\infty}{\|y_{12}\|_\infty} = 0.002465900839 < 1, \\ \alpha_{13} = \frac{\|y_{14}\|_\infty}{\|y_{13}\|_\infty} = 0.0009069435166 < 1, \\ . \\ . \\ . \end{cases} \tag{61}$$

$$\begin{cases} \alpha_{21} = \frac{\|y_{22}\|_\infty}{\|y_{21}\|_\infty} = 0.004660962696 < 1, \\ \alpha_{22} = \frac{\|y_{23}\|_\infty}{\|y_{22}\|_\infty} = 0.002044955392 < 1, \\ \alpha_{23} = \frac{\|y_{24}\|_\infty}{\|y_{23}\|_\infty} = 0.0007566495963 < 1, \\ . \\ . \\ . \end{cases} \tag{62}$$

$$\begin{cases} \alpha_{31} = \frac{\|y_{32}\|_\infty}{\|y_{31}\|_\infty} = 0.005149390057 < 1, \\ \alpha_{32} = \frac{\|y_{33}\|_\infty}{\|y_{32}\|_\infty} = 0.002264178366 < 1, \\ \alpha_{33} = \frac{\|y_{34}\|_\infty}{\|y_{33}\|_\infty} = 0.0008387501091 < 1, \\ . \\ . \\ . \end{cases} \tag{63}$$

All values of α_i's less than one consequently, according to (Corollary 1.) the power series approximate solution converges over the interval [0,5].

Numerical proof of a convergence of MsDTM for nonlinear system

There are two ways to show the convergence of the MsDTM. The first one consists of increasing the number of subintervals. To ensure the absolute error approaches zero, the number of subintervals is increased and the order of approximation is constant. As a result, the absolute error approaches zero gradually as shown in the Tables 2 and 3. The second one is based on the order of the approximation. As the order of the approximation increase gradually, the absolute error approaches zero, while the time step is constant. Two examples are presented for numerical proof. The first example is system (22). In this example, the order of the approximation is a constant. The time step is decreased gradually, where N is the number of subintervals, K is the order of the approximation, h is the time step and the time interval is [0,3].

In the second method, the number of subintervals is a constant, then the order of the approximation is increased regularly as shown in Table 3.

The second example based on system (46). In the first method, the order of the approximation for this system is a constant, then the number of subintervals is increased progressively. Consequently, the absolute error will approach zero for all components of the system as shown in Table 4, where K is the order of the approximation, N is the number of subintervals, h is the time step and the time interval is [0,5].

Table 2: Convergence Analysis by Varying *N* for System (22), K=4

N	h	Error of u_1	Error of u_2	Error of u_3	Time(s)
100	0.03	0.00005	2×10^{-6}	0.000002	1.828
200	0.015	0.000002	1×10^{-7}	1×10^{-7}	2.969
300	0.010	1×10^{-6}	2×10^{-8}	2×10^{-8}	4.625
400	0.0075	1×10^{-7}	2×10^{-8}	2×10^{-7}	5.282
500	0.0060	1×10^{-7}	2.5×10^{-8}	2×10^{-7}	6.422
600	0.0050	1×10^{-7}	2×10^{-9}	1×10^{-8}	7.515

Table 3: Convergence Analysis by Varying K for System. (22), N=100

K	h	Error of u_1	Error of u_2	Error of u_3	Time(s)
4	0.03	0.00005	2×10^{-6}	2×10^{-6}	1.734
6	0.03	0.00002	2×10^{-8}	2×10^{-7}	2.297
8	0.03	1×10^{-5}	2×10^{-8}	2×10^{-7}	2.828
10	0.03	1×10^{-5}	2×10^{-8}	2×10^{-7}	3.547
12	0.03	1×10^{-6}	2×10^{-8}	2×10^{-7}	4.063

Table 4: Convergence Analysis by Varying N for System (46), K=4

N	h	Error of u_1	Error of u_2	Error of u_3	Time(s)
100	0.05	0.00002	0.0001	0.00002	1.797
200	0.025	2×10^{-6}	5×10^{-6}	1×10^{-6}	2.938
300	0.0166	1×10^{-6}	1×10^{-6}	5×10^{-7}	4.062
400	0.0125	1×10^{-6}	1×10^{-6}	2×10^{-7}	5.188
500	0.010	1×10^{-6}	1×10^{-6}	2×10^{-7}	6.344

second method, the order of the approximation is increased while the number of subintervals is a constant as illustrated in Table 5. As the result, the absolute error of every component in system (46) approaches to zero.

Table 5: Convergence Analysis by Varying K for System. (46), N=100

K	h	Error of u_1	Error of u_2	Error of u_3	Time(s)
4	0.05	0.00002	0.0001	0.00002	1.765
6	0.05	0.00001	0.0001	0.00001	2.218
8	0.05	1×10^{-4}	1×10^{-4}	1×10^{-5}	2.843
10	0.05	1×10^{-5}	1×10^{-5}	1×10^{-5}	3.500

Tables 2, 3, 4 and 5 show the MsDTM method accelerates convergence and increases in the accuracy of the approximate solution for all components u1,u2 and u3 over the large time intervals by increasing the order of approximation or the number of subintervals.

Conclusion

In this paper, the new proposed technique is applied to solve nonlinear systems of ODES. The numerical and algebraic proof show the multistage DTM method converges faster than the DTM method over the large time intervals. This feature makes the multistage DTM as effective tool to enlarge the range of convergence rather than the DTM method.

