ISSN: 2754-4753

Journal of Physics & Optics Sciences

Review Article

Open Access

The Iterative Solution of Taylor Formula for Partial Differential Equation

Yang Gao

Department of Science, Binzhou University, Binzhou, China

ABSTRACT

This paper discusses the relation between Taylor's formula and partial differential equation. Taylor formula iteration method can resolve partial differential equation. $u(x,t)$be expanded at $t = 0$ or $t = 1$by Taylor formula.Coefficient of Taylor formula $ut(x,0)$,$utt(x,0)$...can be expressed by partial differential equation. The method can solve nonlinear differential equation.Generalized Taylor's formula can solve fractional partial differential equation. The method is very important way that resolving partial differential equation. The method also can resolve those equations from [1-9]. This article refers to the literature [10]. Taylor formula iteration method belongs to logical thinking.

*Corresponding author

Yang Gao, Department of Science, Binzhou University, Binzhou, China.

Received: September 19, 2025; **Accepted:** September 22, 2025; **Published:** February 27, 2026

Keywords: Taylor's Formula, Iteration Method, Nonlinear, Partial Differential Equation

Introduction

This paper introduce that Taylor formula iteration method resolves partial differential equation. In this paper, six examples are used to introduce Taylor formula iteration method to solve partial differential equation. This paper also introduces that Generalized Taylor's formula can solve fractional partial differential equation. The iterative method of Taylor formula is an important and useful method to solve partial differential equation. The solution of Taylor formula iteration method belongs to C^∞.

Variable Coefficient Problem

We consider equation as following:

$$xu_{tt}(x,t) - (xu_x(x,t))_x = 0, \quad (1)$$

$$u(x,0) = x^2, \quad (2)$$

$$u_t(x,0) = 0. \quad (3)$$

We solve (1) by Taylor formula iteration method as following:

$$xu_{tt}(x,t) - (xu_x(x,t))_x = 0, \quad (4)$$

$$u_{tt} = u_{xx} + \frac{1}{x}u_x, \quad (5)$$

$$u_{tt}(x,0) = u_{xx}(x,0) + \frac{1}{x}u_x(x,0), \quad (6)$$

$$u_{xx}(x,0) = (x^2)''_{xx} = 2, \quad (7)$$

$$u_x(x,0) = (x^2)'_x = 2x, \quad (8)$$

$$u_{tt}(x,0) = 2 + 2 = 4, \quad (9)$$

$$u_{ttt} = u_{txx} + \frac{1}{x}u_{tx}, \quad (10)$$

$$u_{ttt}(x,0) = u_{txx}(x,0) + \frac{1}{x}u_{tx}(x,0), \quad (11)$$

$$u_t(x,0) = 0, \quad (12)$$

$$u_{txx}(x,0) = 0, \quad (13)$$

$$u_{tx}(x,0) = 0, \quad (14)$$

$$u_{ttt}(x,0) = 0, \quad (15)$$

And we have:

$$u_{tttt}(x,0) = u_{ttttt}(x,0) = \ldots = 0. \quad (16)$$

By Taylor's formula, we get as following:

$$u(x,t) = u(x,0) + u_t(x,0)t + u_{tt}(x,0)\frac{t^2}{2!} + u_{ttt}(x,0)\frac{t^3}{3!} + \ldots, \quad (17)$$

$$u(x,t) = x^2 + 2t^2. \quad (18)$$

Solution of equation (1), $,u(x,t) = x^2 + 2t^2$.

