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Conservation of Heat Energy for Derivation of One-Dimensional Heat Equation and its Analytical Solution

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ABSTRACT

This article deals with the derivation of the one-dimensional heat equation and explores its analytical solutions, specifically focusing on Dirichlet boundary value and initial value problems associated with this particular partial differential equation. The primary objective of this study is to investigate the solution for the one-dimensional heat equation, which serves as a prominent example of a partial differential equation. To achieve this, the method of separation of variables is utilized to obtain the analytical solution. The heat equation, renowned for its effectiveness in analyzing the dynamic flow of heat within solids, has stood the test of time as a powerful tool for several centuries. Lastly, we have taken a model example and solve it using the stated method and also mesh plot is displayed using Python software.

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Introduction

In the early 1800s Joseph Fourier's began a mathematical study of heat. A deeper under- standing of heat flows had significant applications in science and within industry. The heat equation describing the conduction of heat in solids occupies a unique position in modern math- ematical physics. In addition to lying at the core of the analysis of problems involving the trans- fer of heat in physical systems, the conceptual-mathematical structure of the heat conduction equation (also known as the heat diffusion equation) has inspired the mathematical formulation of many other physical processes in terms of diffusion [1-4].

Fourier's work on heat conduction and his mathematical formulation of the heat equation laid the foundation for the field of mathematical physics. The heat equation is a partial differential equation that describes how heat spreads and changes over time in a solid or a fluid. It has wide-ranging applications in fields such as engineering, physics, biology, and finance [1,5-10]. The mathematical structure of the heat conduction equation, based on diffusion, has also inspired the study of other physical phenomena that can be described in terms of diffusion processes. Diffusion processes occur in various contexts, such as the spread of pollutants in the environment, the movement of particles in a fluid, and the propagation of information in net- works [11-15].

Fourier's work faced some initial skepticism and took time to gain acceptance among sci- entists of his time. However, his ideas eventually became widely recognized and influential, leading to advancements in the understanding of heat transfer and the development of mathemat- ical techniques that have found applications in numerous scientific and engineering disciplines. many fields besides heat transfer: electricity, chemical diffusion, fluids in porous media, genet- ics, and economics. A study of the conditions that led to the articulation of the heat conduction equation and the reasons why that equation has had such a major influence on scientific thought over nearly two centuries is in itself instructive. At the same time, an examination of how the work was received and accepted by Fourier's peers and successors gives us a glimpse into the culture of science, especially during the nineteenth century in Europe. The present work has been motivated both by the educational and historical importance of Fourier's work [16]. This work studies, a simple type of Partial Differential Equations (PDEs): second order linear PDEs. A partial differential equation (PDE) is any differential equation that contains two or more independent variables and one dependent variable. Therefore, the derivative(s) in the equation are partial derivatives. Thus, the heat equation is the most known important parabolic second order PDE which describes the distribution of heat (or variation in temperature) in a given region. Parabolic PDEs have a natural coupling to system of Ordinary Differential Equation (ODEs). This article solves one-dimensional heat equation analytically using the method of separation of variables. For better understanding of this project, it is very important that to understand the difference between heat and temperature. Heat is a process of energy transfers as a result of temperature difference between the two bodies. Heat is the energy transferred through heating process. Temperature is a physical property of matter that describes the hotness or coldness of an object or environment. Therefore no heat would be exchanged between bodies of the same temperature. We recall that a metal rod with non-uniform temperature, heat (thermal energy) is transferred from regions of higher temperature to regions of lower temperature.

Heat equation in different dimension (1D, 2D, 3D, ...) is a very useful differential equation applied in most of Engineering science, Physics, Chemistry, Heat mechanic, Dynamics, etc. It also has numerous application areas in various fields, such as Material Science and Engineering, Thermal Analysis of Buildings, Heat

Transfer in Electronic Devices, Geothermal Systems, and others. But the derivation of the model is complicated (not easy) [17-19]. The aim of this study is to give an idea how to derive heat equation in 1D and extend this work to 2D and 3D.

