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Counting the Number of Even and Odd AUNU Permutations with Occurrences of 123- Pattern

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ABSTRACT

Permutation groups are important tools in enumerating combinatorial objects. In AUNU permutation, the first element counted as one and its length is a prime number. This paper concerned with the counting system of AUNU permutations which contain certain subsequences to obtain some new algebraic theoretic consequences. The number of even and odd of such permutations is determined and the involutions among the counted elements. Bijections are also determined between the sets of such permutations and other combinatorial objects. A discussion was presented of lattices whose maximum length chains correspond to AUNU permutations. The results in this paper complement previous work by the Authors. We also obtained generating functions $|E_n^r(123)|$ and $|O_n^r(123)|$ for the number of even (respectively odd) AUNU permutations on n letters containing exactly r=1occurrence of 123 avoiding permutation.

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Introduction

An AUNU permutation $\pi \in S_n$,the symmetric group on $[n] = \{I, 2, \dots, n\}$, is said to contain the 3-letter word (or pattern) 123 if there is triple such that π(i)<π(j)<π(k). Similarly, one defines AUNU permutations with r-occurrences of σ-pattern for every $\sigma \in S_3$ [1-3]. We denoted by $F_n^r(\sigma)$ a set of all AUNU permutations of length n that contains exactly r occurrence of a pattern σ. More generally, if $T \subseteq S_3$, let be $F_n^r(T) = \bigcap_{\sigma \in T} F_n^r(\{\sigma\})$ the set of all AUNU permutations in S_n which contains every 3-letter word contained in T . For example, the extreme cases are $F_n(S_n) = S_n$ for all $n \geq 1$, and $F(S) = \varnothing$, if $n \leq 3$, (i.e. $n = 0,1$ or 2) while $F_n(S_3) = S_3$ if $n \geq 3$. Depending on the cardinality of, $T \subseteq S_n$ we shall refer to $F_n(T)$ as a set of single, double, ... , multiple AUNU permutations of length n which contains a subset of every 3-letter word contained in $T \subseteq S_3$ For each $T \subseteq S_3$, let $|F_n(T)|$ denote the number of AUNU permutations of length n which contains a subset of every 3-letter word (pattern) $T \subseteq S_3$ .E.g. There are three AUNU permutations of length five containing exactly five occurrences of 132-patterns: $F_5^4(132) = \{14532, 15342, 15423\}$, hence. $|F_5^4(132)| = 3$. Furthermore, we found that for AUNU permutations of length n,

$$|F_n^1(123)| = |S_n(213)| = |F_\pi^5(132)| = (\mathrm{n}-2), \ \forall \ n \geq 3, |\pi| = 5$$

Of course,

$$T \subseteq T^* \ implies \ F_n^r(T) \subseteq F_n^r(T^*)$$

Permutations with restrictions of this type can be approached from the computer sciences standpoint of sorting problems as well as part of the combinatorial topic of strings with forbidden subworlds [4-8]. Our point of view will be the latter. Previous work by Knuth stablished that

$$S_n(T) = \frac{1}{n+1}\binom{2n}{n}$$

for each R of cardinality one [9-11]. That is, the nth Catalan number, C_n , is the common value for the number of singly restricted permutations, regardless of the specific 3-letter restriction [12-14].

The purpose of this paper is three-fold. First, in the case of singleton set $T \subseteq S_3$, (i.e. $|T| = 1$ we obtain more detailed results concerning the values of $|E_n(T)|$ [resp. $|O_n(T)|$which denotes the number of even (resp. odd) permutations in which contain the singleton set $T \subseteq S_3$ We also give the numbers of even and odd *involutions* in $F_n(T)$ for $|T|=1$ denoted $|EI_n(T)|$ and $|OI_n(T)|$, respectively. The total number of involutions of length n that contain patterns of type $T \subseteq S_3$ is denoted by $|I_n(T)|$ [4,15,16].

Our second goal is to derive the numbers $|F_n(T)|$ for all sets $T \subseteq S_3$ of 3-letter patterns. In Section 4, the number $|F_n(T)|$ is determined, for $|T|=2$.

Finally, we also presented a construction of lattice families $\{L_n(T)\}_{n\geq 1}$, whose maximal chains can be labeled so as to obtain a bijective correspondence between such chains from $L_n(T)$ and the AUNU permutations in $F_n(T)$.

