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Lagrange Interpolation Polynomials with Exponential Weights

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ABSTRACT

Let $W\rho$ be an exponential weight of the form $W\rho\ (t):=|t|^{\rho}\exp(-Q(t))$ on $\mathbb{R} := (-\infty,\infty)$, and let $L_{n,\rho}(F,t)$ for $F \in C(\mathbb{R})$ be the Lagrange interpolation polynomial at the zeros $\{t_{j,n,\rho}\}_{j=1}^{n}$ of the orthonormal polynomial of degree n with respect to $W\rho$. A new L^p-norm convergence of $L_{n,\rho}(F,t)$ is discussed.

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Introduction

This paper gives an improvement of results in [1]. Let W be a weight of the form $W(t)=\exp(-Q(t))$ on $\mathbb{R}: = (-\infty,\infty)$. We always assume that

$$\int_{-\infty}^{\infty} |t|^n W(t)^2 dt < \infty \quad (\forall n = 0, 1, 2, \cdots).$$

We define a class $\mathcal{F}(C^2+)$ of weights on $\mathbb{R}$.

Definition 1.1. Let W be a weight on $\mathbb{R}$ of the form $W(t)=\exp(-Q(t))$.

(1) We write $W \in \mathcal{F}(C^2+)$, if Q: $\mathbb{R}\to\mathbb{R}^+ := [0,\infty)$ is a continuous and even function, and satisfy the following properties:

(A) $Q'(t)$ is continuous in $\mathbb{R}$, with $Q(0)=0$.

(B) $Q''(t)$ exists and is positive in $\mathbb{R}\backslash\{0\}$.

(C) $\lim_{t\to\infty} Q(t)=\infty$.

(D) A function

$$T(t) := \frac{tQ'(t)}{Q(t)}, \quad t \neq 0$$

is quasi-increasing in $(0,\infty)$ (i.e., there exists $C>0$ such that $T(s) \le CT(t)$ for $0< s < t$) with $T(t) \ge \Lambda > 1,\ t \in \mathbb{R}\backslash\{0\}$.

(E) There exists $C_1>0$ such that

$$\frac{Q''(t)}{|Q'(t)|} \le C_1\frac{|Q'(t)|}{Q(t)}, \quad a.e.\ t \in \mathbb{R}\setminus\{0\}.$$

(F) There also exist a compact subinterval $J(\ni 0)$ of $\mathbb{R}$, and $C_2>0$ such that

$$\frac{Q''(t)}{|Q'(t)|} \ge C_2\frac{|Q'(t)|}{Q(t)}, \quad a.e.\ t \in \mathbb{R}\setminus J.$$

(2) If T is bounded, then we call W a Freud-type weight. And, if T is unbounded, then we call W an Erdős-type weight. Note that if W is Erdős-type, $\lim_{|t|\to\infty} T(t) = \infty$ holds.

(3) Let $1<\lambda<3/2$. We write $W \in \mathcal{F}_\lambda(C^3+)$ if $W \in \mathcal{F}(C^2+)$ and $Q\in C^3(\mathbb{R}\backslash\{0\})$, and for $|t|\ge K>0$ (large enough),

$$\left|\frac{Q'''(t)}{Q''(t)}\right| \le C\left|\frac{Q''(t)}{Q'(t)}\right| \quad \text{and} \quad \frac{|Q'(t)|}{Q(t)^\lambda} \le C$$

hold.

(4) Suppose that $\lim_{|t|\to\infty} 1/T(t)$ exists. Then, we say that $W\in\mathcal{F}(C^2+)$ is a regular weight if μ_+ and μ_- are finite and $\mu_+=\mu_-$ holds, where

$$\mu_+ := \limsup_{|t|\to\infty} \frac{Q''(t)}{Q'(t)}\Big/\frac{Q'(t)}{Q(t)} \quad \text{and} \quad \mu_- := \liminf_{|t|\to\infty} \frac{Q''(t)}{Q'(t)}\Big/\frac{Q'(t)}{Q(t)}.$$

Example 1.2. (cf. [4], [6]) (1) Let $Q_\alpha(t) := |t|^\alpha$ for $\alpha>1$. Then $W_\alpha(t) := \exp(-Q_\alpha(t)) \in \mathcal{F}_\lambda(C^3+)$ $(1<\lambda<3/2)$ and it is a regular Freud-type weight.

(2) For $\alpha>1$ and $l\ge 1$ (integer), we define

$$Q_{l,\alpha}(t) := \exp_l(|t|^\alpha) - \exp_l(0),$$

where $\exp_l(t) := \exp(\exp(\exp...\exp(t)))$ (l times). Then $W(t)=\exp(-Q_{l,\alpha}(t))$ is a regular Erdős-type weight and it belongs to $\mathcal{F}_\lambda(C^3+)$ for every $1<\lambda<3/2$.

We recall some notations with respect to $W=\exp(-Q)\in\mathcal{F}(C^2+)$. For $t>0$, the Mhaskar-Rakhmanov-Saff number (MRS number) $a_t=a_t(W)$ of W is defined by a positive root of the equation

$$t = \frac{2}{\pi}\int_0^1 \frac{a_t u Q'(a_t u)}{(1-u^2)^{1/2}} du.$$

We also use the following notations. Let $\delta_u := \{uT(a_u)\}^{-2/3}$ for $u>0$ and

$$\varphi_u(t) := \begin{cases} \dfrac{a_u}{u}\dfrac{1-|t|/a_{2u}}{(1-|t|/a_u+\delta_u)^{1/2}}, & |t|\le a_u; \\ \varphi_u(a_u), & a_u<|t|, \end{cases}$$

Furthermore, for $n\ge 1$, we define

$$\Phi_n(t) := \max\{1-\frac{|t|}{a_n},\ \delta_n\} \quad \text{and} \quad \Phi(t) := \frac{1}{(1+Q(t))^{2/3}T(t)}.$$

Here we note that for any constant $d>0$,

$$\Phi(t) \sim \frac{Q^{1/3}(t)}{tQ'(t)} \quad \text{on} \quad d\le |t|,$$

and there exists a constant $C>0$ such that

$$\Phi(t) \le C\Phi_n(t) \tag{1.1}$$

for $n\ge 1$ and $t\in\mathbb{R}$ (see [5, Lemma 3.3]).

Let $\rho>-1$ and $W\in\mathcal{F}(C^2+)$. We consider a weight

$$W_\rho(t) := |t|^\rho \exp(-Q(t)),\ t\in\mathbb{R}.$$

Let $P_{n,\rho}$ be the orthonormal polynomial of degree n with respect to W_ρ, that is,

$$\int_{-\infty}^{\infty} P_{n,\rho}(t)P_{m,\rho}(t)W_\rho^2(t)dt = \delta_{mn}$$

holds. The zeros of $P_{n,\rho}$ are denoted by

$$-\infty < t_{n,n,\rho} < t_{n-1,n,\rho} < \cdots < t_{2,n,\rho} < t_{1,n,\rho} < \infty.$$

For $F\in C(\mathbb{R})$, the Lagrange interpolation polynomial $L_{n,\rho}(F,t)$ with node $\{t_{j,n,\rho}\}_{j=1}^n$ is defined by

$$L_{n,\rho}(F,t) := \sum_{j=1}^n F(t_{j,n,\rho})\ell_{j,n,\rho}(t), \tag{1.2}$$

where

$$\ell_{j,n,\rho}(t) := \frac{P_{n,\rho}(t)}{(t-t_{j,n,\rho})P'_{n,\rho}(t_{j,n,\rho})}. \tag{1.3}$$

Assumption 1.3. Suppose $W(t) = \exp(-Q(t))$ belongs to $\mathcal{F}_\lambda(C^3+)$ $(1<\lambda<3/2)$, and for any $M\ge 1$ there exists $N_M>0$ such that for $n\ge N_M$

$$Q(a_n/4) \ge (\log Q(a_n))^M \tag{1.4}$$

and

$$T(a_n) \le C(\log n)^2. \tag{1.5}$$

Lemma 1.4. Q_α and $Q_{l,\alpha}$ in Example 1.2 satisfy the conditions in Assumption 1.3.

The proof of Lemma 1.4 will be given in section 5.

We are now in the position to state our main result in this paper.

Theorem 1.5. Suppose that the exponent Q satisfies the conditions in Assumption 1.3. Let $W_\rho(t) := |t|^\rho \exp(-Q(t))$ with $\rho\ge 0$ and $a_n=o(n^{2/3})$, and $1<p<\infty$, $1/p<\beta$. For $F\in C(\mathbb{R})$, there exists a constant $0<c<1$ such that

$$(1+t^2)^{\beta/2}T^{1/2}(t)\Phi^{-1/4}(t)Q^c(t)W_\rho(t)F(t)\in C_0(\mathbb{R}), \tag{1.6}$$

then

$$\lim_{n\to\infty}\|\Phi_n^{(1/2-1/p)^*}W_\rho(L_{n,\rho}(F)-F)\|_{L^p(\mathbb{R})} = 0. \tag{1.7}$$

where $C_0(\mathbb{R}) := \{f\in C(\mathbb{R});\ \lim_{|t|\to\infty} f(t)=0\}$ and

$$(1/2-1/p)^* = \begin{cases} 0, & 1<p<4; \\ 1/2-1/p, & 4\le p<\infty \end{cases} \tag{1.8}$$

This theorem is an improvement of results in [1]. We recall the previous results.