Materials and Methods

Derivation of One-Dimensional heat Equation

In this section, we derive an equation for one dimensional region called one dimensional heat equation that describes the flow of heat in a given region.

In the equation of heat conduction, three physical principles are involved:

The heat energy of a body with uniform property is:

$$\text{Heat energy (H)} = cmu \tag{1}$$

where m is a body mass, c is the specific heat, $u = u(x, t)$ is the temperature.

J. Fourier's law of heat transfer: the rate of heat transfer is proportional to negative tem- perature slope,

$$\frac{\text{Rate of heat transfer}}{\text{area}} = -k\frac{\partial u}{\partial x} \tag{2}$$

where k is the thermal conductivity, unit $[k] = MLT^{-3}U^{-1}$

Conservation of energy: In the heat transfer from the hotter body to the colder body, the heat released from the hotter body is equal to the heat gained by the colder body.

To derive the one-dimensional heat equation, we consider a thin bar that is wrapped around the x-axis, with its ends at x=0 and x=L as shown in Figure 1 below. This bar is composed of a homogeneous material and has a uniform cross-section. Furthermore, we assume that the bar is laterally insulated, preventing heat from passing through it. Due to the thinness of the bar, the temperature u can be treated as constant on any cross-section. Consequently, heat only flows along the x-axis. Thus, the temperature u is solely a function of the position x and the time t, denoted as u=u (x, t). This function represents the temperature flow within the bar.

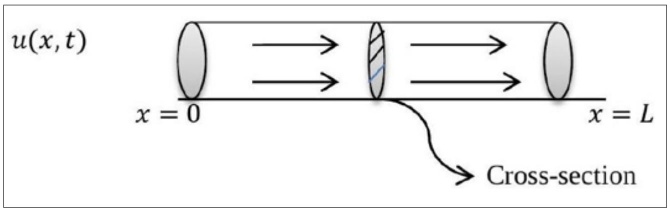

Figure 1: Thin Bar that Transfer Heat

Now, consider a region D of the bar, with ends x0 and x1 as shown in Figure 2 below. Now, the total amount of heat $(H) = H(t)$ in cross-section D is:

$$H(t) = \int_{x_0}^{x_1} c\rho u(x,t)dx \tag{3}$$

Where c is the specific heat and ρ is the density.

Now, differentiating Equation 3 using Leibniz rule we get;

$$\frac{dH}{dt} = \frac{d}{dt}\int_{x_0}^{x_1} c\rho u(x,t)dx = \int_{x_0}^{x_1} c\rho\frac{\partial u(x,t)}{\partial t}dx = c\rho\int_{x_0}^{x_1}\frac{\partial u(x,t)}{\partial t}dx$$

Therefore,

$$\frac{dH}{dt} = c\rho\int_{x_0}^{x_1}\frac{\partial u(x,t)}{\partial t}dx \tag{4}$$

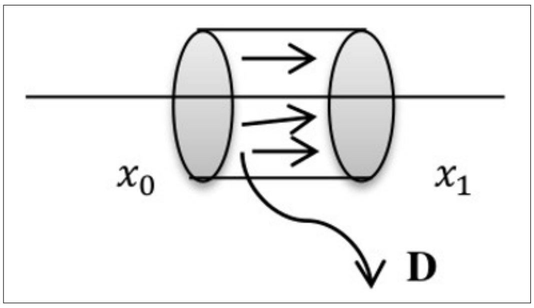

Figure 2: Cross-Section of the thin Bar

Now, the bar is insulated laterally, the only way that heat can flow into or out of D is through the ends x_0 and x_1. From Fourier's law of heat flow, we have a heat flow from hotter to colder regions and the flow rate is proportional to $\frac{\partial u}{\partial t}$.