An Overview of AUNU Numbers / AUNU Permutations

AUNU numbers first emerged out of a study conducted by its founder, Aminu A. Ibrahim (see www.algebragroup.org). AUNU numbers form a sequence of natural numbers that occur in some counting problems, often involving recursively defined objects [7,17,18]. The n^{th} AUNU number is given by the recursive relation $A_n = \frac{p_{n\geq 3}-1}{2}$ [15,18].This result calls for research of suitable prime genering functions. The first AUNU numbers for $n = 3,5,7,\ldots$ are 1,2 ,3,5,6,8,9,11,14,15,18,20,21,23,24,25,26,28,29,… (sequences A119626 and A123367 *in* OEIS).

They are categorized in three groups of Odd, Even and Prime numbers;

Odd numbers include: 1,3,5,9,11,21,23,25,29,…

Even numbers include: 2,6,8,14,18,20,24,26,28,…

Prime numbers include: 2,3,5,11,23,29,….

The entire work in this paper is based on the third category (the prime numbers). Observe that the only numbers which are odd are those for which $n = (2^k - 1)$ where *k* is a natural number.

All others are even numbers. In this paper we shall adopt the notation of symbols $S = \{1,2,3,\ldots,n\}$ for AUNU set (where *n* is prime) and $S_n = \{\alpha \mid \alpha : S \mapsto S\}$ the set of AUNU permutations.

Table 1: Some Special AUNU Permutations of Length n Containing Exactly 1 Occurrence of 123-Pattern

Length of the AUNU permutations (n)	*Number of Aunu permutations in S_n*	*Aunu permutations in S_n with occurrence of exactly one 123 – pattern*	*Number of Aunu permutations in S_n with occurrence of exactly one 123 – pattern* $(\lvert F_n^1(123)\rvert)$
2	1!	None	0
3	2!	123	1
5	4!	14532,15342,15423	3
7	6!	1675432,1756432, 1764532,1765342,1765423	5
11	10!	1(10)(11)98765432,1(11)9(10)8765432 1(11)(10)89765432,1(11)(10)97865432 1(11)(10)98675432,1(11)(10)98756432 1(11)(10)98764532,1(11)(10)98765342 1(11)(10)98765423.	9
⋮	⋮	⋮ ⋮	⋮ ⋮
⋮	⋮	⋮ ⋮	⋮ ⋮
n	$(n-1)!$		$(n-2)$

Let S_n denote the set of all permutations of $[n] = \{1,2,\ldots n\}$ We will view permutations in S_n as words with *n* distinct letters in *[n]*.

A pattern is a permutation $\sigma \in S_k$ and an occurrence of σ in a permutation $\pi = \pi_1\pi_2\ldots\pi_n \in S_n$ is a subsequence of π that is order equivalent to σ. For example, an occurrence of 213 is a subsequence $\pi_i\pi_j\pi_k (1 \le i < j < k \le n)$ of π such that $\pi_j < \pi_i < \pi_k$

Methodology

We denote the permutation $\pi \in S_n$ by the sequence $[\pi(1), \pi(2), \ldots, \pi(n)]$ That is, the set of bijections on $\{1,2,\ldots,n\}$ is denoted by S_n and set of all permutations of length n that contains exactly r occurrence of a pattern σ by $F_n^r(\sigma)$ where $\sigma \in S_k$ *and* $k \le n$ We also denote by $N_\sigma(\pi)$ (and $|N_\sigma(\pi)|$ the set (and respectively number) of occurrences of σ in π. We denote by $|F_n^r(\sigma)|$ the number of permutations in S_n such that $|N_\sigma(\pi)| = r$.

The next step is to consider permutations containing some prescribed number of sequences/sub permutations that have the same relative order as a given pattern. The tools involved are enumeration techniques. Observe that if π contains 123, then any 123-avoiding permutation avoids π as well, so in what follows we always assume that σ≼π if $\sigma^{-1} \preccurlyeq \pi^{-1}$, for instance.

We use one-line notation to express permutations in S_n. However, use cycle notations to express permutations when the need arises. For example, Suppose $\pi \in S_n$ .Since πis viewed as a bijection on the set $[n] = \{1, 2, \ldots, n\}$ This bijection can be represented by listing the elements in the set $[n] = \{1, 2, \ldots, n\}$ in a row with their images under π listed immediately below.

$$\pi : \begin{pmatrix} 1 & 2 \ldots & n \\ (1)\pi & (2)\pi \ldots & (n)\pi \end{pmatrix}$$