Result 1 ([1, Theorem 2.1]). Let $W\in\mathcal{F}_\lambda(C^3+)$ with $1<\lambda<3/2$ and $a_n=o(n^{2/3})$. Let $F\in C(\mathbb{R})$, $\beta>1/2$ and $\rho\ge 0$. If

$$(1+t^2)^{\beta/2}W_\rho(t)F(t)\in C_0(\mathbb{R}),$$

then

$$\lim_{n\to\infty}\|W_\rho(L_{n,\rho}(F)-F)\|_{L^2(\mathbb{R})} = 0.$$

Result 2 ([1, Theorem 2.2]). Let $W\in\mathcal{F}_\lambda(C^3+)$ with $1<\lambda<3/2$ and $a_n=o(n^{2/3})$. Let $F\in C(\mathbb{R})$, $1<p\le 2$, $1/p<\beta$ and $\rho\ge 0$. If

$$(1+t^2)^{\beta/2}T^{1/2}(t)\Phi^{-1/4}(t)W_\rho(t)F(t)\in C_0(\mathbb{R}),$$

then

$$\lim_{n\to\infty}\|W_\rho(L_{n,\rho}(F)-F)\|_{L^p(\mathbb{R})} = 0.$$

Result 3 ([1, Theorem 2.3]). Let $W\in\mathcal{F}_\lambda(C^3+)$ with $1<\lambda<3/2$ and $a_n=o(n^{2/3})$. Further assume that W is a regular weight and for $M\ge 1$ there exists $N_M>0$ such that

$$Q(a_n/4)\ge(\log Q(a_n))^M \text{ for } n\ge N_M.$$

Let $F\in C(\mathbb{R})$, $2<p<\infty$, $1/p<\beta$ and $\rho\ge 0$. If

$$(1+t^2)^{\beta/2}T^{1/2}(t)\Phi^{-3/4+1/p}(t)W_\rho(t)F(t)\in C_0(\mathbb{R}),$$

then

$$\lim_{n\to\infty}\|\Phi^{1/2-1/p}W_\rho(L_{n,\rho}(F)-F)\|_{L^p(\mathbb{R})} = 0.$$

Theorem 1.5 is new on the following three points, that is, we see (i) the connected estimates on $1<p<4$ of a neighborhood of $p=2$.

(ii) the improvement of the weight $\Phi^{1/2-1/p}$ as the weight $\Phi_n^{(1/2-1/p)^*}$ (see (1.1)).

(iii) no requirement for the regularity of the weight W.

This paper is organized as follows: In section 2, we prepare some known estimates, which we will use later. In section 3, we prove the main lemma and then we give a proof of main result. In section

4, we discuss Laguerre-type weights on $[0,\infty)$ and show a new L^p norm convergence of Lagrange interpolation polynomials for Laguerre-type weights. In appendix, we give a proof of Lemma 1.4.

Throughout, $C, C_1, C_2, \ldots$ denote positive constants independent of n, x, t or polynomials $P \in \mathcal{P}_n$. The same symbol does not necessarily denote the same constant in different occurrences. For any nonzero real valued functions $f(x)$ and $g(x)$, if there exist constants $C_1, C_2 > 0$ independent of x such that $C_1 g(x) \le f(x) \le C_2 g(x)$ for all x in the range, then we write $f(x) \sim g(x)$. Similarly, for any two sequences of positive numbers $\{c_n\}_{n=1}^{\infty}$ and $\{d_n\}_{n=1}^{\infty}$ we define $c_n \sim d_n$.

Lemmas for weights belong to class $\mathcal{F}(C^2+)$

Throughout this section, we always assume that $W=\exp(-Q) \in \mathcal{F}(C^2+)$, $\rho > -1/2$ and $W_\rho(t)=|t|^\rho W(t)$. Recall that $a_n, T(t), \varphi_n(t)$ and Φ_n are defined by a weight W. On the other hand, $\{P_{n,\rho}\}_{n=0}^{\infty}$ are orthonormal polynomials with respect to W_ρ. The zeros of $P_{n,\rho}$ are $-\infty < t_{n,n,\rho} < t_{n-1,n,\rho} < \cdots < t_{2,n,\rho} < t_{1,n,\rho} < \infty$. $\mathcal{P}_n$ is the class of all polynomials with degree at most n.

Lemma 2.1. (1) ([6, Lemmas 3.4, 3.5 and 3.8]) (a) Uniformly for $t > 0$, we have

$$Q(a_t) \sim \frac{t}{\sqrt{T(a_t)}}. \tag{2.1}$$

(b) Let $L > 1$. Uniformly for $t > 0$,

$$a_{Lt} \sim a_t, \quad T(a_{Lt}) \sim T(a_t).$$

(2) ([3, Lemma A.1]) (a) Uniformly for n and t

$$\varphi_{n+M}(t) \sim \varphi_n(t), \quad M > 0, \quad \varphi_n(t_{j,n,\rho}) \sim \varphi_n(t_{j+1,n,\rho}), \ 1 \le j \le n-1,$$

and uniformly for $|t| \le \varepsilon a_n$, $0 < \varepsilon < 1$, we have

$$\varphi_{2n}(t) \sim \varphi_n(t) \sim \frac{a_n}{n}.$$

(b) For any fixed $L > 1$ and uniformly for $u > 0$,

$$1 - \frac{a_u}{a_{Lu}} \sim \frac{1}{T(a_u)}.$$

(3) ([3, Theorem 2.2]) (a) For the minimum positive zero $t_{[n/2],n,\rho}$,

$$t_{[n/2],n,\rho} \sim \frac{a_n}{n}$$

and for the maximum zero $t_{1,n,\rho}$,

$$1 - \frac{t_{1,n,\rho}}{a_n} \sim \delta_n, \quad \delta_n := (nT(a_n))^{-2/3}.$$

(b) For $n \ge 1$ and $1 \le j \le n-1$,

$$t_{j,n,\rho} - t_{j+1,n,\rho} \sim \varphi_n(t_{j,n,\rho}).$$

(4) ([3, Lemma 4.7]) Let $P_{n,\rho} = \gamma_{n,\rho} x^n + \cdots$ with $\gamma_{n,\rho} > 0$.

Then

$$\frac{\gamma_{n-1,\rho}}{\gamma_{n,\rho}} \sim a_n.$$

Lemma 2.2 ([3, Theorems 2.3 and 2.4]). Uniformly for $n \ge 1$

$$\sup_{t\in\mathbb{R}} |P_{n,\rho}(t) W(t) (|t| + \frac{a_n}{n})^\rho| |t^2 - a_n^2|^{1/4} \sim 1 \tag{2.2}$$

and

$$\sup_{t\in\mathbb{R}} |P_{n,\rho}(t) W(t) (|t| + \frac{a_n}{n})^\rho| \sim a_n^{-1/2} \delta_n^{-1/4}. \tag{2.3}$$

Lemma 2.3 ([3, Theorem 2.5], [1, Lemma 3.3]).
(a) There exists $n_0 > 0$ such that uniformly for $n \ge n_0$ and $1 \le j \le n$,

$$|P'_{n,\rho}(t_{j,n,\rho})| W(t_{j,n,\rho}) (|t_{j,n,\rho}| + \frac{a_n}{n})^\rho \sim \varphi_n(t_{j,n,\rho})^{-1} |a_n^2 - t_{j,n,\rho}^2|^{-1/4}. \tag{2.4}$$

Then we also see

$$\frac{\Phi_n^{1/4}(t_{j,n,\rho})}{|P'_{n,\rho}(t_{j,n,\rho})| W(t_{j,n,\rho}) (|t_{j,n,\rho}| + \frac{a_n}{n})^\rho} \le C \frac{a_n^{3/2}}{n}. \tag{2.5}$$

(b) ([3, Theorem 2.5 (c)])

$$\max_{t\in\mathbb{R}} |\ell_{j,n,\rho}(t) W(t) (|t| + \frac{a_n}{n})^\rho| W^{-1}(t_{j,n,\rho}) (|t_{j,n,\rho}| + \frac{a_n}{n})^{-\rho} \sim 1, \tag{2.6}$$

where $\ell_{j,n,\rho}$ is the fundamental polynomial (see (1.3)).

Lemma 2.4 ([3, Lemma 3.8]). Let $1 \le p, q \le \infty$. There exists $C > 0$ such that for any $n \in \mathbb{N}$ and any $P \in \mathcal{P}_{n-1}$, if $1 \le q \le p \le \infty$, then

$$\|W(t)(|t| + \frac{a_n}{n})^\rho P(t)\|_{L^q(\mathbb{R})} \le C a_n^{1/q-1/p} \|W(t)(|t| + \frac{a_n}{n})^\rho P(t)\|_{L^p(\mathbb{R})}, \tag{2.7}$$

and

$$\|W_\rho P\|_{L^q(\mathbb{R})} \le C a_n^{1/q-1/p} \|W_\rho P\|_{L^p(\mathbb{R})}. \tag{2.8}$$

If $1 \le p < q \le \infty$, then

$$\|\frac{1}{\sqrt{T(t)}} W(t)(|t| + \frac{a_n}{n})^\rho P(t)\|_{L^q(\mathbb{R})} \le C (\frac{n}{a_n})^{1/p-1/q} \|W(t) (|t| + \frac{a_n}{n})^\rho P(t)\|_{L^p(\mathbb{R})} \tag{2.9}$$

Lemma 2.5 ([3, Theorem 2.7]). Let $W(t) \in \mathcal{F}(C^2+)$ and $0 \le s \le r < \infty$. Assume $p\rho > -1/2$ if $0 < p < \infty$ and $\rho \ge 0$ if $p = \infty$. Then we have for $n \ge 2$,

$$\|\Phi_n^{(r/4-1/p)^+}(t) P_{n,\rho}(t) W(t) (|t| + \frac{a_n}{n})^\rho\|_{L^p(\mathbb{R})}$$

$$\le C \begin{cases} a_n^{1/p-r/2} \log n & \text{if } s = r \text{ or } 4 \le pr < \infty; \\ a_n^{1/p-s/2} & \text{if } s < r \text{ and } 4 \le pr < \infty \text{ or if } pr < 4; \\ a_n^{-s/2} & \text{if } p = \infty, \end{cases}$$

where $(r/4-1/p)^+ := \max\{r/4-1/p, 0\}$. Especialy, for $s = r = 1$ and $1 < p < 4$ we have

$$\|W(t)(|t| + \frac{a_n}{n})^\rho\|_{L^p(\mathbb{R})} \le C a_n^{1/p-1/2},$$

and for $s=1$, $r=2$, $2 \le p$ we have

$$\|\Phi_n^{(1/2-1/p)^+}(t) P_{n,\rho}(t) W(t) (|t| + \frac{a_n}{n})^\rho\|_{L^p(\mathbb{R})} \le C a_n^{1/p-1/2},$$

that is,

$$\|\Phi_n^{(1/2-1/p)^*}(t)P_{n,\rho}(t)W(t)(|t|+\frac{a_n}{n})^\rho\|_{L^p(\mathbb{R})} \le Ca_n^{1/p-1/2}, \quad (2.10)$$

where $(1/2-1/p)^*$ is defined by (1.8).

