ISSN: 2754-6705

Journal of Mathematical & Computer Applications

SCIENTIFIC Research and Community

Research Article

Open Access

Multiplicity of Solutions for Dirichlet Problem of a Superlinear Fractional *p*-Laplacian Equation

Tao Zhang[1], WeiQiang Zhang[2*] and Peihao Zhao[2]

[1]Department of Lanzhou Technology and Business College, Lanzhou, Gansu 730101, People's Republic of China

[2]School of Mathematics and Statistics, Lanzhou University, Lanzhou, Gansu 730000, People's Republic of China

ABSTRACT

This paper is dedicated to studying the following fractional *p*-Laplacian problem

$$\begin{cases} (-\Delta)_p^s u = f(x,u) & \text{in } \Omega \\ u = 0 & \text{in } \Omega^c, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ ($N \geq 2$) is a bounded domain with $C^{1,1}$ boundary, $s \in (0,1)$, $2 \leq p < N/s$, $(-\Delta)_p^s$ is the fractional *p*-Laplacian and *f* is a (*p*-1)-superlinear Carathéodory reaction but does not satisfy the usual Ambrosetti-Rabinowitz condition. By using variational methods and Morse theory, we show that the equation has at least three nontrivial solutions: among one of them is positive and one is negative.

***Corresponding author**

WeiQiang Zhang, School of Mathematics and Statistics, Lanzhou University, Lanzhou, Gansu 730000, People's Republic of China.

E-mail: zhangwq19@lzu.edu.cn

Received: November 18, 2022; **Accepted:** November 21, 2022, **Published:** November 30, 2022

Keywords: Fractional p-Laplacian, Superlinear Reaction, Dirichlet Boundary, Variational Methods, Morse Theory.

Introduction

In recent years, the study of fractional nonlinear boundary value problem driven by *p*-Laplacian has been an interesting topic, which is motivated not only by the significant physical meaning, but also by the mathematical importance in the nonlocal problem of fractional order; see [1]. On some basic properties of the fractional *p*-Laplacian and the corresponding Sobolev spaces, we refer the readers to [1-3].

In this paper, we consider the following fractional nonlinear *p*-Laplacian equation with Dirichlet boundary

$$\begin{cases} (-\Delta)_p^s u = f(x,u) & \text{in } \Omega \\ u = 0 & \text{in } \Omega^c, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ ($N \geq 2$) is a bounded domain with $C^{1,1}$ boundary, $s \in (0,1)$, $2 \leq p < N/s$, the fractional p-Laplacian is defined for all $u: \mathbb{R}^N \to \mathbb{R}$ smooth enough by

$$(-\Delta)_p^s u(x) = 2 \lim_{\epsilon \to 0} \int_{B_\epsilon^c(x)} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{N+ps}} dy$$

and the reaction $f: \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function (i.e. for any $t \in \mathbb{R}$ the map $x \to f(x,t)$ is measurable, and for a.e. $x \in \Omega$ the map $t \to f(x,t)$ is continuous). We assume that *f* is (*p*-1)-superlinear growth and does not fulfill the usual Ambrosetti-Rabinowitz condition (see Section 2 for more details): there exist $\theta > p$ and $M > 0$ such that

$$0 < \theta F(x,t) \leq f(x,t)t \quad \text{for a.e.} \quad x \in \Omega \quad \text{and all} \quad |t| > M. \tag{AR}$$

Note that the condition (AR) plays a very important role in verifying the mountain pass geometry and the boundedness of Palais-Smale sequence or Cerami sequence for the associated variational functional. However, (AR) is a very crucial condition, which implies that *f* is (θ-1)-superlinear growth at infinity and eliminates many nonlinearities.

In the comparison principle and weighted Hölder regularity up to boundary for the degenerate fractional *p*-Laplacian equation (fractional p-Laplacian is degenerate if $p \geq 2$) with Dirichlet boundary have been investigated by the authors [2,3]. Now, let us introduce some recent research results of equation (1.1) in the variational framework. By using Morse theory and variational methods, the multiplicity of solutions has been investigated by Düzgün-Iannizzotto, Iannizzotto-Liu-Perera-Squassina, Iannizzotto-Papageorgiou, Iannizzotto-Livrea and Frassu-Iannizzotto [4-8]. Also, Frassu-Iannizzotto obtained extremal constant sign solutions and nodal solutions by applying variational methods and truncation techniques [9]. At these mentioned works, f is not (p-1)-superlinear growth.

In the celebrated paper, by using Link method and Morse theory, Wang established three solutions for a semilinear elliptic equation under the assumptions that $f \in C^1(\Omega\times\mathbb{R},\mathbb{R})$, has no any symmetry and satisfies the Ambrosetti-Rabinowitz condition [10]. Here our main purpose is to extend Wang's results to equation (1.1) with more mild assumptions on f.