Now, the net rate change of heat H in D is just the rate at which heat enters D minus the rate at which heat leaves from D.

$$\frac{dH}{dt} = -k\frac{\partial u(x_0,t)}{\partial t} - \left(-k\frac{\partial u(x_1,t)}{\partial t}\right) \textit{ where k is the thermal}$$

Conductivity

$$\frac{dH}{dt} = k\frac{\partial u(x_1,t)}{\partial t} - k\frac{\partial u(x_0,t)}{\partial t} = k\left(\frac{\partial u(x_1,t)}{\partial t} - \frac{\partial u(x_0,t)}{\partial t}\right)$$

$$\frac{dH}{dt} = k\left(\frac{\partial u(x_1,t)}{\partial t} - \frac{\partial u(x_0,t)}{\partial t}\right)$$

By applying the fundamental theorem of calculus part II, we have;

$$\frac{dH}{dt} = k\left(\frac{\partial u(x_1,t)}{\partial t} - \frac{\partial u(x_0,t)}{\partial t}\right) = k\int_{x_0}^{x_1}\frac{\partial^2 u(x,t)}{\partial x^2}$$

$$\frac{dH}{dt} = k\int_{x_0}^{x_1}\frac{\partial^2 u(x,t)}{\partial x^2}dx \tag{5}$$

Now, equating the two expressions in Equation 4 and Equation 5 above for $\frac{dH}{dt}$, we get the relationship;

$$\frac{dH}{dt} = c\rho\int_{x_0}^{x_1}\frac{\partial u(x,t)}{\partial t}dx = k\int_{x_0}^{x_1}\frac{\partial^2 u(x,t)}{\partial x^2}dx$$

$$= c\rho\int_{x_0}^{x_1}\frac{\partial u(x,t)}{\partial t}dx = k\int_{x_0}^{x_1}\frac{\partial^2 u(x,t)}{\partial x^2}dx$$

Differentiating both sides with respect to x_1 we have;

$$\frac{d}{dx_1}c\rho\int_{x_0}^{x_1}\frac{\partial u(x,t)}{\partial t}dx = \frac{d}{dx_1}k\int_{x_0}^{x_1}\frac{\partial^2 u(x,t)}{\partial x^2}dx$$

$$c\rho\frac{d}{dx_1}\int_{x_0}^{x_1}\frac{\partial u(x,t)}{\partial t}dx = k\frac{d}{dx_1}\int_{x_0}^{x_1}\frac{\partial^2 u(x,t)}{\partial x^2}dx$$

$$c\rho\frac{\partial u(x,t)}{\partial t} = k\frac{\partial^2 u(x,t)}{\partial x^2}$$ (by Fundamental Theorem of Calculus part I

$$\frac{\partial u(x,t)}{\partial t} = \frac{k}{c\rho}\frac{\partial^2 u(x,t)}{\partial x^2}$$

$$\frac{\partial u(x,t)}{\partial t} = \frac{k}{c\rho}\frac{\partial^2 u(x,t)}{\partial x^2}$$

$$\frac{\partial u(x,t)}{\partial t} = C^2\frac{\partial^2 u(x,t)}{\partial x^2} \qquad (6)$$

Where $\frac{k}{c\rho} = c^2 > 0$, is a thermal diffusivity. Since the above arguments hold true for all intervals from x_0 to x_1 and for all t > 0, it can be concluded that the Partial Differential Equation (PDE) given in equation Equation 6 is satisfied for the specified interval from $x = 0$ to $x = L$ and for all $t > 0$. As a result, Equation 6 is commonly referred to as the one-dimensional heat equation.

Analytical Solution of One-Dimensional Heat Equation

To solve the 1D heat equation, the method of separation of variables in conjunction with Fourier's sine series is utilized.

Separation of Variables

The method of separation of variables is a powerful technique for solving linear Partial Differential Equations (PDEs). It involves breaking down the analysis of PDEs into the analysis of multiple Ordinary Differential Equations (ODEs).