On the other hand, we expressed elements in $[n] = \{1, 2, \ldots, n\}$ as cycles (in cyclic form), let $i_1, i_2, \ldots, i_n$ be distinct elements in the set $[n] = \{1, 2, \ldots, n\} \cdot (i_1 i_2 \ldots i_{n-1} i_n)$ is called a cycle of length nor an n-cycle, and represents the permutation $\pi \in S_n$ that maps $i_1 \mapsto i_2, i_2 \mapsto i_3, \ldots, i_{n-1} \mapsto i_n, i_n \mapsto i_1$, and every other element in s to itself. Every permutation in S_n can be expressed as either a single cycle or a product of disjoint cycles. For example, the permutation $\pi = 345162 \in S_6$ can be expressed as the 6-cycle (135624). Next consider permutation $\alpha = 345612 \in S_6$. To express αusing cycle notation, we must use more than one cycle. For example, we can express αas the following "product" of two 3-cycles: (135) (246). Observe that these cycles contain no element(s) in common, and so they are said to be disjoint. And because they are disjoint, the order in which they are listed does not matter. The permutation αcan also be expressed as (246) (135). In an expression of a permutation as a product of cycles, the cycles need not be *disjoint*. For example, the permutation defined above can also be expressed as the product (13)(15)(16)(12)(14) of 2-cycles. Because these 2-cycles are not disjoint, the order in which they are listed matters.

Counting Occurrences of the Patterns $\sigma \in S_3$ in the Permutations $\pi \in S_n$

The counting of occurrences of a pattern σ in a permutation πis the number of distinct subsequences in the permutation πwhich are order isomorphic to the pattern σ. Let $\pi = a_1a_2\ldots a_t$ $\tau = b_1b_2\ldots b_S$ be finite sequences of integers; then a subsequence of π of the same length as τ is said to be an occurrence of τ if its entries occur in the same relative order as in τ. More precisely, given indices $i_1 < i_2 < \ldots < i_S$, the subsequence $a_{i_1}a_{i_2}\ldots a_{i_S}$ of π is an occurrence of τ if and only if for all *j*, *k* we have $a_{ij} < a_{i_k} \Leftrightarrow b_i < b_j$ Thus for example the patterns in $\sigma \in S_3$ occur in the following selected permutations $\pi \in S_5$ for $\pi = 15423$ there are $\binom{5}{3} = 10$ distinct three-length subsequences in π,

$154, 152, 153, 142, 143, 123, 542, 543, 524, 423$ .Then

$|N_{[123]}(\pi)| = 1, |N_{[132]}(\pi)| = 5, |N_{[213]}(\pi)| = |N_{[231]}(\pi)| = 0, |N_{[312]}(\pi)| = |N_{[321]}(\pi)| = 2$

and of course, $\sum_{\sigma \in S_3} |N_\sigma(\pi)| = 10$ For $\pi = 32451$ 10 three-length Subsequences of $\pi = 32451$ are $324, 325, 321, 345, 341, 351, 245, 241, 251, 451$.

Then

$|N_{[123]}(\pi)| = |N_{[213]}(\pi)| = 2, |N_{[321]}(\pi)| = 1, |N_{[132]}(\pi)| = |N_{[312]}(\pi)| = 0, |N_{[231]}(\pi)| = 5$

For $\pi = 51243$, 10 three-length subsequences of are $512, 514, 513, 524, 523, 543, 124, 123, 143, 243$ .Then

$|N_{[123]}(\pi)| = |N_{[132]}(\pi)| = 2, |N_{[213]}(\pi)| = |N_{[231]}(\pi)| = 0\ |N_{[321]}(\pi)| = 1, |N_{[312]}(\pi)| = 5$

For $\pi = 14532$ 10 three-length subsequences of $\pi = 14532$ are $145, 143, 142, 153, 152, 132, 453, 452, 432, 532$. Then

$|N_{[123]}(\pi)| = 1, |N_{[132]}(\pi)| = 5, |N_{[213]}(\pi)| = |N_{[312]}(\pi)| = 0, |N_{[231]}(\pi)| = |N_{[321]}(\pi)| = 2$

Note that the number $|N_\sigma(\pi)|$ is independent of π for example,

- $for\ |N_\sigma(15423)|\ when\ \sigma = 213, 231, 123, 312, 321, 132,\ we\ have\ 0, 0, 1, 2, 2, 5;$
- $for\ |N_\sigma(32451)|\ when\ \sigma = 132, 312, 321, 123, 213, 231,\ we\ have\ 0, 0, 1, 2, 2, 5;$
- $for\ |N_\sigma(51243)|\ when\ \sigma = 213, 231, 321, 123, 132, 312,,\ we\ have\ 0, 0, 1, 2, 2, 5;$
- $for\ |N_\sigma(14532)|\ when\ \sigma = 213, 312, 123, 231, 321, 132,\ we\ have\ 0, 0, 1, 2, 2, 5.$