Lemma 2.6 ([10, Appendix; Theorem A] , [11, Theorem 4.2]). (1) Let $W \in \mathcal{F}_\lambda(C^3+)$ with $1 < \lambda < 3/2$, and let $\alpha, \beta, \mu, \nu \in \mathbb{R}$. Then we can construct a new weight $W_{\mu,\nu,\alpha,\beta} \in \mathcal{F}(C^2+)$ such that

$$T(t)^\mu(1+t^2)^\nu(1+Q(t))^\alpha(1+|Q'(t)|)^\beta W(t) \sim W_{\mu,\nu,\alpha,\beta}(t)$$

holds on $\mathbb{R}$.

(2) ([11, Theorem 4 and (4.11)]) We put the same assumption in (1) above. Let us denote MRS number for a weight $W_{\mu,\nu,\alpha,\beta}=\exp(-Q_{\mu,\nu,\alpha,\beta})$ by $a_n(Q_{\mu,\nu,\alpha,\beta})$, further we denote the function in Definition 1.1 (d) for a weight $W_{\mu,\nu,\alpha,\beta}=\exp(-Q_{\mu,\nu,\alpha,\beta})$ by $T_{\mu,\nu,\alpha,\beta}$. Then there exist $C_1,C_2>0$ such that

$$a_{C_1n}(Q_{\mu,\nu,\alpha,\beta}) \le a_n(Q) := a_n \le a_{C_2n}(Q_{\mu,\nu,\alpha,\beta})$$

$$\text{and} \quad T_{\mu,\nu,\alpha,\beta}(t) \sim T(t), \quad t \in \mathbb{R}.$$

Lemma 2.7 (cf. [8, Theorem 1.4]). Every weight $W \in \mathcal{F}(C^2+)$ gives a positive answer to Bernstein's problem. That is, for every $f \in C(\mathbb{R})$ with $Wf \in C_0(\mathbb{R})$, there is a sequence of polynomials $\{P_n\}_{n=0}^\infty$ with $P_n \in \mathcal{P}_n$ such that

$$\lim_{n\to\infty}\|W(f-P_n)\|_{L^\infty(\mathbb{R})} = 0.$$

Let $1 \le p < \infty$. L^p-type Christoffel function $\lambda_{n,\rho,p}(W_\rho;t)$ with a weight W_ρ is defined as follows:

$$\lambda_{n,\rho,p}(W_\rho;t) := \inf_{P\in\mathcal{P}_{n-1}}\int_{-\infty}^{\infty}|PW_\rho|^p(u)du/|P(t)|^p.$$

The Christoffel number is defined by

$$\lambda_{j,n,\rho} := \lambda_{n,\rho,2}(W_\rho;t_{j,n,\rho}).$$

Lemma 2.8 ([1, Lemma 4.1], cf. [2,Lemma 2.7]). Let $b\in\mathbb{R}$. Then there exist constants C_1, $C_2 > 0$ such that for an integer $n \ge 1$

$$C_1\int_{-a_n}^{a_n}(1+t^2)^b dt \le \sum_{1\le j\le n}\lambda_{j,n,\rho}W^{-2}(t_{j,n,\rho})(|t_{j,n,\rho}|+\frac{a_n}{n})^{-2\rho}(1+t_{j,n,\rho}^2)^b$$
$$\le C_2\int_{-a_n}^{a_n}(1+t^2)^b dt.$$

Lemma 2.9 ([1, Lemma 5.5]). Let $W_\rho(t)=|t|^\rho W(t)$ ($\rho\ge0$), $1<p\le2$ and let $1/p<\beta$. Let assum that for every $F \in C(\mathbb{R})$

$$(1+t^2)^{\beta/2}T^{1/2}(t)\Phi^{-1/4}(t)W_\rho(t)F(t) \in C_0(\mathbb{R}).$$

Then for large enough n we have

$$\|W_\rho L_{n,\rho}(F)\|_{L^p(\mathbb{R})} \le CM_n \le C\|(1+t^2)^{\beta/2+\rho/2}T^{1/2}(t)$$
$$\Phi^{-1/4}(t)W(t)F(t)\|_{L^\infty(\mathbb{R})},$$

where

$$M_n := \|(1+t^2)^{\beta/2}T^{1/2}(t)\Phi^{-1/4}(t)W(t)(|t|+\frac{a_n}{n})^\rho F(t)\|_{L^p(\mathbb{R})}. \quad (2.11)$$

Lemma 2.10 ([1, Proof of Lemma 5.5]). Let

$$(1+t^2)^{\beta/2}T^{1/2}(t)\Phi^{-1/4}(t)W_\rho(t)F(t) \in C_0(\mathbb{R}).$$

Let $1 < p < \infty$ and F_1 be defined (3.3) below. Then we have

$$\|W_\rho L_{n,\rho}(F_1)\|_{L^p(|t|\le a_n/2)} \le CM_n,$$

and for $1 < p < 4$

$$\|W_\rho L_{n,\rho}(F_1)\|_{L^p(|t|\ge a_n/2)} \le CM_n.$$

(see [1, Remark 5.1]).

Lemma 2.11 ([1, Lemma 6.3]). Let $0 < \alpha < 1$. For $n \in \mathbb{N}$, we put

$$\sum_j(t) := \sum_{|t_{j,n,\rho}|\ge a_{\alpha n}}|\ell_{j,n,\rho}(t)|W^{-1}(t_{j,n,\rho})(|t_{j,n,\rho}|+\frac{a_n}{n})^{-\rho}. \quad (2.12)$$

Then, for $|t| \le a_{\alpha n/2}$ and $|t| \ge a_{2n}$,

$$W(t)(|t|+\frac{a_n}{n})^\rho\sum_j(t) \le C. \quad (2.13)$$

On the other hand, for $a_{\alpha n/2} \le |t| \le a_{2n}$

$$W(t)(|t|+\frac{a_n}{n})^\rho\sum_j(t) \le C\{\log n + a_n^{1/2}|P_{n,\rho}(t)|W(t)$$
$$(|t|+\frac{a_n}{n})^\rho T^{-1/4}(a_n)\}. \quad (2.14)$$

Proof of Theorem 1.5

In this section we assume that the all conditions in Theorem 1.5 fulfill, that is, $W=\exp(-Q) \in \mathcal{F}_\lambda(C^3+)$ $(1<\lambda<3/2)$ with $a_n=o(n^{2/3})$, and (1.4), (1.5) hold. Let $W_\rho(t) := |t|^\rho \exp(-Q(t))$ with $\rho \ge 0$.

We begin with two lemmas. First is the following estimate of Q.

Lemma 3.1. Let $0 < c < 1$ be fixed. For any positive integer k, there exists N_0 such that

$$Q^c(a_n/4) \ge (\log n)^k \quad \text{for} \quad n \ge N_0. \quad (3.1)$$

Proof. We take $M \ge 1$ such that $cM \ge k+1$, then by (1.4), (1.5) and (2.1), we see

$$Q^c(a_n/4) \ge (\log Q(a_n))^{cM} \ge (\log C\frac{n}{\sqrt{T(a_n)}})^{cM} \ge (\log C\frac{n}{\log n})^{cM}$$
$$\ge \{(1/2)\log n\}^{cM} \ge \{(1/2)\log n\}^{k+1} \ge \{\log n\}^k,$$

that is, we have (3.1).

Next is the main lemma for the proof of Theorem 1.5.

Lemma 3.2. Let $1 < p < \infty$, $1/p < \beta$ and $0 < c < 1$. Then there exists $C > 0$ such that for every $F \in C(\mathbb{R})$ and for large enough $n \in \mathbb{N}$,

$$\|\Phi_n^{(1/2-1/p)^*}W_\rho L_{n,\rho}(F)\|_{L^p(\mathbb{R})}$$

$$\leq C\|(1+t^2)^{\beta/2+\rho/2}T^{1/2}(t)\Phi^{-1/4}(t)\ Q^c(t)W(t)F(t)\|_{L^\infty(\mathbb{R})}. \quad (3.2)$$

Proof. We put

$$M^* := \|(1+t^2)^{\beta/2+\rho/2}T^{1/2}(t)\Phi_n^{-1/4}(t)Q^c(t)W(t)F(t)\|_{L^\infty(\mathbb{R})}$$

and

$$M_n^* := \|(1+|t|)^\beta T^{1/2}(t)\Phi^{-1/4}(t)Q^c(t)W(t)$$

$$(|t|+\frac{a_n}{n})^\rho F(t)\|_{L^\infty(\mathbb{R})}.$$

Then $M_n^* \leq M^*$ for large enough n. We may assume $M^* < \infty$.

