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The Strong Sylow Theorem for the Prime p in Simple Locally Finite Groups

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ABSTRACT

This Research Article continues [15]. We begin with giving a profound overview of the structure of arbitrary simple groups and in particular of the simple locally finite groups and reduce their Sylow theory for the prime p to a quite famous conjecture by Prof. Otto H. Kegel (see [44], Theorem 2.4: "Let the p-subgroup P be a p-uniqueness subgroup in the finite simple group S which belongs to one of the seven rank-unbounded families. Then the rank of S is bounded in terms of P.") about the rank-unbounded ones of the 19 well-known families of finite simple groups. We introduce a new scheme to describe the 19 families, the family $\mathcal{T}$ of types, define the rank of each type, and emphasise the great rôle of Kegel covers: Prof. Kegel rediscovered from Prof. Philip Hall (see [46]) that an infinite simple group has a local system consisting of countably infinite simple subgroups (see [45], [46] and [44], Theorem 2.5) (and conversely) and if they are locally finite he discovered groundbreakingly that they have a Kegel cover (see [44], Theorem 2.6), that is, a nested local system $\{G_n\}$ with maximal normal subgroups $M_n \leq G_n$ such that $G_n \cap M_{n+1} = \langle 1 \rangle$ so that G_n embeds into G_{n+1}/M_{n+1}. This part presents a unified picture of known results all of whose proofs are by reference.

Subsequently we apply new ideas to prove the conjecture for the Alternating Groups.

Thereupon we are remembering Kegel covers and ⋆-sequences and the classification of simple locally finite groups according to their Kegel covers. Next we suggest a way 1) and a way 2) how to prove and even how to optimise Kegel's conjecture step-by-step or peu à peu which leads to Conjecture 1, Conjecture 2 and Conjecture 3 thereby unifying *Sylow theory in locally finite simple groups* with *Sylow theory in locally finite and p-soluble groups* whose joint study directs very reliably Sylow theory in (locally) finite groups. For any unexplained terminology we allow us to refer to [15].

We then continue the program begun above to optimise along the way 1) the theorem about the first type Ξ = "$\underline{A}^n$" of infinite families of finite simple groups step-by-step to further types by proving it for the second type Ξ = "A = PSL_n". We apply new ideas to prove Conjecture 2 about the General Linear Groups over locally finite fields, stating that their rank is bounded in terms of their p-uniqueness, and then break down this insight to the Special Linear Groups and to the Projective Special Linear (PSL) Groups over locally finite fields. We close with good suggestions for future research ▶ regarding the remaining rank-unbounded types (the "Classical Groups") and the way 2), ▶ regarding (locally) finite and p-soluble groups, and ▶ regarding Cauchy's and Galois' contributions to Sylow theory in finite groups. We much hope to enthuse group theorists with these suggestions and are ready to support and to cöordinate all related work.

It follows from our two theorems that simple locally finite groups which satisfy the Strong Sylow Theorem for even one Prime p are linear and hence countable if they have a local system of countable simple subgroups each having a Kegel cover "of alternating type" or "of projective special linear type".

We include the beautiful predecessor Research Article [15] as the First Appendix for good reasons. This Research Article had been presented as a slideshow in a Talk at IGT 2024 on April 11. We include its 16 slides as the Second Appendix. Slide 1 to Slide 12 had as well been permanently instaled during IGT 2024 as a Permanent Poster.

The Research Article consists of the following seventeen beautiful Chapters:

● Sketch of proof for $\underline{A}^n$; ● Sketch of proof for A = PSL_n; ① Introduction; ② Proof of Theorem 1;
③ About Kegel covers; ④ Planning future research – Part 1; ⑤ Proof of Theorem 2;
⑥ Proof of Theorem 3; ⑦ Proof of Theorem 4; ⑧ Planning future research – Part 2;
⑨ The First Trilogy and The Second Trilogy and their reviews; ● Acknowledgements;
● Postscript, Luciano De Creszenzo, Felix F. Flemisch, Conflicts of Interest, Pablo Picasso's *La Joie de vivre*;
● About the author in Munich, in Freiburg i.Br., in London, in Weiden i.d.OPf., and in Florence in Tuscany in Italy;
● 75 References; ● Appendix 1 – Reference [15] with MR Review and Zbl Review;
● Appendix 2 – Talk by Felix F. Flemisch at Ischia Group Theory 2024.