Observe that on one hand the permutation π the in second, third and fourth rows (32451, 51243 and 14532) are the reversal, complementation and inversion of the permutation in the first raw (15423). On the other hand, the corresponding patterns, specifically those bearing the same values of $|N_\sigma(\pi)|$ are related similar. For example, the patterns with values of $|N_\sigma(\pi)| = 0$ in 2nd, 3rd, and 4th rows (i.e. $\{312, 132\}, \{231, 213\}, and \{213, 312\}$) are reversal, complementation and inversion respectively of the corresponding patterns in the 1st raw (i.e. $\{213, 231\}$), etc. Similarly, the patterns 321,321 and 123, each bearing the same value $|N_\sigma(\pi)| = 1$, are reversal, complementation and inversion of the corresponding pattern 123 in the 1st raw; the pairs of patterns, each bearing the value $|N_\sigma(\pi)| = 2$ {213,123}, {132,123}, and {231,321} in the 2nd, 3rd, and 4th rows, relate in the same manner to the pair {312,321} in the 1st raw; and,lastly, the patterns 231,312 and 132 in the 2nd, 3rd, and 4th rows relate in the same manner to the pattern 132 in the 1st raw, each bearing $|N_\sigma(\pi)| = 5$ .We summarized the above in table 1 below.

Table 1 (a): Patterns Occurrences in Some Permutations Via Standard Bijections

Permutation $\pi \in S_5$	Pattern $\sigma \in S_3$	$\lvert N_\sigma(\pi)\rvert$ Number of occurrences of σ in permutation π
15423	$\{213, 231\}, \{312, 132\}, \{231, 213\}, \{213, 312\}$	0
32451	$\{123\}, \{321\}$	1
51243	{123,213}, {321,312}, {321,231}	2
14532	$\{132\}, \{231\}, \{312\}$	5

Table 1(a), shows that there are Easy Correspondences which Explain why

$$|F_n(132)| = |F_n(213)| = |F_n(231)| = |F_n(312)|$$

and why

$$|F_n(123)| = |F_n(321)|$$

A tree diagram for Aunu permutations of length 5 containing exactly one Occurrence of 123 pattern

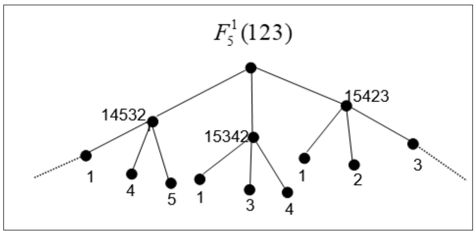

Figure 1: Tree representation for $F_5^1(123)$

Results and Discussion

Enumerative Results

As a consequence to our enumeration so far, and apparently for the first time the relation between even (odd) AUNU permutations and pattern-occurrence problem, we found that

$$|E_n^1(123)| = |O_n^1(123)| = (n-2)\,|sign(\pi)|;\ where\ n \ge 3,\ \pi \in S_n$$

$$and\ sign(\pi) = \begin{cases} 1, & if\ \ \pi\ is\ even, \\ -1, & if\ \pi\ is\ odd. \end{cases}$$

Generally speaking, $|E_n| = |O_n| = \frac{1}{2}n!\ for\ all\ n \ge 2$ The followinglemma holds immediately by definitions.

Lemma 1: $If\ \ \pi \in F_n^r(\sigma),\ then\ \pi^c \in F_n^r(\sigma^c), \forall \sigma \preceq \pi.$

Proof: If, for $1 \le i_1 < i_2 < ... < i_k \le$, a subsequence $\sigma = (\pi(i_1)\pi(i_2)...\pi(i_k))$ is of type τ, then the subsequence $(\pi^c(i_1)\pi^c(i_2)...\pi^c(i_k))$ is isomorphic to the reverse of σ^c. That is, it is of type . The remaining part of the argument is identical in its essential to the previous proof. Hence the proof is complete.

Lemma 2: $If\ \ \pi \in F_n^r(\sigma),\ then\ \pi^{-1} \in F_n^r(\sigma^{-1}), \forall \sigma \preceq \pi.$

Proof: Suppose π has a subsequence of type τ namely $(\pi(i_{\tau(1)})\pi(i_{\tau(2)})...\pi(i_{\tau(k)}))$, where $1 \le i_{\tau(1)} < i_{\tau(2)} < ... < i_{\tau(k)} \le n$ and $(\pi(i_1) < \pi(i_2) < ... < \pi(i_k))$. In light of the last set of inequalities, it is clear that one subsequence of π^{-1} is

$(\pi^{-1}(\pi(i_1)), \pi^{-1}(\pi(i_2)), ..., \pi^{-1}(\pi(i_k))) = (i_1 i_2 ... i_k)$. This is a subsequence of type τ^{-1} .Since π^{-1} contains a subsequence of type τ^{-1} , precisely when π contains the pattern τ, the inverse permutation π^{-1} contains τ^{-1} , Precisely when π contains τ.