Notation: In this text, for any $A\subset\mathbb{R}^N$, let A^c denote $\mathbb{R}^N\setminus A$. For every $r\in[1,\infty]$, $\|\cdot\|_r$ denotes the standard norm of $L^r(A)$ (or $L^r(\mathbb{R}^N)$). Let $C, C_1, C_2, \ldots$ denote some fixed constants possibly different in different places, and let $\to$ and $\rightharpoonup$ denote strong convergence and weak convergence in the corresponding space respectively.

This paper is organized as follows. In Section 2, we introduce some properties of spaces and fractional p-Laplacian. In Section 3, we demonstrate the existence of constant solutions, and in the last section, we show the existence of three nontrivial solutions.

Preliminaries

We start with some basic preliminary results on the Morse theory. Let X be a Banach space. We denote by $\langle\cdot,\cdot\rangle$ the duality bracket for the pari (X^*,X).

Given $\varphi\in C^1(X)$ and $c\in\mathbb{R}$, we introduce the following notations:

$$\varphi^c=\{u\in X:\varphi(x)\le c\},\quad K_\varphi=\{u\in X:\varphi'(u)=0\},$$

and $$K_\varphi^c=\{u\in K_\varphi:\varphi(u)=c\}.$$

Let (X_1,X_2) be a topological pair such that $X_2\subset X_1\subset X$. For every integer $k\ge 0$, we denote by $H_k(X_1,X_2)$ the k^{th} -relative singular homology group with integer coefficients for the topological pair (X_1,X_2).

Assume that $u\in X$ is an isolated critical point of φ with $\varphi(u)=c$. Then the critical group of φ at u is defined by

$$C_k(\varphi,u):=H_k(\varphi^c\cap U,\varphi^c\cap U\setminus\{u\})\quad\text{for all}\quad k\ge 0,$$

where U is a neighborhood of u such that $K\varphi\cap\varphi^c\cap U=\{u\}$. By the excision property of singular homology group, we know the definition of critical group at u is independent of the choice of the neighborhood of u.

Recall that $\{u_n\}\subset X$ is said to be a Cerami sequence at the level c ($(C)_c$ sequence for short) if $\varphi(u_n)\to c$ and $(1+\|u_n\|)\varphi'(u_n)\to 0$, and φ is said to satisfy the Cerami condition ($(C)_c$ condition) if any $(C)_c$ sequence possesses a convergent subsequence.

Let φ satisfy the $(C)_c$ condition and $\inf\ \varphi(K_\varphi)> -\infty$. Take $c<\inf\varphi(K_\varphi)$. Then the critical group of φ at infinity is defined by

$$C_k(\varphi,\infty):=H_k(X,\varphi^c)\quad\text{for all}\quad k\ge 0.$$

The second deformation theorem shows that the above definition is independent of the choice of the number $c<\inf\varphi(K_\varphi)$.

If K_φ is finite, we can define that

$$M(t,u)=\sum_{k\ge 0}\operatorname{rank}C_k(\varphi,u)t^k\quad\text{for all}\quad t\in\mathbb{R},$$

and

$$P(t,\infty)=\sum_{k\ge 0}\operatorname{rank}C_k(\varphi,\infty)t^k\quad\text{for all}\quad t\in\mathbb{R}.$$

The Morse relation says that

$$\sum_{u\in K_\varphi}M(t,u)=P(t,\infty)+(1+t)Q(t)\quad\text{for all}\quad t\in\mathbb{R},$$

where $Q(t)=\sum_{k\ge 0}\beta_k t^k$ is a formal series with nonnegative integer coefficients [11].

Next, we introduce the following fractional Sobolev space

$$W_0^{s,p}(\Omega)=\left\{u\in L^p(\mathbb{R}^N):\iint_{\mathbb{R}^{2N}}\frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}dxdy<\infty\right.$$
$$\left.\text{and}\quad u=0\quad\text{in}\quad\Omega^c\right\},$$

and consider the weighted Hölder-type spaces with weight $d_\Omega^s(x)=\operatorname{dist}(x,\Omega^c)^s$ endowed with their norms:

$$C_s^0(\overline{\Omega})=\{u\in C^0(\overline{\Omega}):u/d_\Omega^s\in C^0(\overline{\Omega})\},\quad \|u\|_{0,s}=\|u/d_\Omega^s\|_\infty,$$

and for all $\alpha\in(0,1)$

$$C_s^\alpha(\overline{\Omega})=\{u\in C^0(\overline{\Omega}):u/d_\Omega^s\in C^\alpha(\overline{\Omega})\},$$
$$\|u\|_{\alpha,s}=\|u\|_{0,s}+\sup_{x\ne y}\frac{|u(x)/d_\Omega^s(x)-u(y)/d_\Omega^s(y)|}{|x-y|^\alpha}.$$

The embedding $C_s^\alpha(\overline{\Omega})\hookrightarrow C_s^0(\overline{\Omega})$ compact for all $\alpha\in(0,1)$.