Two-Dimensional Heat Conduction Equation Solution
We study the equation as following:

$$u_t - tx(u_{xx} + u_{yy}) = t^2, \quad (19)$$

$$u(x, y, 0) = xy + y^3. \quad (20)$$

Next, we solve (19) by Taylor formula iteration method,

$$u_t = tx(u_{xx} + u_{yy}) + t^2, \quad (21)$$

Let t = 0 on (21),

$$ut(x,y,0) = 0, \quad (22)$$

On equation (21), finding 1-order partial derivative of *t* on both sides,

We have:

$$u_{tt} = x(u_{xx} + u_{yy}) + tx(u_{txx} + u_{tyy}) + 2t, \quad (23)$$

$$u_{xx}(x, y, 0) = 0, \quad (24)$$

$$u_{yy}(x, y, 0) = 6y, \quad (25)$$

Next, t = 0 on (23),

$$u_{tt}(x, y, 0) = 6xy, \quad (26)$$

On equation (23), finding 1-order partial derivative of t on both sides, We have:

$$u_{ttt} = 2x(u_{txx} + u_{tyy}) + tx(u_{ttxx} + u_{ttyy}) + 2, \quad (27)$$

$$u_{ttxx}(x, y, 0) = (6xy)''_{xx} = 0, \quad (28)$$

$$u_{txx}(x, y, 0) = 0, \quad (29)$$

$$u_{tyy}(x, y, 0) = 0, \quad (30)$$

Next, t = 0 on (27),

$$u_{ttt}(x, y, 0) = 2, \quad (31)$$

$$u_{tttt} = 3x(u_{ttxx} + u_{ttyy}) + tx(u_{tttxx} + u_{tttyy}), \quad (32)$$

$$utttt\ (x,y,0) \quad (33)$$

So we have :

$$u_{ttttt}(x, y, 0) = u_{tttttt}(x, y, 0) = \ldots = 0, \quad (35)$$

By Taylor's formula, we get as following:

$$u(x, y, t) = u(x, y, 0) + u_t(x, y, 0)t + u_{tt}(x, y, 0)\frac{t^2}{2!} + u_{ttt}(x, y, 0)\frac{t^3}{3!} + \ldots, \quad (36)$$

$$u(x, y, t) = (3t^2 + 1)xy + y^3 + \frac{t^3}{3}. \quad (37)$$

Solution of (19) $u(x, y, t) = (3t^2 + 1)xy + y^3 + \frac{t^3}{3}$.

The Third Problem with Boundary Values
We consider following equation:

$$u_t - 4u_{xx} = \cos t, \quad (38)$$

$$u(x, 0) = \cos x, \quad (39)$$

$$u_x(0, t) = u_x(1, t) = 0. \quad (40)$$

By Taylor formula iteration method,we have:

$$u_t = 4u_{xx} + \cos t, \quad (41)$$

$$u_t(x, 0) = 4u_{xx}(x, 0) + 1, \quad (42)$$

$$u_t(x, 0) = -4\cos x + 1, \quad (43)$$

$$u_{tt} = 4u_{txx} - \sin t, \quad (44)$$

$$u_{tt}(x, 0) = 4u_{txx}(x, 0), \quad (45)$$

$$u_{tt}(x, 0) = 4^2 \cos x, \quad (46)$$

$$u_{ttt} = 4u_{ttxx} - \cos t, \quad (47)$$

$$u_{ttt}(x, 0) = 4u_{ttxx}(x, 0) - 1, \quad (48)$$

$$u_{ttt}(x, 0) = -4^3 \cos x - 1, \quad (49)$$

$$u_{tttt} = 4u_{tttxx} + \sin t, \quad (50)$$

$$u_{tttt}(x, 0) = 4u_{tttxx}(x, 0), \quad (51)$$

$$u_{tttt}(x, 0) = 4^4 \cos x, \quad (52)$$

$$u_{ttttt} = 4u_{ttttxx} + \cos t, \quad (53)$$

$$u_{ttttt}(x, 0) = -4^5 \cos x + 1, \quad (54)$$

(55)

By Taylor's formula, we get as following:

$$u(x, t) = u(x, 0) + u_t(x, 0)t + u_{tt}(x, 0)\frac{t^2}{2!} + u_{ttt}(x, 0)\frac{t^3}{3!} + \ldots, \quad (56)$$

$$u(x,t)=\sum_{n=0}^{\infty}\frac{(-4)^n}{n!}t^n\cos x+\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}t^{2n+1}, \tag{57}$$

$$\sin t=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}t^{2n+1}, \tag{58}$$