Boundary and Initial Conditions

In our case, we consider a perfectly insulated homogeneous rod of finite length L, with both ends maintained at zero temperature. This gives us the boundary conditions:

$$u(0,t) = u(L,t) = 0 \qquad (7)$$

Additionally, if the initial temperature distribution in the rod is described by a function f(x), we have the initial condition:

$$u(x,0) = f(x) \qquad (8)$$

Solution by Separation of Variables Method

To solve the heat equation in Equation 6 using separation of variables method, first assume that the function of the two variables has a solution of the form:

$u(x, t) = F(x)G(t)$, $F(x)$ and $G(t)$ are not identicaly zero.

Differentiating the above gives:

$$\frac{\partial u}{\partial t} = F(x)G'(t),\ \frac{\partial^2 u}{\partial x^2} = F''(x)G(t),\ \text{where } G' = \frac{\partial G}{\partial t}, F'' = \frac{\partial^2 F}{\partial x^2}$$

So that,

$$\frac{\partial u(x,t)}{\partial t} = C^2\frac{\partial^2 u(x,t)}{\partial x^2} \Rightarrow F(x)G'(t) = C^2F''(x)G(t)$$

Now, using separation of variable, we can write equivalently as:

$$\frac{G'(t)}{c^2G(t)} = \frac{F''(x)}{F(x)} = \lambda$$

where recall λ is constant of separation. Now, we can write this in two homogenous ODEs as:

$$F''(x) - \lambda F(x) = 0 \text{ and } G'(x) - \lambda C^2G(t) = 0$$

Solving for $F(x)$

Now, we have to solve; $F''-\lambda F = 0$. But the solution depends on λ, that means for $\lambda = 0$, $\lambda > 0$, and $\lambda < 0$; we check each one by one.

Case 1. $\lambda = 0$, then $F'' - \lambda F = 0 \Rightarrow F'' = 0 \Rightarrow F' = A \Rightarrow F(x) = Ax + B$

$\therefore F(x) = Ax + B$ is a general solution for constants A and B.Now, using boundary conditions

$u(0, t) = 0 \Rightarrow F(0)G(t) = 0, \Rightarrow F(0) = 0$ (Because $G(t)$ is not identically zero) and $u(L, t) = 0 \Rightarrow F(L)G(t) = 0, \Rightarrow F(L) = 0$

Therefore, $F(x) = Ax + B$

$F(0) = 0 \Rightarrow A.0 + B = 0 \Rightarrow B = 0$ and $F(L) = 0 \Rightarrow A.L + B = 0 \Rightarrow A = 0$

$\Rightarrow F(x) = Ax + B = 0$ is a trivial solution. But we are not interested in trivial one. So $\lambda = 0$
cannot be our choice.

Case 2: $\mu^2 = \lambda > 0$, then $F'' - \lambda F = 0 \Rightarrow F'' - \mu^2F = 0$; solving this using the method of
characteristics equation, we get the general solution is:

$F(x) = Ae^{\mu x} + Be^{-\mu x}$ with constants A and B

Applying the boundary conditions on this we have;

$F(0) = 0 \Rightarrow A + B = 0 \Rightarrow A = -B$

$F(L) = 0 \Rightarrow Ae^{\mu L} + Be^{-\mu L} = 0$. But $A = -B$, so we have:

$= -Be^{\mu L} + Be^{-\mu L} = 0$

$= -B(e^{\mu L} + e^{-\mu L}) = 0$, but $e^{\mu L} + e^{-\mu L} \neq 0$

So, $B = 0$ implies, $A = 0$. Again in this case we get $F(x) = 0$, which is a trivial solution.

Case 3: $-p^2 = \lambda < 0$, then $F'' - \lambda F = 0 \Rightarrow F'' + p^2F = 0$; solving this using the method of characteristics equation is:

$$r^2 + p^2 = 0 \Rightarrow r^2 = -p^2 \Rightarrow r_{1,2} = \pm pi$$

Thus the general solution is:

$F(x) = A\cos(px) + B\sin(px)$ is the general solution.