Besides, $C_s^0(\overline{\Omega})$ owns a positive cone $C_s^0(\overline{\Omega})_+=\{u\in C_s^0(\overline{\Omega}):u\ge 0$ in $\overline{\Omega}\}$, which admits a nonempty interior given by

$$\operatorname{int}(C_s^0(\overline{\Omega})_+)=\{u\in C_s^0(\overline{\Omega}):u(x)/d_\Omega^s(x)>0\quad\text{in}\quad\overline{\Omega}\}.$$

Letting $p\in(1,\infty)$, then for any $u\in W_0^{s,p}(\Omega)$, $(-\Delta)_p^s u\in W^{-s,p'}(\Omega)$ is in the sense for any $v\in W^{s,p}(\Omega)$

$$\langle(-\Delta)_p^s u,v\rangle=\iint_{\mathbb{R}^{2N}}\frac{|u(u)-u(y)|^{p-2}(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+ps}}dxdy.$$

Lemma 2.1. ([9], Lemma 2.1]) $(-\Delta)_p^s:W_0^{s,p}(\Omega)\to W^{-s,p}$ is a monotone, continuous and $(S)_+$-operator.

We list the following regularity and maximum principle.

Lemma 2.2. ([8], Propsition 2.5, Proposition 2.6]) Let **H**(i) hold and u be a solution of equation (1.1). Then

$$u\in L^\infty(\Omega)\ \textit{with}\ \|u\|_\infty\le C\ \textit{for some}\ C=C(\|u\|)>0$$
$$\textit{and}\ u\in C_s^\alpha(\overline{\Omega})\ \textit{for some}\ \alpha\in(0,1).$$

Lemma 2.3. ([12], Theorem 1,5]) Let $c\in C(\overline{\Omega})+$ and $u\in W_0^{s,p}(\Omega)\setminus\{0\}$. If u is a weak supersolution of the equation

$$\begin{cases}(-\Delta)_p^s u=-c(x)|u|^{p-2}u & in\quad\Omega\\ u=0 & in\quad\Omega^c,\end{cases}$$

then $\inf_{x\in\overline{\Omega}}\frac{u(x)}{d_\Omega^s(x)}>0$. *Moreover, if* $u\in C_s^0(\overline{\Omega})$, *then* $u\in int(C_s^0(\overline{\Omega})_+)$.

Recall that the fractional p-Laplacian has first eigenvalue λ_1, which is positive, isolated and simple. λ_1 also satises the following variational characterization:

$$\lambda_1 = \inf\left\{\frac{\|u\|^p}{\|u\|_p^p} : u \in W_0^{s,p}(\Omega)\setminus\{0\}\right\},$$

see [13, Proposition 3.4]. We denote by e_1 the L^p-normalized (i.e. $\|e_1\|_p=1$) eigenfunction associated to λ_1. Clearly, we have $e_1 \in \operatorname{int}(C_s^0(\overline{\Omega})_+)$.

Constant Sign Solutions

We make the following basic hypotheses on f:

H: $f: \Omega\times\mathbb{R}\to\mathbb{R}$ is a Caratheodory function such that $f(x,0)=0$ for a.e. $x\in\Omega$ and

(i) there exist $C>0$ and $r\in(p,p^*_s)$ ($p_s^* = \frac{Np}{N-ps}$ is the critical exponent) such that

$$|f(x,t)| \le C(1+|t|^{r-1}) \quad \text{for a.e.} \quad x\in\Omega \quad \text{and all} \quad t\in\mathbb{R};$$

(ii) $\lim_{|t|\to\infty}\frac{F(x,t)}{|t|^p} = +\infty$ uniformly for a.e. $x\in\Omega$;

(iii) let $\sigma(x,t) = f(x,t)t - pF(x,t)$. There exists $K>0$ such that $\sigma(x,t_1)\le\sigma(x,t_2)+K$ for a.e. $x\in\Omega$ and all $0\le t_1\le t_2$ or all $t_2\le t_1\le 0$;

(iv) there exist a function $\eta_1\in L^\infty(\Omega)_+$, $\eta_1\le\lambda_1$, $\eta_1\ne\lambda_1$ and a constant $\eta_2>0$ such that

$$\limsup_{t\to 0}\frac{f(x,t)}{|t|^{p-2}t} \le \eta_1(x) \text{ uniformly for a.e } x\in\Omega$$

and

$$\liminf_{t\to 0}\frac{f(x,t)}{|t|^{p-2}t} \ge -\eta_2 \text{ uniformly for a.e } x\in\Omega$$

It is clear from the assumptions on f that $f(x,\cdot)$ is $(p-1)$- superlinear near $\pm\infty$ and does not satisfy the condition (AR).

We consider the following positive and negative truncations on f:

$$f_+(x,t) = f(x,t^+) \text{ and } f_-(x,t) = f(x,t^-) \text{ for a.e. } x\in\Omega, \text{ for all } t\in\mathbb{R},$$

where $t^+ = \max\{t,0\}$ and $t^- = \min\{t,0\}$. Obviously, f_+ and f_- are Carathéodory functions.