References

1. Benhammouda B, Vazquez Leal H (2016) A new multi-step technique with differential transform method for analytical solution of some nonlinear variable delay differential equations, SpringerPlus 5: 1723.
2. Mohyud Din ST, Usman M, Wang W, Hami M (2018) A study of heat transfer analysis for squeezing flow of a Casson fluid via differential transform method, Neural Computing and Applications 30: 3253-3264.
3. Sheikholeslami M, Ganji DD (2015) Nanofluid flow and heat transfer between parallel plates considering Brownian motion using DTM, Computer Methods in Applied Mechanics and Engineering 283: 651-663.
4. Ayaz F (2004) Solutions of the system of differential equations by differential transform method, Applied Mathematics and Computation 147: 547-567.
5. Biazar J, Asadi MA, Salehi F (2015) Rational homotopy perturbation method for solving stiff systems of ordinary differential equations, Applied Mathematical Modelling 39: 1291-1299.
6. Ganji D, Sadighi A (2006) Application of he's homotopy-perturbation method to nonlinear coupled systems of reaction-diffusion equations, International Journal of Nonlinear Sciences and Numerical Simulation 7: 411-418.
7. Hosseinzadeh H, Jafari H, Gholami M, Ganji D (2017) Solving a class of boundary value problems in structural engineering and fluid mechanics using homotopy perturbation and adomian decomposition methods, Italian J. Pure Appl. Math 37: 687-698.
8. Rach R, Wazwaz AM, Duan JS (2015) The Volterra integral form of the lane-emden equation: new derivations and solution by the adomian decomposition method,Journal of Applied Mathematics and Computing 47: 365-379.
9. J. Zhou, Differential transformation and its applications for electrical circuits (1986).
10. Odibat ZM, Bertelle C, Aziz Alaoui M, Duchamp GH (2010) A multi-step differential transform method and application to non-chaotic or chaotic systems, Computers,Mathematics with Applications 59: 1462-1472.
11. Benhammouda B, Vazquez Leal H (2010) Analytical solution of a nonlinear index-three daes system modelling a slider-crank mechanism, Discrete Dynamics in Nature and Society 2015: 1-14
12. Gokdogan A, Merdan M (2013) Adaptive multi-step differential transformation method to solve ode systems, Kuwait Journal of Science 40: 35-55.
13. Odibat ZM, Kumar S, Shawagfeh N, Alsaedi A, Hayat T (2017) A study on the convergence conditions of generalized differential transform method, Mathematical Methods in the Applied Sciences 40: 40-48.
14. Ayaz F (2004) Applications of differential transform method to differential-algebraic equations, Applied Mathematics and Computation 152: 649-657.
15. Ibis B, Bayram M, Agargun AG (2011) Applications of fractional differential transform method to fractional differential-algebraic equations, European Journal of Pure

and Applied Mathematics 4: 129-141.
16. Fatoorehchi H, Abolghasemi H (2013) Improving the differential transform method: a novel technique to obtain the differential transforms of nonlinearities by the ado250 mian polynomials, Applied Mathematical Modelling 37: 6008-6017.
17. Nourifar M, Sani AA, Keyhani A (2017) Efficient multi-step differential transform method: Theory and its application to nonlinear oscillators, Communications in Nonlinear Science and Numerical Simulation 53: 154-183.
18. El Zahar ER (2015) Applications of adaptive multi step differential transform method to singular perturbation problems arising in science and engineering, Applied Mathematics and Information Sciences 9: 223-232.
19. Abuteen E, Freihat A, Al-Smadi M, Khalil H, Khan RA (2016) Approximate series solution of nonlinear, fractional klein-gordon equations using fractional reduced differential transform method 1-11.
20. Arikoglu A, Ozkol I (2007) Solution of fractional differential equations by using differential transform method, Chaos, Solitons and Fractals 34: 1473-1481.
21. Wang S, Yu Y (2012) Application of multistage homotopy-perturbation method for the solutions of the chaotic fractional order systems, International Journal of Nonlinear Science 13: 3-14.
22. Motsa S, Dlamini P, Khumalo M (2013) A new multistage spectral relaxation method for solving chaotic initial value systems, Nonlinear Dynamics 72: 265-283.
23. Vazquez Leal H (2014) Generalized homotopy method for solving nonlinear differential equations, Computational and Applied Mathematics 33: 275-288.
24. Jafari H, Alipour M, Tajadodi H (2012) Convergence of homotopy perturbation method for solving integral equations, Thai Journal of Mathematics 8: 511-520.
25. Heris JM, Aghdaei MF (2013) Equivalent hpm with adm and convergence of the hpm to a class of nonlinear integral equations, Journal of Mathematical Extension 7: 33-49.
26. Hosseini MM, Nasabzadeh H (2006) On the convergence of adomian decomposition method, Applied mathematics and computation 182: 536-543.
27. Kaya D, Inan IE (2005) A convergence analysis of the adm and an application, Applied Mathematics and computation 161: 1015-1025.
28. Duan JS, Rach R, Wang Z (2013) On the effective region of convergence of the decomposition series solution, Journal of Algorithms and Computational Technology 7: 227-247.
29. Ray SS (2014) New approach for general convergence of the adomian decomposition method, World Applied Sciences Journal 32: 2264-2268.
30. Erturk VS, Momani S (2010) On the generalized differential transform method: application to fractional integro-differential equations, Studies in Nonlinear Sciences 1: 118-126.
31. Oke A (2017) Convergence of differential transform method for ordinary differential equations, Journal of Advances in Mathematics and Computer Science 24: 1-17.
32. Biazar J, Babolian E, Islam R (2004) Solution of the system of ordinary differential equations by adomian decomposition method, Applied Mathematics and Computation 147: 713-719.
33. Kolebaje O, Ojo M, Ojo O, Omoliki A (2014) On the application of the multistage differential transform method to the rabinovich-fabrikant system, The African Review of Physics 9: 140-144.