We have:

$$u(x,t)=e^{-4t}\cos x+\sin t. \tag{59}$$

We take the best of Fourier expansion:

$$a_n(t)=2\int_0^1(e^{-4t}\cos x)\cos(n\pi x)dx, \tag{60}$$

We get the solution of (38):

$$u(x,t)=\sum_{n=1}^{\infty}a_n(t)\cos(n\pi x)+e^{-4t}\sin 1+\sin t. \tag{61}$$

Fractional Partial Differential Equation

$$\frac{\partial^\alpha u(x,y,t)}{\partial t^\alpha} = D_x^\beta u(x,y,t)+D_y^\gamma u(x,y,t)+u(x,y,t), \tag{62}$$

$$u(x,y,0) = q(x,y). \tag{63}$$

where q(x,y)is known integral polynomial.
The definition of Caputo fractional derivative about t:
$q_1(x;y)$is known function.

We consider $\frac{\partial^{2\alpha}u(x,y,0)}{\partial t^{2\alpha}}$. On equation (62),nding -order partial derivative of t on both sides,

$$\frac{\partial^{2\alpha}u(x,y,t)}{\partial t^{2\alpha}} = D_x^\beta\frac{\partial^\alpha u(x,y,t)}{\partial t^\alpha}+D_y^\gamma\frac{\partial^\alpha u(x,y,t)}{\partial t^\alpha}+\frac{\partial^\alpha u(x,y,t)}{\partial t^\alpha}, \tag{64}$$

$$\frac{\partial^{2\alpha}u(x,y,0)}{\partial t^{2\alpha}} = D_x^\beta\frac{\partial^\alpha u(x,y,0)}{\partial t^\alpha}+D_y^\gamma\frac{\partial^\alpha u(x,y,0)}{\partial t^\alpha}+\frac{\partial^\alpha u(x,y,0)}{\partial t^\alpha}, \tag{65}$$

$$\frac{\partial^{2\alpha}u(x,y,0)}{\partial t^{2\alpha}} = D_x^\beta q_1(x,y)+D_y^\gamma q_1(x,y)+q_1(x,y), \tag{66}$$

where

$$\frac{\partial^{2\alpha}u(x,y,0)}{\partial t^{2\alpha}}=q_2(x,y), \tag{67}$$

$q_2(x;y)$is known function.

We consider $\frac{\partial^{3\alpha}u(x,y,0)}{\partial t^{3\alpha}}$. On equation (62),nding 2-order partial derivative of *t* on both sides, We have:

$$\frac{\partial^{3\alpha}u(x,y,t)}{\partial t^{3\alpha}} = D_x^{\beta}\frac{\partial^{2\alpha}u(x,y,t)}{\partial t^{2\alpha}} + D_y^{\gamma}\frac{\partial^{2\alpha}u(x,y,t)}{\partial t^{2\alpha}} + \frac{\partial^{2\alpha}u(x,y,t)}{\partial t^{2\alpha}}, \tag{68}$$

$$\frac{\partial^{3\alpha}u(x,y,0)}{\partial t^{3\alpha}} = D_x^{\beta}\frac{\partial^{2\alpha}u(x,y,0)}{\partial t^{2\alpha}} + D_y^{\gamma}\frac{\partial^{2\alpha}u(x,y,0)}{\partial t^{2\alpha}} + \frac{\partial^{2\alpha}u(x,y,0)}{\partial t^{2\alpha}}, \tag{69}$$

$$\frac{\partial^{3\alpha}u(x,y,0)}{\partial t^{3\alpha}} = D_x^{\beta}q_2(x,y) + D_y^{\gamma}q_2(x,y) + q_2(x,y), \tag{70}$$

where

$$\frac{\partial^{3\alpha}u(x,y,0)}{\partial t^{3\alpha}} = q_3(x,y), \tag{71}$$

$q_3(x;y)$is known function. We consider $\frac{\partial^{4\alpha}u(x,y,0)}{\partial t^{4\alpha}}$. On equation (62),nding 3-order partial derivative of t on both sides,