Let φ be the energy functional of equation (1.1) defined by

$$\varphi(u) = \frac{1}{p}\|u\|^p - \int_\Omega F(x,u)dx,$$

and $\varphi\pm: W_0^{s,p}(\Omega)\to\mathbb{R}$ be defined by

$$\varphi_\pm(u) = \frac{1}{p}\|u\|^p - \int_\Omega F_\pm(x,u)dx,$$

where $F_\pm(x,t) = \int_0^t f(x,\tau)d\tau$.

H (i) implies $\varphi\in C^1(W_0^{s,p}(\Omega),\mathbb{R})$ and $\varphi_\pm\in C^1(W_0^{s,p}(\Omega),\mathbb{R})$. We say that $u\in W_0^{s,p}(\Omega)$ is a solution (weak solution) of equation (1.1) if u satisfies $\langle\varphi'(u),v\rangle=0$ for all $\upsilon\in W_0^{s,p}(\Omega)$.

Lemma 3.1. *If* **H**(i), (ii), (iii) hold, then φ and $\varphi\pm$ satisfy the $(C)_c$ condition.

Proof. We only show the results to φ. By a similar way, the consequences also hold to $\varphi\pm$. Let $\{u_n\}$ be a $(C)_c$ sequence, namely

$$\varphi(u_n)\to c \quad\text{and}\quad (1+\|u_n\|)\varphi'(u_n)\to 0.$$

Then there exist $C_1>0$ and $\epsilon_n\to 0^+$ such that

$$\|\varphi(u_n)\|\le C_1 \quad\text{and}\quad |\langle\varphi'(u_n),h\rangle| \le \frac{\epsilon_n\|h\|}{1+\|u_n\|} \quad \text{for all } h\in W_0^{s,p}(\Omega). \tag{3.1}$$

We claim that $\{u_n\}$ is bounded in $W_0^{s,p}(\Omega)$. If the claim is false, then up to a subsequence $\|u_n\|\to\infty$. Let $v_n = \frac{u_n}{\|u_n\|}$.

Then $\|v_n\|=1$ and we may assume that

$$(v_n\rightharpoonup v \text{ in } W_0^{s,p}(\Omega) \text{ and } v_n\to v \text{ in } L^r(\Omega).) \tag{3.2}$$

Now, we show $\upsilon\equiv 0$ in Ω. Assume by contradiction that $\upsilon\not\equiv 0$ in Ω. Take $A=\{x\in\Omega:\upsilon(x)\ne 0\}$. Then "meas"(A) > 0. Further we deduce that $u_n=\|u_n\|\upsilon_n\to\infty$ a.e. in A. By **H** (iii), one has

$$\frac{F(x,u_n)}{\|u_n\|^p} = \frac{F(x,u_n)}{|u_n|^p}|v_n|^p\to\infty \quad \text{for a.e. } x\in A.$$

Following from the Fatou's lemma, we have

$$\liminf_{n\to\infty}\int_\Omega\frac{F(x,u_n)}{\|u_n\|^p}dx = \infty. \tag{3.3}$$

(3.1) implies that there exists $C_2>0$ such that

$$\int_\Omega\frac{F(x,u_n)}{\|u_n\|^p}dx \le \frac{C_1}{\|u_n\|^p}+\frac{1}{p}\le C_2. \tag{3.4}$$

Combining with (3.3) and (3.4), we obtain

$$\infty = \liminf_{n\to\infty}\int_\Omega\frac{F(x,u_n)}{\|u_n\|^p}dx\le C_2.$$

This is impossible. Hence $\upsilon=0$.

Define $\gamma_n:[0,1]\to\mathbb{R}$ with $\gamma_n(t)=\varphi(tu_n)$. Let $\gamma_n(t_n)=\max_{t\in[0,1]}\{\gamma_n(t):t\in[0,1]\}$. For any $i>0$, since $\|u_n\|\to\infty$, then we may assume that $\frac{l}{\|u_n\|}\in(0,1)$.

From [3.2], we know that $\upsilon_n\to 0$ in $L^r(\Omega)$. By **H** (i), according Lebesgue dominated convergence theorem, we conclude that

$$\int_\Omega F(x,lv_n)dx\to 0 \quad \text{as } n\to\infty.$$

Using $\frac{l}{\|u_n\|}\in(0,1)$, one hold $\gamma_n(t_n)\ge\gamma_n(\frac{l}{\|u_n\|})$. Thereby

$$\varphi(t_nu_n)\ge\varphi(lv_n)\ge\frac{l^p}{p}-\int_\Omega F(x,lv_n)dx\ge\frac{l^p}{2p},$$

which implies

$$\varphi(t_nu_n)\to\infty. \tag{3.5}$$

Due to $t_n \in(0,1)$, then $\gamma_n'(t_n)=0$. So we have

$$\langle \varphi'(t_n u_n), t_n u_n \rangle = 0. \tag{3.6}$$

Putting together (3.5) and (3.6), we obtain

$$\int_\Omega \sigma(x, t_n u_n)dx \le p\varphi(t_n u_n) - \langle \varphi'(t_n u_n), t_n u_n\rangle \to \infty. \tag{3.7}$$