Applying the boundary conditions to this, we get:

$F(0) = 0 \Rightarrow A\cos(p.0) + B\sin(p.0) = 0 \Rightarrow A = 0$
$F(L) = 0 \Rightarrow A\cos(p.L) + B\sin(p.L)$. but $A = 0$, so that we get; $B \sin(p.L) = 0$.

Now, to obtain the non-trivial solution, we take non-zero arbitrary B, then we get:

$$\sin(p.L) = 0 \Leftrightarrow pL = n\pi \Rightarrow p = \frac{n\pi}{L}$$

$$\Rightarrow -p^2 = \frac{-(n\pi)^2}{L^2} = \lambda \qquad (9)$$

$$\Rightarrow F(x) = \sin\frac{n\pi}{L}x$$

$$\Rightarrow F_n(x) = \sin\frac{n\pi}{L}x \quad n = 1, 2, 3, \ldots\ldots\ldots\ldots \qquad (10)$$

Solving for $G(t)$
To solve for G(t), we shall solve the ODE: $G' - \lambda C^2 G = 0$. But from Equation 10, $\lambda = -\frac{n^2\pi^2}{L^2}$

$$G' - \lambda C^2 G = 0 \Rightarrow G' + \frac{C^2 n^2 \pi^2}{L^2} G = 0$$

Thus, using the method of characteristics equation, we get the general solution is:

$$G(t) = \mathrm{D}e^{-\frac{c^{222}}{L^2}t}$$

Where D is constant and, then

$$G_n(t) = D_n e^{-\frac{c^2 n^2 \pi^2}{L^2}t} \qquad (11)$$

for arbitrary Dn and n = 1, 2, 3, ...

Full solution of $u(x, t)$
From our supposition, $u(x, t) = F(x)G(t)$ is a solution for Equation 6, implies that;

$$un(x,t) = F_n(x)Gn(t)$$

is also a solution for n = 1, 2, 3, ...
Again, since the PDE given in Equation 6 is linear, by supper position principle we have;

$$u_n(x,t) = \sum_{n=1}^{\infty} F_n(x) G_n(t)$$

$$u(x,t) = \sum_{n=1}^{\infty}\left(D_n e^{\frac{-C^2 n^2 \pi^2}{L}t} \sin\frac{n\pi}{L}x\right) \qquad (12)$$

Equation 12 is a general solution of equation 6. Applying the initial conditions, Equation 12 can be rewrite as;

$$u(x,0) = \sum_{n=1}^{\infty}\left(D_n e^{\frac{-C^2 n^2 \pi^2}{L}\cdot 0} \sin\frac{n\pi}{L}x\right) = \sum_{n=1}^{\infty}\left(D_n \sin\frac{n\pi}{L}x\right)$$

$$u(x,0) = f(x) = \sum_{n=1}^{\infty}\left(D_n \sin\frac{n\pi}{L}x\right) \qquad (13)$$

which is the fourier sine series.
To obtain D_n, we shall use the Fourier's sine series and the value is:

$$D_n = \tfrac{2}{L}\int_0^L \left(f(x)\sin\tfrac{n\pi}{L}\right) dx$$

Therefore, the particular solution of 1-D heat equation in Eq. (6) is

$$u(x,t) = \sum_{n=1}^{\infty}\left(D_n e^{\frac{-C^2 n^2 \pi^2}{L}t} \sin\frac{n\pi}{L}x\right) \qquad (14)$$

$$D_n = \frac{2}{L}\int_0^L \left(f(x)\sin\frac{n\pi}{L}\right) dx \qquad (15)$$

Steady State Solution of 1D Heat Equation
Definition 1: Let us consider the function $u(x, t)$ satisfying the partial differential equation $\frac{\partial u}{\partial t} = c^2\frac{\partial^2 u}{\partial x^2}$. We define a steady-state solution of this equation as a solution where the partial derivative with respect to time $\frac{\partial u}{\partial t}$ is equal to zero. In other words, the temperature distribution u remains constant and does not vary with time, making it time independent.,