We have:

$$\frac{\partial^{4\alpha}u(x,y,t)}{\partial t^{4\alpha}} = D_x^{\beta}\frac{\partial^{3\alpha}u(x,y,t)}{\partial t^{3\alpha}} + D_y^{\gamma}\frac{\partial^{3\alpha}u(x,y,t)}{\partial t^{3\alpha}} + \frac{\partial^{3\alpha}u(x,y,t)}{\partial t^{3\alpha}}, \tag{72}$$

$$\frac{\partial^{4\alpha}u(x,y,0)}{\partial t^{4\alpha}} = D_x^{\beta}\frac{\partial^{3\alpha}u(x,y,0)}{\partial t^{3\alpha}} + D_y^{\gamma}\frac{\partial^{3\alpha}u(x,y,0)}{\partial t^{3\alpha}} + \frac{\partial^{3\alpha}u(x,y,0)}{\partial t^{3\alpha}}, \tag{73}$$

$$\frac{\partial^{3\alpha}u(x,y,0)}{\partial t^{3\alpha}} = D_x^{\beta}q_2(x,y) + D_y^{\gamma}q_2(x,y) + q_2(x,y), \tag{74}$$

where

$$\frac{\partial^{3\alpha}u(x,y,0)}{\partial t^{3\alpha}} = q_3(x,y), \tag{75}$$

$q_3(x;y)$is known function. We consider $\frac{\partial^{4\alpha}u(x,y,0)}{\partial t^{4\alpha}}$.On equation (62),nding 3-order partial derivative of t on both sides,

We have:

$$\frac{\partial^{4\alpha}u(x,y,t)}{\partial t^{4\alpha}} = D_x^{\beta}\frac{\partial^{3\alpha}u(x,y,t)}{\partial t^{3\alpha}} + D_y^{\gamma}\frac{\partial^{3\alpha}u(x,y,t)}{\partial t^{3\alpha}} + \frac{\partial^{3\alpha}u(x,y,t)}{\partial t^{3\alpha}}, \tag{76}$$

$$\frac{\partial^{4\alpha}u(x,y,0)}{\partial t^{4\alpha}} = D_x^{\beta}\frac{\partial^{3\alpha}u(x,y,0)}{\partial t^{3\alpha}} + D_y^{\gamma}\frac{\partial^{3\alpha}u(x,y,0)}{\partial t^{3\alpha}} + \frac{\partial^{3\alpha}u(x,y,0)}{\partial t^{3\alpha}}, \tag{77}$$

$$\frac{\partial^{4\alpha}u(x,y,0)}{\partial t^{4\alpha}} = D_x^{\beta}q_3(x,y) + D_y^{\gamma}q_3(x,y) + q_3(x,y), \tag{78}$$

where

$$\frac{\partial^{4\alpha}u(x,y,0)}{\partial t^{4\alpha}} = q_4(x,y), \tag{79}$$

$q_4(x;y)$is known function. We have:

$$\frac{\partial^{n\alpha}u(x,y,0)}{\partial t^{n\alpha}} = q_n(x,y), \tag{80}$$

$q_n(x; y)$is known function.

By Generalized Taylor's formula:

$$u(x,y,t) = \sum_{j=0}^{N} \frac{t^{j\alpha}}{\Gamma(j\alpha+1)} \frac{\partial^{j\alpha} u(x,y,0)}{\partial t^{j\alpha}} + \dots. \quad (81)$$

So we have the solution of (62):

$$u(x,y,t) = q(x,y) + q_1(x,y)\frac{t^{\alpha}}{\Gamma(\alpha+1)} + q_2(x,y)\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \dots. \quad (82)$$