Let $h = u_n$ in [3.1]. Then there exists $C_3>0$ such that

$$\int_\Omega \sigma(x, u_n)dx \le p\varphi(u_n) - \langle \varphi'(u_n), u_n\rangle \le C_3. \tag{3.8}$$

Thanks to $0\le t_n u_n^+ \le u_n+$ and $u_n^- \le t_n u_n^- \le 0$, by using **H** (iii), we observe

$$\int_\Omega \sigma(x, t_n u_n)dx \le \int_\Omega \sigma(x, u_n)dx + K\text{meas}(\Omega). \tag{3.9}$$

In the light of (3.7), (3.8) and (3.9), we deduce

$$\infty = \liminf_{n\to\infty} \int_\Omega \sigma(x, t_n u_n)dx \le \liminf_{n\to\infty} \int_\Omega \sigma(x, u_n)dx + K\text{meas}(\Omega) \le C_3 + K\text{meas}(\Omega).$$

We get a contradiction. Then $\{u_n\}$ is bounded. Thus we have

$$u_n \rightharpoonup u \text{ in } W_0^{s,p}(\Omega) \quad \text{and} \quad u_n \to u \text{ in } L^r(\Omega). \tag{3.10}$$

By using **H** (i), (3.10) and Lebesgue dominated convergence theorem, we have

$$\int_\Omega f(x, u_n)(u_n - u)dx \to 0.$$

Texting (3.1) with $u_n - u$, then $\limsup_{n\to\infty} \langle (-\Delta)^s_p u_n, u_n - u\rangle \le 0$. Since $(-\Delta)^s_p$ is a $(S)_+$-operator (see Lemma (2.1), then $u_n \to u$ in $W_0^{s,p}(\Omega)$.

Lemma 3.2. *If* **H**(i), (iv) hold, then u=0 is a strict local minimizer for the functionals φ and $\varphi\pm$.

Proof. We only prove the conclusion to φ, the proof for $\varphi\pm$ being similar.
According to **H**(i), (iv), for any $\epsilon > 0$, we have

$$F(x,t) = \int_0^t f(x,\tau)d\tau \le \int_0^{t} ((\eta_1(x)+\epsilon)|\tau|^{p-2}\tau + \frac{C_\epsilon}{r}|\tau|^{r-2}\tau)d\tau. \tag{3.11}$$

Using (3.11) and the meaning of λ_1, we deduce that for any $u \in W_0^{s,p}(\Omega)$

$$\begin{aligned}\varphi(u) &= \frac{1}{p}\|u\|^p - \int_\Omega F(x,u)dx \\ &\ge \frac{1}{p}\|u\|^p - \frac{1}{p}\int_\Omega \eta_1(x)|u|^p dx - \frac{\epsilon}{p}\int_\Omega |u|^p dx - C_\epsilon \int_\Omega |u|^r dx \\ &\ge \frac{C_1}{p}\|u\|^p - \frac{\epsilon}{p\lambda_1}\|u\|^p - C_\epsilon C_2 \|u\|^r \\ &\ge C_3\|u\|^p(1 - C_4\|u\|^{r-p}).\end{aligned}$$

Since $p<r$, then there exists $\rho > 0$ such that

$$\varphi(u) > \varphi(0) = 0 \quad \text{for any } u \in W_0^{s,p}(\Omega)\setminus\{0\} \text{ and } \|u\| \le \rho,$$

and there exists $\delta > 0$ such that

$$\inf_{u\in\partial B_\rho(0)} \varphi(u) \ge \delta > \varphi(0) = 0.$$

Lemma 3.3. If H holds, then equation (1.1) has at least two nontrivial constant sign solutions

$$u_+ \in \text{int}\,(C_s^0(\overline{\Omega})_+) \text{ and } u_- \in -\text{int}\,(C_s^0(\overline{\Omega})_+).$$

Proof. Arguing as in the proof of Lemma 3.2, there exist $\rho_+ > 0$ and $\delta_+ > 0$ such that

$$\varphi_+(u) \ge \delta_+ \ge \varphi_+(0) = 0 \quad \text{for all } u \in \partial B_{\rho_+}(0). \tag{3.12}$$

From **H**(i), (ii), we deduce that there exists $C > 0$ such that

$$F(x,t) \ge \frac{2\lambda_1}{p}t^p - C \quad \text{for a.e. } x\in\Omega \text{ and all } t \ge 0.$$

Then

$$\begin{aligned}\varphi_+(te_1) &= \frac{t^p}{p}\lambda - \int_\Omega F(x, te_1)dx \\ &\le \frac{\lambda_1}{p}t^p - \frac{2\lambda_1}{p}t^p + C\,\text{meas}\,(\Omega) \\ &= -\frac{\lambda_1}{p}t^p + C\,\text{meas}\,(\Omega).\end{aligned}$$