We can take $1/2<\alpha<1$ such that $a_n/4\leq a_{\alpha n}$ for all $n \in \mathbb{N}$. In fact, by [6, Lemma 3.11 (3.50)],

$$\left|1-\frac{a_{\alpha n}}{a_n}\right| \leq \frac{C}{T(a_{\alpha n})}(1-\alpha) \leq C(1-\alpha).$$

If we take α such that $C(1 - \alpha)<3/4$, then $a_n/4 \leq a_{\alpha n}$. Define

$$F_1(t) := \begin{cases} F(t), & |t| \leq a_n/4; \\ 0, & a_n/4 < |t|, \end{cases} \quad (3.3)$$

$$F_2(t) := \begin{cases} 0, & |t| \leq a_n/4; \\ F(t), & a_n/4 \leq |t| \leq a_{\alpha n}; \\ 0, & a_{\alpha n} < |t|, \end{cases}$$

and

$$F_3(t) := F(t) - F_1(t) - F_2(t).$$

Then we have

$$\|\Phi_n^{(1/2-1/p)^*}W_\rho L_{n,\rho}(F)\|_{L^p(\mathbb{R})}$$

$$\leq \|\Phi_n^{(1/2-1/p)^*}W_\rho L_{n,\rho}(F_1)\|_{L^p(|t|\leq a_n/2)}$$

$$+\|\Phi_n^{(1/2-1/p)^*}W_\rho L_{n,\rho}(F_1)\|_{L^p(|t|\geq a_n/2)}$$

$$+\|\Phi_n^{(1/2-1/p)^*}W_\rho L_{n,\rho}(F_2)\|_{L^p(|t|\leq a_{2\alpha n})}$$

$$+\|\Phi_n^{(1/2-1/p)^*}W_\rho L_{n,\rho}(F_2)\|_{L^p(|t|\geq a_{2\alpha n})}$$

$$+\|\Phi_n^{(1/2-1/p)^*}W_\rho L_{n,\rho}(F_3)\|_{L^p(\mathbb{R})}$$

$$=: J_1+J_2+J_3+J_4+J_5.$$

The condition

$$(1+t^2)^{\beta/2}T^{1/2}(t)\Phi^{-1/4}(t)Q^c(t)W_\rho(t)F(t) \in C_0(\mathbb{R})$$

means

$$(1+t^2)^{\beta/2}T^{1/2}(t)\Phi^{-1/4}(t)W_\rho(t)F(t) \in C_0(\mathbb{R}).$$

Let

$$M_n := \|(1+|t|)^\beta T^{1/2}(t)\Phi^{-1/4}(t)W(t)$$

$$(|t|+\frac{a_n}{n})^\rho F(t)\|_{L^\infty(\mathbb{R})}.$$

Since $\Phi_n(t)\leq 1$, by Lemma 2.10 we have for $1 < p < \infty$

$$J_1 = \|\Phi_n^{(1/2-1/p)^*}W_\rho L_{n,\rho}(F_1)\|_{L^p(|t|\leq a_n/2)}$$

$$\leq \|W_\rho L_{n,\rho}(F_1)\|_{L^p(|t|\leq a_n/2)} \leq CM_n \leq CM_n^* \leq CM^*, \quad (3.4)$$

for $1 < p < 4$

$$J_2 \leq \|W_\rho L_{n,\rho}(F_1)\|_{L^p(|t|\leq a_n/2} \leq CM_n \leq CM_n^* \leq CM^* \quad (3.5)$$

and by Lemma 2.9, for $1< p \leq 2$

$$J_3, J_4, J_5 \leq CM_n \leq CM_n^* \leq CM^*. \quad (3.6)$$

Hence we may show the estimate of J_2 for $4 \leq p <\infty$, and the estimates of J_3,J_4, J_5 for $2 < p < \infty$.

To estimate J_2 for $4 \leq p < \infty$, we use the Lagrange interpolation formula in the form

$$L_{n,\rho}(F_1,t) = P_{n,\rho}(t)\frac{\gamma_{n-1,\rho}}{\gamma_{n,\rho}}\sum_{k=1}^n \lambda_{k,n,\rho}P_{n-1,\rho}(t_{k,n,\rho})\frac{F_1(t_{k,n,\rho})}{t-t_{k,n,\rho}} \quad (3.7)$$

(see [9, p.243]). Let $|t_{k,n,\rho}| \leq a_n/4$ and $|t| \geq a_n/2$. Then we see $|t-t_{k,n,\rho}| \geq a_n/4$. Using Lemma 2.1 (4) and Lemma 2.2 (2.2), we have from (3.7)

$$|L_{n,\rho}(F_1,t)| \leq Ca_n|P_{n,\rho}(t)| \sum_{|t_{k,n,\rho}|\leq a_n/4} \lambda_{k,n,\rho}|P_{n-1,\rho}(t_{k,n,\rho})|$$

$$|\frac{F_1(t_{k,n,\rho})}{t-t_{k,n,\rho}}| \leq CM_n a_n^{-1/2}|P_{n,\rho}(t)|$$

$$\times\sum_{k=1}^n \lambda_{k,n,\rho}\{W(t_{k,n,\rho})(|t_{k,n,\rho}|+a_n/n)^\rho\}^{-2}(1+t_{k,n,\rho}^2)^{-\beta/2}T^{-1/2}(t_{k,n,\rho})$$

$$\Phi^{1/4}(t_{k,n,\rho}).$$

Using Lemma 2.8 and the Hölder inequality and $\beta >1/p$, $1/p+1/q=1$, we have

$$\sum_{k=1}^n \lambda_{k,n,\rho}\{W(t_{k,n,\rho})(|t_{k,n,\rho}|+\frac{a_n}{n})^\rho\}^{-2}(1+t_{k,n,\rho}^2)^{-\beta/2}T^{-1/2}(t_{k,n,\rho})\Phi^{1/4}(t_{k,n,\rho})$$

$$\leq C\int_{-a_n}^{a_n}(1+t^2)^{-\beta/2}dt$$

$$\leq C(\int_{-a_n}^{a_n}(1+t^2)^{-p\beta/2}dt)^{1/p}(\int_{-a_n}^{a_n}1dt)^{1/q} \leq Ca_n^{1/q},$$

which implies

$$|L_{n,\rho}(F_1,t)| \le Ca_n^{-1/2+1/q}|P_{n,\rho}(t)|M_n.$$

Therefore, by Lemma 2.5 (2.10) and $1/p+1/q=1$, we have

$$J_2 = \|\Phi_n^{(1/2-1/p)^*}(t)W_\rho(t)L_{n,\rho}(F_1,t)\|_{L^p(|t|\ge a_n/2)}$$
$$\le Ca_n^{-1/2+1/q}M_n\|\Phi_n^{(1/2-1/p)^*}(t)W_\rho(t)P_{n,\rho}(t)\|_{L^p(|t|\ge a_n/2)}.$$
$$\le Ca_n^{-1+1/q+1/p}M_n \le CM_n^*,$$

and we have for $4 < p < \infty$

$$J_2 \le CM_n^* \le CM^*. \tag{3.8}$$

Next, we estimate J_3 and J_4. If $a_n/4 \le |t| \le a_{\alpha n}$, then $(1+|t|)^{-\beta} \sim a_n^{-\beta}$ for large $n \in \mathbb{N}$. We see

$$|\Phi_n^{(1/2-1/p)^*}(t)W(t)(|t|+\frac{a_n}{n})^\rho L_{n,\rho}(F_2,t)|$$
$$\le a_n^{-\beta}\Phi_n^{(1/2-1/p)^*}(t)W(t)(|t|+\frac{a_n}{n})^\rho M_n^* \sum_{a_n/4\le|t_{k,n,\rho}|\le a_{\alpha n}} J(k,t), \tag{3.9}$$

where

$$J(k,t) := T^{-1/2}(t_{k,n,\rho})\Phi^{1/4}(t_{k,n,\rho})Q^{-c}(t_{k,n,\rho})W^{-1}(t_{k,n,\rho})$$
$$(|t_{k,n,\rho}|+\frac{a_n}{n})^{-\rho}|\ell_{k,n,\rho}(t)|.$$

For J_3, we fix $|t| \le a_{2\alpha n}$ and take m such that $t \in [t_{m+1,n,\rho}, t_{m,n,\rho}]$. Then, by Lemma 2.3 (b) we see

$$|\ell_{j,n,\rho}(t)W(t)(|t|+\frac{a_n}{n})^\rho W^{-1}(t_{j,n,\rho})(|t_{j,n,\rho}|+\frac{a_n}{n})^{-\rho}| \le C,$$
$$j=m,m+1.$$

Therefore, we have

$$|\Phi_n^{(1/2-1/p)^*}(t)W(t)(|t|+\frac{a_n}{n})^\rho(J(m+1,t)+J(m,t))|$$
$$\le CT^{-1/2}(a_n/4)\Phi^{1/4}(a_n/4)Q^{-c}(a_n/4)$$
$$\times\{|\ell_{m+1,n,\rho}(t)|W(t)(|t|+\frac{a_n}{n})^\rho W^{-1}(t_{m+1,n,\rho})(|t_{m+1,n,\rho}|+\frac{a_n}{n})^{-\rho}$$
$$+|\ell_{m,n,\rho}(t)|W(t)(|t|+\frac{a_n}{n})^\rho W^{-1}(t_{m,n,\rho})(|t_{m+1,n,\rho}|+\frac{a_n}{n})^{-\rho}\}$$
$$\le CT^{-1/2}(a_n/4)\Phi^{1/4}(a_n/4)Q^{-c}(a_n/4). \tag{3.10}$$

Suppose $k \ne m$, $m+1$. For $|t| \le a_{2\alpha n}$ and $a_n/4 \le |t_{k,n,\rho}| \le a_{\alpha n}$, we see

$$|t-t_{k,n,\rho}| \sim \sum_{k\le j\le m-1 \text{ or } m+2\le j\le k}\varphi_n(t_{j,n,\rho})$$
$$\ge C|k-m|\frac{a_n}{n}\min\{\sqrt{1-\frac{|t|}{a_n}},\sqrt{1-\frac{|t_{k,n,\rho}|}{a_n}}\}.$$