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Dedicated to ***Prof. Otto H. Kegel*** *on the occasion of his 90th birthday on 20 July 2024 – Ischia Group Theory 2024 from April 8 to April 13*

(see https://www.advgrouptheory.com/GTArchivum/Pictures/gtphotos/OttoKegel.jpg)

Talk presented at IGT 2024 on 11 April 2024, that is,

on the 120th birthday of ***Prof. Philip Hall***

(see https://mathshistory.st-andrews.ac.uk/Biographies/Hall/)

Keywords • singular (Sylow) p-subgroup • (very) good Sylow p-subgroup • p-uniqueness subgroup • minimal p-unique subgroup • **very beautiful** (numerical) Sylow p-invariant p-uniqueness a_p • locally finite group satisfying the Strong Sylow Theorem for the Prime p, equivalently, the Strong Sylow p-Theorem • simple group • nested local system • family $\mathcal{T}$ of types of known finite simple groups • simple locally finite group *of type* $\Xi \in \mathcal{T}$, *of alternating type* and *of projective special linear type* • rank of a locally finite simple group • classification of the transitive G-sets • **beautiful** Kegel cover • ⋆-sequence • Kegel sequence • simple locally finite group which is finitary, of 1-type, of p-type, and of ∞-type • P-invariant Sylow p-subgroup • conjugacy class • P-isomorphic P-orbit • **beautiful** p-length of a p-soluble finite group • classical Hall-Higman Theory • locally finite field $\mathcal{F}$ • algebraic closure of the **beautiful** prime field in characteristic p • General Linear Group • Special Linear Group • Projective Special Linear (PSL) Group • G-module over some (locally finite) field $\mathcal{F}$ • irreducibility • complete reducibility • (non-)modular G-module • G-isomorphic G-modules • Jordan normal form • Classical Group • Group of Lie type • twisted Chevalley Group

Note – The **rank** of a known locally finite simple group is defined below. For PSL(n,$\mathcal{F}$), and hence for GL(n,$\mathcal{F}$) and SL(n,$\mathcal{F}$), it is simply n = dim($\mathcal{F}^n$). So we have a **rather simple** concept of rank of a linear group which, however, does not contradict any of the elaborate concepts of rank in the excellent book [13].

Let p be a prime: 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97, 101, 103, 107, 109, 113, 127, 131, 137, 139, 149, 151, 157, 163, 167, 173, 179, 181, 191, 193, 197, 199, 211, 223, 227, 229, 233, 239, 241, 251, 257, 263, 269, 271, 277, 281, 283, 293, 307, 311, 313, 317, 331, 337, 347, 349, 353, 359, 367, 373, 379, 383, 389, 397, 401, 409, 419, 421, 431, 433, 439, 443, 449, 457, 461, 463, 467, 479, 487, 491, 499, 503, 509, 521, 523, 541, 547, 557, 563, 569, 571, 577, 587, 593, 599, 601, 607, 613, 617, 619, 631, 641, 643, 647, 653, 659, 661, 673, 677, 683, 691, 701, 709, 719, 727, 733, 739, 743, 751, 757, 761, 769, 773, 787, 797, 809, 811, 821, 823, 827, 829, 839, 853, 857, 859, 863, 877, 881, 883, 887, 907, 911, 919, 929, 937, 941, 947, 953, 967, 971, 977, 983, 991, 997, 1009, 1013, 1019, 1021, 1031, 1033, 1039, 1049, 1051, 1061, 1063, … ☺