Hence we have that

$$\varphi+(te_1) \to -\infty \text{ as } t\to\infty.) \tag{3.13}$$

By (3.12), (3.13) and the mountain pass theorem (see [14], Theorem 5.4.6]), there exists $u_+ \in W_0^{s,p}(\Omega)$ such that

$$\varphi_+(u_+) \ge \delta_+ \text{ and } \varphi'_+(u_+) = 0.$$

Since $\langle \varphi_+'(u_+), u_+^-\rangle=0$, it holds

$$\|u_+^-\|^p \le \langle (-\Delta)^s_p u_+, u_+^-\rangle = \int_\Omega f_+(x,u_+)u_+^- dx = 0,$$

from which we obtain $u_+ \ge 0$. By Lemma 2.2, there holds $u_+ \in L^\infty(\Omega)_+ \cap C_s^0(\overline{\Omega})$. Using H(i), (iv), we deduce that there exists a constant $\eta > \eta_2$ such that

$$f(x,t) \ge -\eta t^{p-1} \quad \text{for a.e. } x\in\Omega \text{ and all } t\in[0, \|u_+\|_\infty],$$

which implies

$$(-\Delta)^s_p u_+ + \eta u_+^{p-1} = f(x,u_+) + \eta u_+^{p-1} \ge 0. \tag{3.14}$$

In view of Lemma 2.3 and (3.14), we conclude $u_+ \in \text{int}(C_s^0(\overline{\Omega})_+)$. By a similar way, we can show that there exists a solution u_ of equation (1.1) and $u_- \in -\text{int}(C_s^0(\overline{\Omega})_+)$.

Three Solutions

In this section, we assume that K_φ is finite.

Lemma 4.1. If the hypothesises **H** hold, then $C_k(\varphi,\infty)=0$ for all $k \ge 0$.

Proof. By **H**(ii), one has that for any $u \in W_0^{s,p}(\Omega) \setminus \{0\}$

$$\varphi(t_u) \to -\infty \text{ as } t \to \infty. \quad (4.1)$$

Choosing $c < 0$ small enough, we show that for any $v \in \varphi^c$

$$\langle \varphi'(v), v\rangle < 0. \quad (4.2)$$

Actually, from **H**(iii), we have

$$0 \le \sigma(x,t) + K \text{ for any } t \in \mathbb{R}. \quad (4.3)$$

Using (4.3), we deduce that for any

$$\begin{aligned} & v \in W_0^{s,p}(\Omega) \setminus \{0\} \\ & \langle \varphi'(v), v\rangle = \|v\|^p - \int_\Omega f(x,v)v\,dx \\ & \quad = p\varphi(u) - \int_\Omega \sigma(x,v)\,dx \\ & \quad \le p\varphi(v) + K \operatorname{meas}(\Omega). \end{aligned}$$

Let $c < \min\{-\frac{K}{p}\operatorname{meas}(\Omega), \inf_{\|u\|\le 1}\varphi(u)\}$. Then (4.2) holds. Take

$S=\{u \in W_0^{s,p}(\Omega): \| u \|=1\}$. By (4.2), for any $(t,u) \in (0,\infty)\times S$ with $\varphi(tu)=c$, we have

$$\frac{\partial}{\partial t}\varphi(tu) = \frac{\langle \varphi'(tu), tu\rangle}{t} < 0$$

From (4.1) and the implicit function theorem (see [11, Theorem 1.2.1]), we know that there exists a continuous map $\beta : S \to (1,\infty)$ such that $\varphi(\beta(u)u)=c$. Also, for any $(t,u) \in (1,\infty)\times S$, we have

$$\varphi(tu) \begin{cases} > c & t < \beta(u) \\ = c & t = \beta(u) \\ < c & t > \beta(u). \end{cases}$$

Hence $\varphi^c=\{tu: u \in S \text{ and } t \in [\beta(u),\infty)\}$. Let $E=\{tu: u \in S \text{ and } t \ge 1\}$. We define a continuous deformation $h:[0,1] \times E \to E$ such that for all $(s,tu) \in [0,1]\times E$

$$h(s,tu) = \begin{cases} (1-s)tu + s\beta(u)u & t < \beta(u) \\ tu & t \ge \beta(u). \end{cases}$$

Then

$$\varphi^c \text{ is a strong deformation restrict of } E. \quad (4.4)$$

Further we consider a continuous deformation $\hat{h} : [0,1] \times E \to E$

defined for any $(s,tu) \in [0,1]\times E$ by

$$\hat{h}(s,tu) = (1 -|s)tu + su. \quad (4.5)$$

We obtain that

$$S \text{ is a strong deformation restrict of } E.$$

On the other hand, since $W_0^{s,p}(\Omega)$ is infinite dimension, then S is contractible in itself. In the light of the choice of c, (4.4), (4.5) and [14] , we have for all $k \ge 0$

$$\begin{aligned} C_k(\varphi,\infty) = H_k(W_0^{s,p}(\Omega),\varphi^c) = H_k(W_0^{s,p}(\Omega),E) = \\ H_k(W_0^{s,p}(\Omega),S) = 0. \end{aligned}$$

By a similar way, we can get a same result to $\varphi\pm$.