Hence, with Lemma 2.3 (2.5) we have for $a_n/4 \le t_{k,n,\rho} \le a_{\alpha n}$ and $k \ne m$, $m+1$

$$|\Phi_n^{(1/2-1/p)^*}(t)W(t)(|t|+\frac{a_n}{n})^\rho J(k,t)|$$
$$=\Phi_n^{(1/2-1/p)^*}(t)|P_{n,\rho}(t)|W(t)(|t|+\frac{a_n}{n})^\rho$$
$$|\frac{T^{-1/2}(t_{k,n,\rho})\Phi^{1/4}(t_{k,n,\rho})Q^{-c}(t_{k,n,\rho})}{(t-t_{k,n,\rho})P'_{n,\rho}(t_{k,n,\rho})W(t_{k,n,\rho})(|t_{k,n,\rho}|+\frac{a_n}{n})^\rho}|$$
$$\le C\Phi_n^{(1/2-1/p)^*}(t)|P_{n,\rho}(t)|W(t)(|t|+\frac{a_n}{n})^\rho|\frac{a_n^{3/2}}{n}\frac{T^{-1/2}(t_{k,n,\rho})}{Q^c(t_{k,n,\rho})(t-t_{k,n,\rho})}|$$
$$\le Ca_n^{1/2}\Phi_n^{(1/2-1/p)^*}(t)|P_{n,\rho}(t)|W(t)(|t|+\frac{a_n}{n})^\rho\frac{T^{-1/2}(t_{k,n,\rho})}{Q^c(t_{k,n,\rho})|k-m|}$$
$$\times\max\{(1-\frac{|t|}{a_n})^{-1/2},(1-\frac{|t_{k,n,\rho}|}{a_n})^{-1/2}\}$$
$$\le Ca_n^{1/2}\Phi_n{}^{(1/2-1/p)^*}(t)|P_{n,\rho}(t)|W(t)(|t|+\frac{a_n}{n})^\rho\frac{1}{Q^c(t_{k,n,\rho})|k-m|}$$
$$\times T^{1/2}(t)T^{-1/2}(t)T^{-1/2}(t_{k,n,\rho})\max\{(1-\frac{|t|}{a_n})^{-1/2},(1-\frac{|t_{k,n,\rho}|}{a_n})^{-1/2}\}$$
$$\le Ca_n^{1/2}\Phi_n{}^{(1/2-1/p)^*}(t)|P_{n,\rho}(t)|W(t)(|t|+\frac{a_n}{n})^\rho(\log n)\frac{1}{Q^c(a_n/4)|k-m|}$$

Here we used the facts; if $|t| \le a_{2\alpha n}$, then

$$T^{-1/2}(t)T^{-1/2}(t_{k,n,\rho})\max\{(1-\frac{|t|}{a_n})^{-1/2},(1-\frac{|t_{k,n,\rho}|}{a_n})^{-1/2}\} \le C$$

(Lemma 2.1 (2-b)). Hence with (3.10),

$$\le Ca_n^{1/2}\Phi_n{}^{(1/2-1/p)^*}(t)|P_{n,\rho}(t)|W(t)(|t|+\frac{a_n}{n})^\rho Q^{-c}(a_n/4)(\log n)$$
$$+C\Phi_n^{(1/2-1/p)^*}(t)T^{-1/2}(a_n/4)\Phi^{1/4}(a_n/4)Q^{-c}(a_n/4).$$

Thus by Lemma 2.5 (2.10) and Assumption 1.3, we have

$$J_3 \le \|\Phi_n^{(1/2-1/p)^*}(t)W(t)(|t|+\frac{a_n}{n})^\rho L_{n,\rho}(F_2,t)\|_{L^p(|t|\le a_{2\alpha n})}$$
$$\le CM_n^*a_n^{1/2-\beta}\|\Phi_n{}^{(1/2-1/p)^*}(t)P_{n,\rho}(t)W(t)$$
$$(|t|+\frac{a_n}{n})^\rho\|_{L^p(|t|\le a_{2\alpha n})}Q^{-c}(a_n/4)(\log n)^2$$
$$+CM_n^*a_n^{-\beta}T^{-1/2}(a_n/4)\Phi^{1/4}(a_n/4)Q^{-c}(a_n/4)\|1\|_{L^p(|t|\le a_{2\alpha n})}$$
$$\le CM_n^*Q^{-c}(a_n/4)(\log n)^2a_n^{1/p-\beta}$$
$$+CM_n^*a_n^{-\beta}T^{-1/2}(a_n/4)\Phi^{1/4}(a_n/4)Q^{-c}(a_n/4)a_n^{1/p}$$
$$\le CM_n^*a_n^{1/p-\beta}Q^{-c}(a_n/4)(\log n)^2$$
$$\le CM_n^* \le CM^*, \tag{3.11}$$

because of $1/p-\beta<0$ and Lemma 3.1 with $k=2$.

Next, we consider J_4. Let $a_n/4 \le |t_{k,n,\rho}| \le a_{\alpha n}$ and $|t| \ge a_{2\alpha n}$. First we note

$$|t-t_{k,n,\rho}| \ge C(a_{2\alpha n}-a_{\alpha n}) \sim \frac{a_n}{T(a_n)}. \tag{3.12}$$

From (3.9) we see

$$\Phi_n^{(1/2-1/p)^*}(t)W(t)(|t|+\frac{a_n}{n})^\rho J(k,t)$$
$$=\Phi_n^{(1/2-1/p)^*}(t)|P_{n,\rho}(t)W(t)(|t|+\frac{a_n}{n})^\rho$$
$$\frac{T^{-1/2}(t_{k,n,\rho})\Phi^{1/4}(t_{k,n,\rho})Q^{-c}(t_{k,n,\rho})}{(t-t_{k,n,\rho})P'_{n,\rho}(t_{k,n,\rho})W(t_{k,n,\rho})(|t_{k,n,\rho}|+a_n/n)^\rho}|.$$

By Lemma 2.3 (2.5) and (3.12), we have

$$\left|\frac{T^{-1/2}(t_{k,n,\rho})\Phi^{1/4}(t_{k,n,\rho})Q^{-c}(t_{k,n,\rho})}{(t-t_{k,n,\rho})P'_{n,\rho}(t_{k,n,\rho})W(t_{k,n,\rho})(|t_{k,n,\rho}|+a_n/n)^\rho}\right|$$
$$\le C\frac{a_n^{3/2}}{n}\frac{T(a_n)}{a_n}T^{-1/2}(t_{k,n,\rho})Q^{-c}(t_{k,n,\rho})$$

$$\le C\frac{a_n^{1/2}}{n}T(a_n)T^{-1/2}(a_n/4)Q^{-c}(a_n/4) \le C\frac{a_n^{1/2}}{n}T(a_n)Q^{-c}(a_n/4)$$

$$\le C\frac{a_n^{1/2}}{n}\frac{T(a_n)}{Q^c(a_n/4)}.$$

From this fact, we have

$$|\Phi_n^{(1/2-1/p)^*}(t)W(t)(|t|+\frac{a_n}{n})^\rho \sum_{a_n/4\le|t_{k,n,\rho}|\le a_{\alpha n}} J(k,t)|$$

$$\le C\Phi_n^{(1/2-1/p)^*}(t)W(t)(|t|+\frac{a_n}{n})^\rho|P_{n,\rho}(t)| \sum_{a_n/4\le|t_{k,n,\rho}|\le a_{\alpha n}} \frac{a_n^{1/2}}{n}\frac{T(a_n)}{Q^c(a_n/4)}$$

$$\le Ca_n^{1/2}\Phi_n^{(1/2-1/p)^*}(t)W(t)(|t|+\frac{a_n}{n})^\rho|P_{n,\rho}(t)|\frac{T(a_n)}{Q^c(a_n/4)}.$$

Therefore, from (3.9), Lemma 2.5 (2.10) and Assumption 1.3 we have

$$J_4 \le \|\Phi_n^{(1/2-1/p)^*}(t)W(t)(|t|+\frac{a_n}{n})^\rho L_{n,\rho}(F_2,t)\|_{L^p(|t|\ge a_{2\alpha n})}$$
$$\le CM_n^* a_n^{1/2-\beta}T(a_n)Q^{-c}(a_n/4)\|\Phi_n^{(1/2-1/p)^*}(t)P_{n,\rho}(t)W(t) (|t|+\frac{a_n}{n})^\rho\|_{L^p(|t|\ge a_{2\alpha n})}$$

$$\le CM_n^* a_n^{1/p-\beta}(\log n)^2\frac{1}{(\log n)^2}$$
$$\le CM_n^* \le CM^*. \quad (\text{by } 1/p<\beta) \tag{3.13}$$

Finally, we estimate J_5. Let $n\in\mathbb{N}$ and $t\in\mathbb{R}$. In the following, we may assume $|t_{k,n,\rho}|\ge a_{\alpha n}$, because $F_3(t_{k,n,\rho})=0$ if $|t_{k,n,\rho}|\le a_{\alpha n}$. Note that $(1+|t_{k,n,\rho}|)^{-\beta}\le Ca_n^{-\beta}$. Then using Lemma 2.3 (2.5), we have

$$|\Phi_n^{(1/2-1/p)^*}(t)W(t)(|t|+\frac{a_n}{n})^\rho L_{n,\rho}(F_3,t)|$$
$$=|\Phi_n^{(1/2-1/p)^*}(t)W(t)(|t|+\frac{a_n}{n})^\rho \sum_{|t_{k,n,\rho}|\ge a_{\alpha n}} F(t_{k,n,\rho})\ell_{k,n,\rho}(t)|$$
$$\le C|\Phi_n^{(1/2-1/p)^*}(t)P_{n,\rho}(t)W(t)(|t|+\frac{a_n}{n})^\rho|M_n^* a_n^{-\beta}T^{-1/2}(a_{\alpha n})Q^{-c}(a_{\alpha n})$$
$$\times\sum_{|t_{k,n,\rho}|\ge a_{\alpha n}}\left|\frac{\Phi^{1/4}(t_{k,n,\rho})}{(t-t_{k,n,\rho})P'_{n,\rho}(t_{k,n,\rho})W(t_{k,n,\rho})(|t_{k,n,\rho}|+a_n/n)^\rho}\right|$$

$$\le CM_n^* a_n^{-\beta}T^{-1/2}(a_{\alpha n})Q^{-c}(a_{\alpha n})\sum_{|t_{k,n,\rho}|\ge a_{\alpha n}}\frac{a_n^{3/2}}{n}\left|\frac{1}{t-t_{k,n,\rho}}\right|$$
$$\times|\Phi_n^{(1/2-1/p)^*}(t)P_{n,\rho}(t)W(t)(|t|+\frac{a_n}{n})^\rho|. \tag{3.14}$$