Nonlinear *KdV* Equation

We consider the wave equation as following:

$$u_t - 6uu_x + u_{xxx} = 0, \quad (83)$$

$$u(x,0) = x. \quad (84)$$

We have:

$$u_t = 6uu_x - u_{xxx}, \quad (85)$$

$$u_t(x,0) = 6u(x,0)u_x(x,0) - u_{xxx}(x,0), \quad (86)$$

$$u(x,0) = x, \quad (87)$$

$$u_x(x,0) = 1, \quad (88)$$

$$u_{xxx}(x,0) = 0, \quad (89)$$

$$u_t(x,0) = 6x, \quad (90)$$

On equation (88), finding 1-order partial derivative of t on both sides, we have:

$$u_{tt} = 6(u_t u_x + uu_{tx}) - u_{txxx}, \quad (91)$$

$$u_{tt}(x,0) = 6(u_t(x,0)u_x(x,0) + u(x,0)u_{tx}(x,0)) - u_{txxx}(x,0), \quad (92)$$

$$u_{tx}(x,0) = 6, \quad (93)$$

$$u_{txxx}(x,0) = 0, \quad (94)$$

$$u_{tt}(x,0) = 2\cdot 6^2 x, \quad (95)$$

On equation (94),nding 1-order partial derivative of t on both sides,
We have:

$$u_{ttt} = 6(u_{tt}u_x + 2u_t u_{tx} + uu_{ttx}) - u_{ttxxx}, \quad (96)$$

$$u_{ttt}(x,0) = 6(u_{tt}(x,0)u_x(x,0) + 2u_t(x,0)u_{tx}(x,0) + u(x,0)u_{ttx}(x,0)) - u_{ttxxx}(x,0),$$

$$u_{tx}(x,0) = 6, \quad (97)$$

$$u_{ttx}(x,0) = 2\cdot 6^2, \quad (98)$$

$$u_{ttt}(x,0) = 6^4 x. \quad (99)$$

By Taylor's formula ,we get as following:

$$u(x,t) = u(x,0) + u_t(x,0)t + u_{tt}(x,0)\frac{t^2}{2!} + u_{ttt}(x,0)\frac{t^3}{3!} + \dots, \quad (100)$$

We have:

$$u(x,t) = x + 6xt + 6^2xt^2 + 6^3xt^3 + 6^4xt^4 + \dots, \quad (104)$$

$$u(x,t) = \frac{x}{1-6t}. \quad (105)$$

So we can get the solution of (86),$u(x,t) = \frac{x}{1-6t}$.

Nonlinear sine-Gordon Equation

We consider following

On equation (114),nding 1-order partial derivative of t on both sides,
We have:

$$u_{tttt} = c^2 u_{ttxx} - \alpha u_{tt}\cos u + \alpha u_t^2 \sin u, \quad (106)$$

$$u_{tttt}(x,0) = c^2 u_{ttxx}(x,0) - \alpha u_{tt}(x,0)\cos u(x,0) + \alpha(u_t(x,0))^2 \sin u(x,0),$$

$$u_{ttxx}(x,0) = \alpha \sin x, \quad (107)$$

$$u_{tttt}(x,0) = \alpha c^2 \sin x + \alpha^2 \sin x \cos x + \alpha \sin x, \quad (108)$$

uttttt(x; 0),utttttt(x; 0) : : :is known function.

By Taylor's formula ,we get as following:

$$u(x,t) = u(x,0) + u_t(x,0)t + u_{tt}(x,0)\frac{t^2}{2!} + u_{ttt}(x,0)\frac{t^3}{3!} + u_{tttt}(x,0)\frac{t^4}{4!} + \dots \quad (109)$$

$$u_{ttxx}(x,0) = \alpha \sin x, \quad (110)$$

$$u_{tttt}(x,0) = \alpha c^2 \sin x + \alpha^2 \sin x \cos x + \alpha \sin x, \quad (111)$$

uttttt(x; 0),utttttt(x; 0) : : :is known function.

By Taylor's formula ,we get as following:

$$u(x,t) = u(x,0) + u_t(x,0)t + u_{tt}(x,0)\frac{t^2}{2!} + u_{ttt}(x,0)\frac{t^3}{3!} + u_{tttt}(x,0)\frac{t^4}{4!} + \dots \quad (112)$$

So, we can get the solution of (107).

Iterative solution of partial differential equations by Taylor formula is important and good method that solve linear and nonlinear partial differential equations. And the method also can solve fractional partial differential equations.

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