Proposition 4.1. *If* **H** holds, then $C_k(\varphi\pm,\infty)=0$ for all $k \ge 0$.

Since 0 is a local minimum of φ and $\varphi\pm$, then

$$C_k(\varphi,0) = \delta_{k,0}\mathbb{Z} \quad \text{and} \quad C_k(\varphi_\pm,0) = \delta_{k,0}\mathbb{Z}. \quad (4.6)$$

Lemma 4.2. Let **H** hold. *If* $K_{\varphi+} = \{0,u_+,u\_\}$, then $C_k(\varphi_+,u_+) = C_k(\varphi\_,u\_) = \delta k,_1\,\mathbb{Z}$, where δ denotes the usual Kronecker symbol.

Proof. We only prove the lemma to φ_+. Notice $K_{\varphi+} = \{0,u_+\}$. Take $a, b \in \mathbb{R}$ such that $a<\varphi_+(0)< b < \varphi_+(u_+)$. Since $\varphi^a_+ \subset \varphi^b_+ \subset W_0^{s,p}(\Omega)$ then we have the following long exact sequence of singular homology group

$$\begin{aligned} \cdots \longrightarrow H_k(W_0^{s,p}(\Omega),\varphi_+^a) \xrightarrow{i_*} H_k(W_0^{s,p}(\Omega),\varphi_+^b) \\ \xrightarrow{\partial_*} H_{k-1}(\varphi_+^b,\varphi_+^a) \longrightarrow \cdots \quad (4.7) \end{aligned}$$

By the choices of a, b and [11,Theorem 5.1.27], one has

$$\begin{aligned} 0 = C_k(\varphi,\infty) = H_k(W_0^{s,p}(\Omega),\varphi_+^a) \quad \text{and} \quad H_k(\varphi_+^b,\varphi_+^a) \\ = C_k(\varphi_+,0) = \delta_{k,0}\mathbb{Z}. \quad (4.8) \end{aligned}$$

By a standard way, we deduce

$$C_k(\varphi_+,u_+) = H_k(W_0^{s,p}(\Omega),\varphi_+^b). \quad (4.9)$$

(Since $W_0^{s,p}(\Omega)=\varphi_+^\infty$, then $H_k(W_0^{s,p}(\Omega),\varphi_+^b) = H_k(\varphi_+^\infty,\varphi_+^b)$. Let $c = \varphi_+(u_+)$. Due to $K_{\varphi+} \cap \varphi_+^\infty \setminus \varphi_+^a=\varnothing$, by the second deformation theorem, we have that φ_+^c is a strong deformation restrict of φ_+^∞. Hence $H_k(\varphi_+^\infty,\varphi_+^b) = H_k(\varphi_+^\infty,\varphi_+^b) = C_k(\varphi_+^\infty,\varphi_+^b)$. From (4.7), (4.8) and (4.9), we know that the sequence

$$0 \xrightarrow{i_*} C_k(\varphi_+,u_+) \xrightarrow{\partial_*} \delta_{k,1}\mathbb{Z}$$

is exact. Then

$$\begin{aligned} \operatorname{rank} C_k(\varphi_+,u_+) &= \operatorname{rank}\ker\partial_* + \operatorname{rank}\operatorname{im}\partial_* \\ &= \operatorname{rank}\ker i_* + \operatorname{rank}\operatorname{im}\partial_* \\ &= \operatorname{rank}\operatorname{im}\partial_* \le \delta_{k,1}. \end{aligned}$$

From this fact, we obtain $C_k(\varphi_+,u_+) = 0$ if $k \neq 1$ and "rank" $C_1(\varphi_+,u_+) \le 1$. Since u_+ is a mountain pass type solution, then $C_1(\varphi_+,u_+) \neq 0$. Thereby $C_1(\varphi_+,u_+) = \mathbb{Z}$.

Lemma 4.3. Let **H** hold. *If* $K_\varphi=\{0,u_+,u\_\}$, then $C_k(\varphi,u\pm) = \delta_{k,1}\,\mathbb{Z}$.

We consider u_+ ($u\_$ is analogous). Now, we shall demonstrate

$$C_k(\varphi,u_+) = C_k(\varphi_+,u_+).$$

For any $t \in [0,1]$, we dene the map $h_t(u)=(1-t)\varphi(u)+t\varphi+(u)$. Then $h_t \in C^1(W_0^{s,p}(\Omega),\mathbb{R})$ for any $t \in [0,1]$ and $u_+ \in K_{ht}$) for any $t \in [0,1]$.