We first estimate the integral on $[-a_{\alpha n/2},a_{\alpha n/2}]$. Let $|t|\le a_{\alpha n/2}$. From $|t-t_{k,n,\rho}|\ge Ca_n/T(a_n)$ and Lemma 2.1 (1-a) we have

$$|\Phi_n^{(1/2-1/p)^*}(t)W(t)(|t|+\frac{a_n}{n})^\rho L_{n,\rho}(F_3,t)|$$
$$\le CM_n^* a_n^{-\beta}T^{-1/2}(a_{\alpha n})Q^{-c}(a_{\alpha n})\sum_{|t_{k,n,\rho}|\ge a_{\alpha n}}\frac{a_n^{3/2}}{n}\frac{T(a_n)}{a_n}$$
$$\times|\Phi_n^{(1/2-1/p)^*}(t)P_{n,\rho}(t)W(t)(|t|+\frac{a_n}{n})^\rho|$$
$$\le CM_n^* a_n^{1/2-\beta}(\log n)\left(\frac{1}{Q^c(a_{\alpha n})}\right)|\Phi_n{}^{(1/2-1/p)^*}(t)P_{n,\rho}(t) W(t)(|t|+\frac{a_n}{n})^\rho|$$
$$\le CM_n^* a_n^{1/2-\beta}(\log n)\left(\frac{n}{\sqrt{T(a_n)}}\right)^{-c}|\Phi_n{}^{(1/2-1/p)^*}(t)P_{n,\rho}(t) W(t)(|t|+\frac{a_n}{n})^\rho|.$$

Therefore, we have for $|t|\le a_{\alpha n/2}$

$$\|\Phi_n^{(1/2-1/p)^*}(t)W(t)(|t|+\frac{a_n}{n})^\rho L_{n,\rho}(F_3,t)\|_{L^p(|t|\le a_{\alpha n/2})}$$
$$\le CM_n^* a_n^{1/2-\beta}(\log n)\left(\frac{\sqrt{T(a_n)}}{n}\right)^c\|\Phi_n{}^{(1/2-1/p)^*}(t)P_{n,\rho}(t) W(t)(|t|+\frac{a_n}{n})^\rho\|_{L^p(|t|\le a_{\alpha n/2})}$$

$$\le CM_n^* a_n^{1/p-\beta}(\log n)\left(\frac{\log n}{n}\right)^c\le CM_n^*, \tag{3.15}$$

by Assumption 1.3. Next we estimate the integral on $(-\infty,-a_{2n}]\cup[a_{2n},\infty)$. Noting Lemma 2.1 (1-a), (2-b), from (3.14) we see for $|t|\ge a_{2n}$,

$$|\Phi^{(1/2-1/p)^*}(t)W(t)(|t|+\frac{a_n}{n})^\rho L_{n,\rho}(F_3,t)|$$
$$\le CM_n^* a_n^{-\beta}T^{-1/2}(a_{\alpha n})Q^{-c}(a_{\alpha n})\sum_{|t_{k,n,\rho}|\ge a_{\alpha n}}\frac{a_n^{1/2}}{n}\frac{1}{1-a_n/a_{2n}}$$
$$\times|\Phi_n^{(1/2-1/p)^*}(t)P_{n,\rho}(t)W(t)(|t|+\frac{a_n}{n})^\rho|$$
$$\le CM_n^* a_n^{1/2-\beta}T^{-1/2}(a_{\alpha n})T(a_n)Q^{-c}(a_{\alpha n}) |\Phi_n^{(1/2-1/p)^*}(t)P_{n,\rho}(t)W(t)(|t|+\frac{a_n}{n})^\rho|$$
$$\le CM_n^* a_n^{1/2-\beta}\,T^{1/2}(a_n)\,Q^{-c}(a_{\alpha n})|\Phi_n^{(1/2-1/p)^*}(t)P_{n,\rho}(t) W(t)(|t|+\frac{a_n}{n})^\rho|.$$

Therefore

$$\|\Phi_n^{(1/2-1/p)^*}(t)W(t)(|t|+\frac{a_n}{n})^\rho L_{n,\rho}(F_3,t)\|_{L^p(|t|\ge a_{2n})}$$

$$\leq CM_n^* a_n^{1/2-\beta}\, T^{1/2}(a_n)\; Q^{-c}(a_{\alpha n})\|\Phi_n^{(1/2-1/p)^*}(t)$$

$$P_{n,\rho}(t)W(t)(|t|+\frac{a_n}{n})^\rho\|_{L^p(|t|\geq a_{2n})}$$

$$\leq CM_n^* a_n^{1/p-\beta}(\log n)Q^{-c}(a_{\alpha n}) \leq CM_n^* \qquad (3.16)$$

Here we used $1/p < \beta$ and $(\log n)\, Q^{-c}(a_{\alpha n}) \leq (\log n)\, Q^{-c}(a_n/4) \leq C$ by Lemma 3.1 with $k = 1$. Finally we estimate the integral on $[-a_{2n},-a_{\alpha n/2}] \cup [a_{\alpha n/2}, a_{2n}]$. Since $Q^{-c}(a_{\alpha n}) \leq Q^{-c}(a_n/4) \leq (\log n^{-2})$ by Lemma 3.1 with $k=2$, for $a_{\alpha n/2} \leq |t| \leq a_{2n}$, Lemma 2.11 (2.14) gives us

$$|\Phi_n^{(1/2-1/p)^*}(t)W(t)(|t|+\frac{a_n}{n})^\rho L_{n,\rho}(F_3,t)|$$
$$\leq CM_n^*\Phi_n^{(1/2-1/p)^*}(t)W(t)(|t|+\frac{a_n}{n})^\rho$$

$$\times \sum_{|t_{k,n,\rho}|\geq a_{\alpha n}} (1+t_{k,n,\rho}^2)^{-\beta/2}T^{-1/2}(t_{k,n,\rho})Q^{-c}(t_{k,n,\rho})$$

$$W^{-1}(t_{k,n,\rho})(|t_{k,n,\rho}|+\frac{a_n}{n})^{-\rho}\times|\ell_{k,n,\rho}(t)|$$

$$\leq CM_n^* a_n^{-\beta}T^{-1/2}(a_{\alpha n})Q^{-c}(a_{\alpha n})\Phi_n^{(1/2-1/p)^*}(t)W(t)(|t|+\frac{a_n}{n})^\rho$$

$$\times \sum_{|t_{k,n,\rho}|\geq a_{\alpha n}} W^{-1}(t_{k,n,\rho})(|t_{k,n,\rho}|+\frac{a_n}{n})^{-\rho}|\ell_{k,n,\rho}(t)|$$
$$\leq CM_n^* a_n^{-\beta}T^{-1/2}(a_{\alpha n})Q^{-c}(a_{\alpha n})\Phi_n^{(1/2-1/p)^*}(t)$$
$$\times\{\log n + a_n^{1/2}|P_{n,\rho}(t)|W(t)(|t|+\frac{a_n}{n})^\rho T^{-1/4}(a_n)\}$$

$$\leq CM_n^* a_n^{-\beta}(\log n)^{-2}\Phi_n^{(1/2-1/p)^*}(t)\{\log n + a_n^{1/2}|P_{n,\rho}(t)|$$
$$W(t)(|t|+\frac{a_n}{n})^\rho\}$$

$$\leq CM_n^* a_n^{-\beta}(\log n)^{-2}\{\log n + a_n^{1/2}|\Phi_n^{(1/2-1/p)^*}(t)P_{n,\rho}(t)|$$
$$W(t)(|t|+\frac{a_n}{n})^\rho\}.$$

Therefore, we have by Lemma 2.5 (2.10),

$$\|\Phi_n^{(1/2-1/p)^*}(t)W(t)(|t|+\frac{a_n}{n})^\rho L_{n,\rho}(F_3,t)\|_{L^p(a_{\alpha n/2}\leq|t|\leq a_{2n})}$$
$$\leq CM_n^* a_n^{-\beta}(\log n)^{-2}\{\log n\|1\|_{L^p(a_{\alpha n/2}\leq|t|\leq a_{2n})}$$
$$+a_n^{1/2}\|\Phi_n^{(1/2-1/p)^*}(t)P_{n,\rho}W(t)(|t|+\frac{a_n}{n})^\rho\|_{L^p(a_{\alpha n/2}\leq|t|\leq a_{2n})}\}$$
$$\leq CM_n^* a_n^{-\beta}(\log n)^{-2}\{(\log n)\; a_n^{1/p}$$
$$+a_n^{1/2}\|\Phi^{(1/2-1/p)^*}(t)P_{n,\rho}\,W(t)(|t|+\frac{a_n}{n})^\rho\|_{L^p(a_{\alpha n/2}\leq|t|\leq a_{2n})}\}$$
$$\leq CM_n^*(\log n)^{-2}(\{a_n^{1/p-\beta}\log n + a_n^{1/p-\beta}\} \leq CM_n^*$$

(note Assumption 1.3). From (3.15)-(3.17) we have

$$J_5 \leq \|\Phi_n^{(1/2-1/p)^*}(t)W(t)(|t|+\frac{a_n}{n})^\rho L_{n,\rho}(F_3,t)\|_{L^p(\mathbb{R})}$$

$$\leq CM_n^* \leq CM^*. \qquad (3.18)$$

Consequently, from (3.4), (3.8), (3.11), (3.13) and (3.18) we have for $n = 1, 2,\cdots$,

$$\|\Phi_n^{(1/2-1/p)^*}W_\rho L_{n,\rho}(F)\|_{L^p(\mathbb{R})} \leq CM^*$$

and hence we conclude (3.2).