As a further observation of the relation between the three operations of reveral, complementation and inversion, note that $(\pi^C)^{-1} = (\pi^{-1})^R$

For example, consider a permutation $\pi = 14532 \in S_5$ and a pattern. $\tau = 123 \in S_3$ Then it is clear that

$$\pi^r = 23541\ and\ \tau^r = 321,$$
$$\pi^c = 52134\ and\ \tau^c = 321,$$
$$\pi^{-1} = 15423\ and\ \tau^{-1} = 123.$$

Here, the pattern $\tau = 123 \in S_3$ has exactly one occurrence in the permutation $\pi = 14532 \in S_5$, in its sunsequence $\pi_1\pi_2\pi_3 = 145$, while the pattern 213 has zero occurrences in π (i.e. π avoids 213 pattern); the pattern $\tau^r = 321$ has exactly one occurrence in the permutation $\pi^r = 23541$, in its subsequence $\pi_3^r\pi_4^r\pi_5^r = 321$; the pattern $\tau^c = 321$ has exactly one occurrence in the permutation $\pi^c = 52134$, in its subsequence $\pi_1^c\pi_2^c\pi_3^c = 321$; finally, the pattern $\tau^{-1} = 123$ has also one occurrence in the permutation $\pi^{-1} = 15423$, in its subsequence $\pi_1^{-1}\pi_4^{-1}\pi_5^{-1} = 123$.

It is therefore easy to see that π avoids the subsequences 312, and 213. It follows that avoids the subsequences 213 and 312, that avoids the subsequences 132 and 231, and that avoids the subsequences 231 and 213. The following proposition follows from Simion and Schmidt [19].

Proposition 1: We have that Ω_p is isomorphic to the dihedral group D_8.

Proof: It is easy to see that $r^2 = c^2 = (r \circ c)^2 = 1, c \circ r = r \circ c, i^2 = (r \circ i)^4 = (c \circ i)^4 = 1$ and $i \circ r \circ i = c$ So, Ω_p is isomorphic to D_8.

More generally, for a set of patterns T, we define $g(T) = \{g(\tau) \mid \tau \in T\}$ for any $g \in \Omega_p$. For example, if $T = \{123,132\}$ and $g = r$, then $g(T) = \{321, 231\}$ The following proposition was given by Simion and Schnidt [29].

Proposition 2: A permutation that consists of exactly one even cycle is odd. A permutation that consists ofexactly one odd cycle is even.

Proof: We prove the claim by induction on the length n of the only cycle of our permutation π. For $n = 1$ and $n = 2$ the statement is trivially true. Now let $n \geq 3$, and consider the cycle $(i_1 i_2 \ldots i_{n-1} i_n)$. It is straightforward to verify that $(i_1 i_2 \ldots i_{n-1} i_n)\ (i_1 i_2 \ldots i_{n-1} i_n) = (i_1 i_2 \ldots i_{n-1})(i_{n-1} i_n)$. The multiplication by $(i_{n-1} i_n)$ at the end simply swapsthe last two entries of $(i_1 i_2 \ldots i_{n-1})$, and therefore, either increases the number ofinversions by one, or decreases it by one. So in either case, it changes the parity of the number of inversions. The proof is then immediate by the induction hypothesis.

Theorem 1: Any permutation π, $sign(\pi) = (-1)^{21(|\pi|)}$, |π | is the length of π.

Proof: Considering the propositions 1.1 & 1.2 above we obtain

$$|E_n^1(123)| = |O_n^1(123)| = (n-2)\,|sign(\pi)|;\ where\ n \geq 3,\ \pi \in S_n$$

$$and\ sign(\pi) = \begin{cases} 1, & if\ \pi\ is\ even, \\ -1, & if\ \pi\ is\ odd. \end{cases}$$

Conclusion

This paper has no doubt explore the basic methods in the counting the number of even and odd AUNU permutations with occurrences of 123- pattern. The number of even and odd of such permutations is determined and the involutions among them are counted. Bijections are obtained between sets of such permutations and other combinatorial objects. We also obtained generating functions $|E_n^r(123)|$ and $|O_n^r(123)|$ for the number of even and odd permutation. AUNU permutations on n letters containing exactly occurrence of 123 avoiding permutation were also found.

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