Now we show that u_+ is an isolated critical point of h_t uniformly for $t \in [0,1]$. Assume by contradiction that there exist $\{u_n\} \subset W_0^{s,p}(\Omega) \setminus \{u_+\}$ with $u_n \to u_+$ in $W_0^{s,p}(\Omega)$ and $t_n \in [0,1]$ such that $h_{tn}'(u_n)=0$, i.e.

$$\begin{cases}(-\Delta)_p^s u_n = (1-t_n)f(x,u_n) + t_n f_+(x,u_n) & \text{in } \Omega \\ u_n = 0 & \text{in } \Omega^c.\end{cases} \quad (4.10)$$

By Lemma 2.2 we deduce $u_n \in C_s^\alpha(\overline{\Omega})$ and $\|u_n\|_{0,\alpha} \leq C$ for all integer $n \geq 0$. Since $C_s^\alpha(\overline{\Omega}) \hookrightarrow C_s^0(\overline{\Omega})$ is compact, then $u_n \to u_+$ in $C_s^0(\overline{\Omega})$. Using $u_+ \in \text{int}(C_s^0(\overline{\Omega})_+)$, one has $u_n \in \text{int}(C_s^0(\overline{\Omega})_+)$ in the sense of subsequence. From (4.10), we conclude that $\{u_n\}$ is a sequence of solution of equation (1.1), which is a contradiction. According to [15], Theorem 5.6], there holds $C_k(\varphi, u_+) = C_k(\varphi_+, u_+)$.

Last, we show our main results as follows.

Theorem 4.1. If hypothesises **H** hold, then equation (1.1) has at least three nontrivial solutions

$$u_+ \in \text{int}(C_s^0(\overline{\Omega})_+),\ u_- \in -\text{int}(C_s^0(\overline{\Omega})_+) \text{ and } \tilde{u} \in C_s^0(\overline{\Omega}).$$

Proof. By Lemma 3.3, there exist two solutions

$$u_+ \in \text{int}\,(C_s^0(\overline{\Omega})_+) \text{ and } u_- \in -\text{int}\,(C_s^0(\overline{\Omega})_+).$$

If $K_\varphi = \{0, u_+, u\_\}$, by Lemma 4.1, Lemma 4.3 and Morse relation, we have

$$2(-1)^1 + (-1)^0 = 0.$$

This is impossible. Then equation has a solution $\tilde{u} \notin \{0, u_+, u_-\}$.

By Lemma 2.2, we know $\tilde{u} \in C_s^0(\overline{\Omega})$.

References

1. E Di Nezza, G Palatucci, E Valdinoci (2012) Hitchhiker's guide to the fractional Sobolev spaces. Bull Sci Math. 136: 521-573.
2. A Iannizzotto, S Mosconi, M Squassina (2016) Global Ho¨lder regularity for the fractional p-Laplacian. Rev Mat Iberoam. 32: 1353-1392.
3. A Iannizzotto, S Mosconi, M Squassina (2020) Fine boundary regularity for the degenerate fractional p-Laplacian. J Funct Anal. 279: 108659.
4. F G Duzgun, A Iannizzotto (2018) Three nontrivial solutions for nonlinear fractional Laplacian equations. Adv Nonlinear Anal. 7: 211-226.
5. A Iannizzotto, S B Liu, K Perera, M Squassina (2016) Existence results for fractional p-Laplacian problems via Morse theory. Adv Calc Var. 9: 101-125.
6. A Iannizzotto, N S Papageorgiou (2018) Existence and multiplicity results for resonant fractional boundary value problems. Discrete Contin Dyn Syst Ser. 11: 511-532.
7. A Iannizzotto, R Livrea (2021) Four solutions for fractional p-Laplacian equations with asymmetric reactions. Mediterr J Math. 18: 220.
8. S Frassu, A Iannizzotto (2021) Five solutions for the fractional p-Laplacian with noncoercive energy. arXiv. https://arxiv.org/abs/2110.10098?context=math.
9. S Frassu, A Iannizzotto (2021) Extremal constant sign solutions and nodal solutions for the fractional p-Laplacian. J Math Anal Appl. 501: 124205.
10. Z Q Wang (1991) On a superlinear elliptic equation, Ann. Inst H Poincar´e C Anal Non Lin´eaire 8: 43-57.
11. K C Chang (2005) Methods in nonlinear analysis, Springer Monographs in Mathematics. Springer Berlin. https://link.springer.com/book/10.1007/3-540-29232-2.
12. L M Del Pezzo, A Quaas (2017) A Hopf's lemma and a strong minimum principle for the fractional p-Laplacian. J Differential Equations. 263: 765-778.
13. S Frassu, E Rocha, V Staicu (2022) The Obstacle Problem at Zero for the Fractional p-Laplacian. Set-Valued. Var. Anal. 30: 207-231.
14. N S Papageorgiou, V D Radulescu, D Repovs (2019) Nonlinear Analysis-Theory and Methods. Springer Monographs in Mathematics. Springer Berlin. Cham. https://link.springer.com/book/10.1007/978-3-030-03430-6.
15. K C Chang (1993) Infinite-dimensional Morse theory and multiple solution problems. Birkhauser, Boston. https://link.springer.com/book/10.1007/978-1-4612-0385-8.