Proof of Theorem 1.5. Let $F \in C(\mathbb{R})$ with

$$(1+t^2)^{\beta/2}T^{1/2}(t)\Phi^{-1/4}(t)Q^c(t)W_\rho(t)F(t) \in C_0(\mathbb{R}). \qquad (3.19)$$

By Lemma 2.6, There exists $W^* \in \mathcal{F}(C^2+)$ such that

$$(1+t^2)^{(\rho+\beta)/2}T^{1/2}(t)\Phi^{-1/4}(t)Q^c(t)W(t) \sim W^*(t).$$

Since $W^* F \in C_0(\mathbb{R})$ by (3.19), Lemma 2.7 implies that there exist $P_{n-1} \in \mathcal{P}_{n-1}$, $n = 1,2,\cdots$ such that

$$\lim_{n\to\infty}\|(1+t^2)^{(\rho+\beta)/2}T^{1/2}(t)\Phi^{-1/4}(t)Q^c(t)W(t)$$

$$(F(t)-P_{n-1}(t))\|_{L^\infty(\mathbb{R})} = 0. \qquad (3.20)$$

Since $L_{n,\rho}(P_{n-1}) = P_{n-1}$, we have

$$\|\Phi_n^{(1/2-1/p)^*}(t)W_\rho(t)(F(t)-L_{n,\rho}(F,t))\|_{L^p(\mathbb{R})}$$
$$\leq \|\Phi_n^{(1/2-1/p)^*}(t)W_\rho(t)(F(t)-P_{n-1}(t))\|_{L^p(\mathbb{R})}$$
$$+\|\Phi_n^{(1/2-1/p)^*}(t)W_\rho(t)L_{n,\rho}(F-P_{n-1},t)\|_{L^p(\mathbb{R})} =: K_1+K_2 \qquad (3.21)$$

By Lemma 3.2 and (3.20), we see $K_2 \to 0$ as $n \to \infty$. On the other hand

$$K_1 = \|\Phi_n^{(1/2-1/p)^*}(t)W_\rho(t)(F(t)-P_{n-1}(t))\|_{L^p(\mathbb{R})}$$

$$\leq \|(1+t^2)^{\beta/2}T^{1/2}(t)\Phi^{-1/4}(t)Q^c(t)W_\rho(t)(F(t)-P_{n-1}(t))\|_{L^\infty(\mathbb{R})}$$

$$\times\|(1+t^2)^{-\beta/2}T^{-1/2}(t)\Phi^{1/4}(t)Q^{-c}(t)\|_{L^p(\mathbb{R})}$$
$$\leq\|(1+t^2)^{\beta/2+\rho/2}T^{1/2}(t)\Phi^{-1/4}(t)Q^c(t)W(t)(F(t)-P_{n-1}(t))\|_{L^\infty(\mathbb{R})}$$

$$\times\|(1+t^2)^{-\beta/2}T^{-1/2}(t)\Phi^{1/4}(t)Q^{-c}(t)\|_{L^p(\mathbb{R})}.$$

Since $1/p - \beta < 0$, we have

$$\|(1+t^2)^{-\beta/2}T^{-1/2}(t)\Phi^{1/4}(t)Q^{-c}(t)\|_{L^p(\mathbb{R})}$$

$$\leq \|(1+|t|)^{-\beta}\|_{L^p(\mathbb{R})} < C,$$

so that (3.17) again shows $K_1 \to 0$ as $n \to \infty$. Consequently from (3.21) we have (1.7).

Weighted Lagrange Interpolation Theory on [0,∞)

In this section we consider a Laguerre-type weight w of the form $w(x) = \exp(-R(x))$ on $\mathbb{R}^+ := [0,\infty)$. We always assume that

$$\int_0^\infty x^n w(x)^2 dx < \infty \quad (\forall n = 0,1,2,\cdots).$$

We begin with the definition of a class $\tilde{\mathcal{L}}(C^2)$.

Definition 4.1.(1) We write $w \in \tilde{\mathcal{L}}(C^2)$, if a function R has the following properties:

(a) $\sqrt{x}R'(x)$ is continuous in $\mathbb{R}^+=[0,\infty)$ with limit 0 at 0 and $R(0)=0$;
(b) $R''(x)>0$ exists in $\mathbb{R}^+\setminus\{0\}$;
(c) $\lim_{x\to\infty} R(x)=\infty$.
(d) A function

$$T^*(x) := \frac{xR'(x)}{R(x)}$$

is quasi-increasing in $\mathbb{R}^+\setminus\{0\}$ (i.e. There exists $C>0$ such that $T^*(x)\le CT^*(y)$ for $0<x<y$) with $T^*(x)\ge\Lambda^*>1/2$ on $\mathbb{R}^+\setminus\{0\}$.
(e) There exists $C>0$ such that

$$\frac{R''(x)}{R'(x)} \le C\frac{R'(x)}{R(x)}, \quad a.e.\ t\in\mathbb{R}^+\setminus\{0\}.$$

(2) We call $w\in\tilde{\mathcal{L}}(C^2)$ a Freud-type weight if T^* is bounded, and otherwise, w is called an Erdős-type weight. Note that if w is Erdős-type, then $\lim_{x\to\infty} T^*(x)=\infty$ holds.

(3) We write $w\in\tilde{\mathcal{L}}(C^2+)$, if $w\in\tilde{\mathcal{L}}(C^2)$ further satisfies

$$\frac{R''(x)}{R'(x)} \ge C\frac{R'(x)}{R(x)}, \quad a.e.\ x\in\mathbb{R}^+\setminus J^*,$$

with $C>0$ and some compact subinterval $J^*\ni 0$ of $\mathbb{R}^+$.

(4) Let $1<\lambda<3/2$. We write $w\in\tilde{\mathcal{L}}_\lambda(C^3+)$ if $w\in\tilde{\mathcal{L}}_\lambda(C^2+)$ and $R\in C^3(\mathbb{R}^+\setminus\{0\})$, and for $x\ge K>0$ (large enough),

$$\frac{|R'''(x)|}{|R''(x)|} \le C\frac{|R''(x)|}{R'(x)} \text{ and } \frac{\sqrt{x}R'(x)}{R(x)^\lambda} \le C$$

hold.

Example 4.2. (1) For $R(x)=x^\beta$, $\beta>1/2$, $w(x)=\exp(-R(x))$ belongs to $\tilde{\mathcal{L}}(C^2+)$. Since $T^*(x)=\beta$, w is a Freud-type weight. Moreover it belongs to $\tilde{\mathcal{L}}_\lambda(C^3+)$.

(2) ([7]) Let $l\ge1$ be an integer, and let $\beta>1/2$. Then we define $R_{l,\beta}(x) := \exp_l(x^\beta) - \exp_l(0)$, where $\exp_l(x) = \exp(\exp(\exp\cdots\exp(x)))$ (l times). Then $w(x)=\exp(-R_{l,\beta}(x))$ belongs to $\tilde{\mathcal{L}}(C^2+)$ and it is an Erdős-type weight. Moreover it belongs to $\tilde{\mathcal{L}}_\lambda(C^3+)$.

To describe our result, we prepare some notations. For $w=\exp(-R)\in\tilde{\mathcal{L}}(C^2+)$, the Mhaskar-Rakhmanov-Saff number a_x^* ($x>0$) is defined as the positive root of the equation

$$x = \frac{1}{\pi}\int_0^1 \frac{a_x^* u R'(a_x^* u)}{\sqrt{u(1-u)}}du.$$

We also use for $x>0$

$$\Phi_n^*(x) := \max\{1-\frac{x}{a_n^*},\ \delta_n^*\},\quad \delta_n^* := \frac{1}{(nT^*(a_n^*))^{2/3}},$$

and

$$\Phi^*(x) := \frac{1}{T^*(x)(1+R(x))^{2/3}}\left(=\frac{R(x)}{xR'(x)(1+R(x))^{2/3}}\right),$$

Finally, for $w\in\tilde{\mathcal{L}}(C^2+)$ and $\rho\ge-1/2$, we always use the notation

$$w_\rho(x) := x^\rho w(x) \tag{4.1}$$

and for $p>1$,

$$\rho^* := \rho + \frac{1}{2p} - \frac{1}{4}. \tag{4.2}$$

Note that $\rho^*=\rho$ in the case $p=2$.

The n th order Lagrange interpolation polynomial $L^*_{n,\rho^*}(f,x)$ for a function f on $\mathbb{R}^+$ with a weight w_{ρ^*} is defined as follows:

$$L^*_{n,\rho^*}(f,x) := \sum_{k=1}^n f(x_{k,n,\rho^*})\frac{p_{n,\rho^*}(x)}{(x-x_{k,n,\rho^*})p'_{n,\rho^*}(x_{k,n,\rho^*})}, \tag{4.3}$$

where $\{p_{n,\rho^*}\}_{n=0}^\infty$ are orthonormal ploynomials with respect to the weight w_{ρ^*} and $\{x_{k,n,\rho^*}\}_{k=1}^n$ are the zeros of p_{n,ρ^*} with $0<x_{n,n,\rho^*}<\cdots<x_{2,n,\rho^*}<x_{1,n,\rho^*}<\infty$.

By the same argument as in [1, section 7], the following theorem is deduced from Theorem 1.5.

Theorem 4.3. Assume that $w\in\tilde{\mathcal{L}}_\lambda(C^3+)$ with $1<\lambda<3/2$ and $a_n^*=o(n^{4/3})$. Further we assume that for $M>0$ there exists $N_M\in\mathbb{N}$ such that for any $n\ge N_M$, $R(a_n^*/16)\ge(\log R(a_n^*))^M$ and $T^*(a_n^*)\le C(\log n)^2$ hold. Let $f\in C(\mathbb{R}+)$, $1<p<\infty$, $1/p<\beta$ and $-1/p\le2\rho$. If for $0<c<1$

$$(1+x)^{\beta/2+1/4}T^{*1/2}(x)\Phi^{*-1/4}(x)R^c(x)w_{\rho^*}(x)f(x)\in C_0(\mathbb{R}^+), \tag{4.4}$$

then

$$\lim_{n\to\infty}\|w_\rho(L^*_{n,\rho^*}(f)-f)\|_{L^p(\mathbb{R}^+)}=0,\ 1<p<4, \tag{4.5}$$

and

$$\lim_{n\to\infty}\|\Phi_n^{*(1/2-1/p)}w_\rho(L^*_{n,\rho^*}(f)-f)\|_{L^p(\mathbb{R}^+)}=0,\ 4\le p<\infty. \tag{4.6}$$

To prove the Theorem 4.3, we recall the relations between $\tilde{\mathcal{L}}(C^2+)$ and $\mathcal{F}(C^2+)$. Let $\rho>-1/2$, $w\in\tilde{\mathcal{L}}(C^2+)$ and put

$$W(t) := w(t^2).$$

and we put

$$\rho^\# := 2\rho+1/2.$$

Then

$$W_{\rho^\#}(t) = |t|^{\rho^\#}W(t) = |t|^{2\rho+1/2}w(t^2).$$

Lemma 4.4 ([1, Lemma 7.5]). Let $\{p_{n,\rho}\}$ and $\{p_{n,\rho+1/2}\}$ be orthonormal polynomials with respect to weights $w_\rho(x)=x^\rho w(x)$ and $w_{\rho+1/2}(x)=x^{\rho+1/2}w(x)$, respectively. We set

$$P_{2n}(t)=p_{n,\rho}(t^2),\quad P_{2n+1}(t)=tp_{n,\rho+1/2}(t^2)\quad(n=0,1,2,\cdots).$$

Then $\{P_n\}$ are orthonormal polynomials with respect to a weight $W_{\rho^\#}$.