Therefore, $u(x, t) = u(x)$ is a steady-state solution to the 1D heat equation. We can observe the following result:

$$\frac{\partial u}{\partial t} = 0 \Rightarrow c^2\frac{\partial^2 u}{\partial x^2} = 0 \Rightarrow \frac{\partial^2 u}{\partial x^2} = 0 \Rightarrow \frac{\partial u}{\partial x} = A \Rightarrow u(x) = Ax + B$$

where A and B are constants.
This represents the general steady-state solution of the 1D heat equation. The steady-state solution can help us deal with inhomogeneous Dirichlet boundary conditions. If the Dirichlet boundary conditions are given as follows:

$$u(0, t) = T_1,\ u(L, t) = T_2$$

we can determine the particular steady-state solution by setting $B = T_1$ and $A = \frac{T_2 - T_1}{L}$. Therefore, the particular steady-state solution becomes:

$$u(x) = \frac{T_2 - T_1}{L}x + B$$

where B is the constant term.
It is important to note that this steady-state solution can be applied when the given boundary conditions are non-zero.

Numerical Results
Example 1: solve the initial boundary value problem

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2}, \quad 0 < x < \pi, t > 0 \qquad (16)$$

$$u(x,0) = \frac{\pi}{2}, \quad 0 < x < \pi \qquad (17)$$

$$u(0,t) = u(L,t) = 0, t > 0 \qquad (18)$$

Thus, Equation 16 is 1-D heat equation with, $c = 1$, $L = \pi$ and $f(x) = \frac{\pi}{2}$, so the solution of 16 is given by:

$$u(x,t)=\sum_{n=1}^{\infty}\left(D_n e^{\frac{-C^2 n^2 \pi^2}{L}t}\sin\frac{n\pi}{L}x\right)$$

where $D_n=\frac{2}{L}\int_0^L\left(f(x)\sin\frac{n\pi}{L}\right)dx$

Using the given conditions, we find out the coefficient C_n.

$$D_n=\frac{2}{L}\int_0^L\left(f(x)\sin\frac{n\pi}{L}\mathrm{x}\right)\mathrm{dx}$$

$$\Rightarrow C_n=\frac{2}{\pi}\int_0^{\pi}\left(\frac{\pi}{2}\sin\frac{n\pi}{\pi}\mathrm{x}\right)\mathrm{dx}$$

$$\Rightarrow C_n=\int_0^{\pi}\sin(nx)\mathrm{dx}$$

$$=-\frac{1}{n}(\cos n\pi-\cos 0)$$

$$=-\frac{1}{n}(\cos n\pi-1)=\begin{cases}0 \text{ for } n \text{ is even}\\ \frac{2}{n} \text{ for } n \text{ is odd}\end{cases}$$

Therefore, the non-trivial solution of Equation 16.

$$u(x,t)=\sum_{n=1}^{\infty}\frac{2}{n}e^{-n^2\pi t}\sin(nx)$$

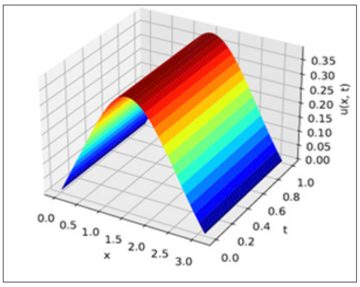

Figure 3: Surf Plot for the Analytical Solution of 1D Heat Equation Using Example 1.

Conclusion

This article derived the time-dependent partial differential equation known as the 1D heat equation and investigated its solutions, which were observed to exhibit variation in both time and space. The primary emphasis of the discussion revolved around obtaining analytical solutions for the one-dimensional heat equation through the implementation of the method of separation of variables. Furthermore, the research explored the steady-state solution of the 1D heat equation while considering non-zero boundary conditions, and a practical test example was presented to demonstrate the applicability of the derived method.

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