Lemma 4.4 shows that

$$x_{j,n,\rho} = t^2_{j,2n,\rho^\#} = t^2_{2n-j+1,2n,\rho^\#} \quad (j = 1, 2, \cdots, n),$$

where $0 < x_{n,n,\rho} < x_{n-1,n,\rho} < \cdots < x_{1,n,\rho}$ are zeros of $p_{n,\rho}$ and $t_{2n,2n,\rho^\#} < t_{2n+1,2n,\rho^\#} < \cdots < t_{1,2n,\rho^\#}$ are zeros of $P_{2n,\rho^\#}$. We note that

$$t_{j,2n,\rho^\#} = -t_{2n-j+1,2n,\rho^\#} \quad (j = 1, 2, \cdots, n). \tag{4.9}$$

We transform a function f on $\mathbb{R}^+$ into an even function F on $\mathbb{R}$ as follows;

$$F(t) = f(t^2), \quad t \in \mathbb{R}.$$

Note that $L^*_{n,\rho}(f,x)$ is the nth order Lagrange interpolation polynomial with respect to a weight w_ρ and $L_{2n,\rho^\#}(F,t)$ is the $2n$th order Lagrange interpolation polynomial with respect to a weight $W_{\rho\#}$. We summarize the fundamental facts. Recall that for $x = t^2$, $f(x) = F(t)$ and $w(x) = W(t)$.

(1) ([1, Lemma 7.1]) $w \in \tilde{\mathcal{L}}(C^2+) \Rightarrow W \in \mathcal{F}(C^2+)$.
(2) ([1, Lemmas 7.2 and 7.3])

$$a^*_n = a^2_{2n}, \quad T^*(x) = \tfrac{1}{2}T(t), \quad \Phi^*(x) = 2\Phi(t).$$

(3) ([1, Lemma 7.4 (1)]) $a_n^* = o(n^{4/3}) \Rightarrow a_n = o(n^{2/3})$.

(4) $\Phi^*_n(x) \sim \Phi_{2n}(t)$. In fact, since $\delta^*_n = 2^{2/3}\delta_{2n}$, we see

$$\Phi^*_n(x) = \max\{1 - \frac{x}{a^*_n}, \delta^*_n\} = \max\{(1 - \frac{|t|}{a_{2n}})(1 + \frac{|t|}{a_{2n}}), 2^{2/3}\delta_{2n}\}$$
$$\sim \max\{(1 - \frac{|t|}{a_{2n}}), \delta_{2n}\} = \Phi_{2n}(t).$$

(5) ([1, Lemmas 7.5 and 7.6]) For $\rho^\# = 2\rho+1/2$,

$$P_{2n,\rho\#}(t) = p_{n,\rho}(x) \text{ and } L^*_{n,\rho}(f,x) = L_{2n,\rho\#}(F,t)$$

(6) ([1, Lemma 7.4 (2)]) For $1<\lambda<3/2$, $w \in \tilde{\mathcal{L}}_\lambda(C^3+) \Rightarrow W \in \mathcal{F}_\lambda(C^3+)$.

(7) (cf. [1, (VIII)) Let $2\rho + 1/p \geq 0$ and $\rho^* = \rho + 1/(2p) - 1/4$. Then for $0 < c < 1$,

$$(1+x)^{\beta/2+1/4}T^{*1/2}(x)\Phi^{*-1/4}(x)w_{\rho^*}(x)R^c(x)f(x) \in C_0(\mathbb{R}^+)$$
$$\Longrightarrow (1+t^2)^{\beta/2}T^{1/2}(t)\Phi^{-1/4}(t)W_{2\rho+1/p}(t)Q^c(t)F(t) \in C_0(\mathbb{R}).$$

(8) (cf. [1, Lemma 7.4 (4)) For $M > 0$ there exists $N_M \in \mathbb{N}$ such that for any $n \geq N_M$, $R(a^*_n/16) \geq (\log R(a^*_n))^M$ and $T^*(a^*_n) \leq C(\log n)^2$. Then W satisfies the Assumption 1.3.

Proof of Theorem 4.3. Let $1 < p < \infty$ and $2\rho + 1/p \geq 0$. Then we see $(\rho^*)^\# = 2\rho^*+1/2 = 2\rho + 1/p \geq 0$. We suppose (4.4). Using (1) ~ (8) and Theorem 1.5 for $2\rho+1/p$ instead of ρ there, we have for $1 < p < 4$

$$\|w_\rho(L^*_{n,\rho^*}(f) - f)\|^p_{L^p(\mathbb{R}^+)}$$
$$= \int_0^\infty |x^\rho w(x)(L^*_{n,\rho^*}(f,x) - f(x))|^p dx$$
$$= 2\int_0^\infty |W_{2\rho}(t)(L_{2n,(\rho^*)^\#}(F,t) - F(t))|^p t dt$$
$$= \int_{-\infty}^\infty |W_{2\rho+1/p}(t)(L_{2n,2\rho+1/p}(F,t) - F(t))|^p dt \to 0$$

and for $4 \leq p < \infty$ we have

$$\|\Phi_n^{*(1/2-1/p)}w_\rho(L^*_{n,\rho^*}(f) - f)\|^p_{L^p(\mathbb{R}^+)}$$
$$= \int_0^\infty |\Phi_n^{*(1/2-1/p)}(x)x^\rho w(x)(L^*_{n,\rho^*}(f,x) - f(x))|^p dx$$
$$\leq 2C\int_0^\infty |\Phi_{2n}^{1/2-1/p}(t)W_{2\rho}(t)(L_{2n,(\rho^*)^\#}(F,t) - F(t))|^p t dt$$
$$= C\int_{-\infty}^\infty |\Phi_{2n}^{1/2-1/p}(t)W_{2\rho+1/p}(t)(L_{2n,2\rho+1/p}(F,t) - F(t))|^p dt$$
$$\to 0 \quad \text{as } n \to \infty.$$

Thus we have the result.

Appendix

In this section we show that weights in Example 1.2 satisfy the conditions in Assumption 1.3. Note that they belong to $\mathcal{F}_\lambda(C^3+)$ with $1 < \lambda < 3/2$ (cf. [4], [6]).

Lemma 5.1 ([6, Example 1, p.27]). Let $Q(t) = Q_\alpha(t) = |t|^\alpha$ for $\alpha > 1$. Then $T(t) = \alpha$ and $a_n \sim n^{1/\alpha}$.

This lemma shows (1.4) and (1.5) for Q_α immediately.

Lemma 5.2 ([6, Example 2, pp.29-30]). Let $Q(t) = Q_{l,\alpha}(t) = \exp_l(|t|^\alpha) - \exp_l(0)$ for $\alpha > 1$, $l \geq 1$, where $\exp_l(t) := \exp(\exp(\exp\cdots\exp(t)))$ (l times). Then

$$a_n = (\log_l(n))^{1/\alpha}(1 + o(1)), \tag{5.1}$$

where $\log_l(t) := \log(\log(\log\cdots\log(t)))$ (l times), and

$$T(t) := \frac{a_n Q'_{l,\alpha}(a_n)}{Q_{l,\alpha}(a_n)} \sim \Pi^l_{j=1}\log_j n. \tag{5.2}$$

Since $\Pi^l_{j=1}\log_j n \leq C(\log n)^2$, (1.5) for $Q_{l,\alpha}$ follows from (5.2). We discuss (1.4) for $Q_{l,\alpha}$. Let $M \geq 1$ and $C \geq 2\cdot 4^\alpha$ be constants. We may assume that $M \geq 2$. Then there exists $K_{C,M,l} \geq M$ depending on C,M,l such that

$$\log_l(t) \geq C\log_l((\log t)^M) \text{ for } t \geq K_{C,M,l}. \tag{5.3}$$

In fact, for $l=1$ we see that if t is large enough, then $C\log((\log t^M) = CM\log(\log t) \leq \log t$. Inductively, we suppose (5.3) for $l \geq 1$. Put $K_{C,M,l+1} := e^{K_{C,M,l}}$ and $t \geq K_{C,M,l+1}$. Then $\log t \geq K_{C,M,l}$, (5.3) for l and $(\log(\log t))^M \geq \log((\log t)^M)$ show

$$\log_{l+1}(t) = \log_l(\log t) \geq C\log_l((\log(\log t))^M)$$
$$\geq C\log_{l+1}((\log t)^M),$$

which implies (5.3) for $l+1$. To show (1.4), we remark $2\log_l(t) \geq \log_l(2t)$ and $\log_l(\exp_l(t)) = t$ hold for sufficiently large $t>0$. Hence (5.3) gives us

$$2\log_l(Q_{l,\alpha}(a_n/4)) \geq \log_l(Q_{l,\alpha}(a_n/4) + \exp_l(0)) =$$
$$(1/4)^\alpha a_n^\alpha = (1/4)^\alpha\log_l(\exp_l(a_n^\alpha))$$
$$= (1/4)^\alpha\log_l(Q_{l,\alpha}(a_n) + \exp_l(0)) \geq (1/4)^\alpha C\log_l((\log Q_{l,\alpha}(a_n))^M)$$
$$\geq 2\log_l((\log Q_{l,\alpha}(a_n))^M),$$

which shows (1.4) for $Q_{l,\alpha}$.

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