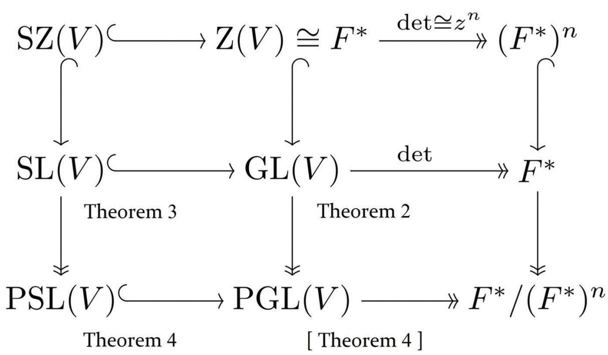
In this paper we prove **Kegel**'s conjecture for $\underline{A}^n$ and for A = PSL$_n$. It continues [15] F.F. FLEMISCH: "**Characterising Locally Finite Groups Satisfying the Strong Sylow Theorem for the Prime p**", *Adv. Group Theory Appl.* **13** (June 2022), 13-39 (see MR4441631 and Zbl 1496.20065). We included that **beautiful** predecessor paper completely as an **Appendix**, although it is open access, since the current paper cannot be understood without that predecessor paper – so one needs to have it present when reading the current paper – and included as well the MR Review and the Zbl Review and an important comment ☺.

Sketch of proof for $\underline{A}^n$

Let the finite p-group P act on $\underline{A}^n$. Let α be a point and let $P_\alpha := \{x \in P \mid \alpha^x = \alpha\} \subseteq P$ be the stabiliser of α. We denote by **U(P)** the set of all subgroups of P and for every $U \in$ U(P) by R(P,U) := $\{Ux \mid x \in P\}$ the set of all right cosets of U in P. Then P operates by multiplication from the right for every $U \in$ U(P) transitively on R(P,U) with $\mathrm{Cor}_P U := \{U^x \mid x \in P\}$ as the kernel. The classification of transitive P-sets reads as follows: *Every transitive P-set $\Omega \neq \emptyset$ is P-isomorphic to* R(P,P_α) *for all $\alpha \in \Omega$, and for any $U, V \in$* U(P) *the two sets* R(P,U) *and* R(P,V) *are P-isomorphic if and only if U and V are conjugate in P.* Hence for the action of P we have a bijection between the class $\mathcal{J}$**(P)** of all P-isomorphism types of transitive P-sets and the set of all conjugacy classes (in P) of subgroups of P, and therefore $|\mathcal{J}(P)|$ = $\mathbf{g}_p(|P|)$:= the number of conjugacy classes of subgroups of P. Therefore for every P-set Ω the class $\mathcal{J}$**(P,Ω)** of P-isomorphism types of P-orbits on Ω has at most $g_p(|P|)$ elements and since every subgroup of P is a subset containing 1, we can now summarising deduce $|\mathcal{J}(P,\Omega)| \leq g_p(|P|) \leq |\mathrm{U}(P)| \leq 2^{|P|-1}$. If P is a p-subgroup of $\underline{S}^n$ which is contained in exactly k $\in \mathbb{N}$ Sylow p-subgroups of $\underline{S}^n$ and if m := k + p + 1, then n ≤ m • $|P|$ • $g_p(|P|)$ - 1 and n ≤ (p + 2) • $2^{|P|-1}$ - 1 for k = 1 (see **Page 5**), whence, if not so, P has at least m many P-isomorphic P-orbits on Ω := {1, 2, …, n} (see **Page 5**). We are then able to deduce from this fact the central observation that $\{S \in \mathrm{Syl}_p\underline{S}^\Omega \mid S$ is P-invariant$\}$ =: $\mathrm{Syl}_p(\underline{S}^\Omega,P) \geq |\mathrm{Syl}_p\underline{S}^m| \geq$ m - 2 ≥ k + 1 by using **beautiful new ideas** (see **Page 6**). □

Sketch of proof for A = PSL$_n$

We are applying a three-stage-approach whilst **first** proving the theorem for the **General Linear Groups** over (commutative) locally finite fields (**Theorem 2**), **then** for the **Special Linear Groups** over locally finite fields (**Theorem 3**) and **finally** for the **Projective Special Linear (PSL) Groups** over locally finite fields (**Theorem 4**), thereby using that GL(n,F) = SL(n,F) • F^* and PSL(n,F) = SL(n,F)/Z(SL(n,F)) (see **Page 11** and **Page 12**). This can be shown with a **very beautiful** diagram:



The major work is required for the **General Linear Groups** with two different and both **very beautiful** approaches for characteristic $\neq p$ and characteristic p. In characteristic $\neq p$ we use that, if for a finite p-group P which is operating on a finite-dimensional vector space V over a locally finite field and a direct decomposition of V into irreducible P-submodules, there are k many of the P-submodules P-isomorphic, then at least $|\mathrm{Syl}_p\underline{\mathrm{S}}^{\mathrm{k}}|$ Sylow p-subgroups of $\mathrm{GL}(V)$ are P-invariant (see **Proposition 7 a)**). In characteristic p we use that, if k is the dimension of the P-submodule $\mathrm{C}_V(P) := \{v \in V \mid v^x = v \text{ for all } x \in P\}$ of a non-trivial modular P-module V, then again there are at least $|\mathrm{Syl}_p\underline{\mathrm{S}}^{\mathrm{k}}|$ many P-invariant Sylow p-subgroups of $\mathrm{GL}(V)$ (see **Proposition 7 b)**). We then are able to argue that from **Proposition 7** follows that $\mathrm{n} \leq (p+2) \bullet |P|^2 - 1$ for a p-uniqueness subgroup P of $\mathrm{GL}(\mathrm{n},\mathcal{F})$ (see **Lemma 2** on **Page 11**). For the transition from $\mathrm{GL}(\mathrm{n},\mathcal{F})$ to $\mathrm{SL}(\mathrm{n},\mathcal{F})$ we are using that a p-uniqueness subgroup of $\mathrm{SL}(\mathrm{n},\mathcal{F})$ is a p-uniqueness subgroup of $\mathrm{GL}(\mathrm{n},\mathcal{F})$ as well. For the transition from $\mathrm{SL}(\mathrm{n},\mathcal{F})$ to $\mathrm{PSL}(\mathrm{n},\mathcal{F})$ we use that $P := Q \bullet \mathrm{D}(\mathrm{SL}(\mathrm{n},\mathcal{F})) / \mathrm{D}(\mathrm{SL}(\mathrm{n},\mathcal{F}))$ is a p-uniqueness subgroup of $\mathrm{PSL}(\mathrm{n},\mathcal{F})$ when Q is a p-uniqueness subgroup of $\mathrm{SL}(\mathrm{n},\mathcal{F})$, and conversely, together with the **Proposition 4** and the **Proposition 6** to get the lower bound $p+2$ whence P lies in at least $|\mathrm{Syl}_p\underline{\mathrm{S}}^{p+2}|$ Sylow p-subgroups of $\mathrm{PSL}(\mathrm{n},\mathcal{F})$. □

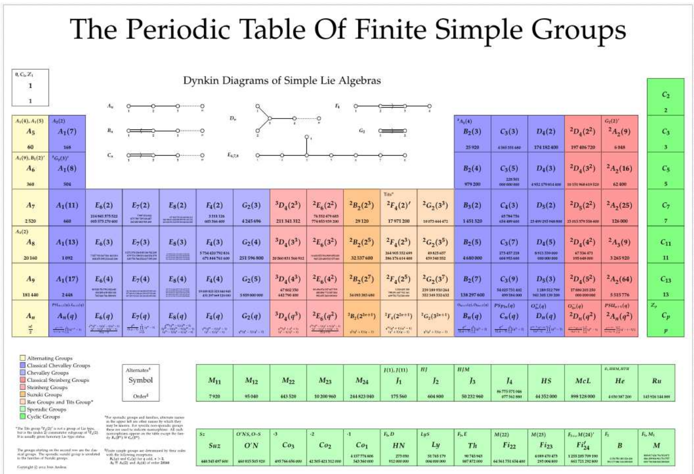
1. Introduction

For any unexplained notation we refer to [15].

Bring to mind that a group is called *simple* if itself and <1> are its sole normal subgroups and that a *local system for a group G* is a family Σ of subgroups such that every element of G lies in a Σ-group and for every two Σ-groups there exists another Σ-group which contains both. The local system Σ for the group G is said to be *nested* if there exists a sequence $\{U_{\mathrm{n}} \mid \mathrm{n} \in \mathbb{N}\}$ of subgroups of G such that $U_{\mathrm{n}} \subseteq U_{\mathrm{n}+1}$ for all $\mathrm{n} \in \mathbb{N}$ and $\Sigma = \{U_{\mathrm{n}} \mid \mathrm{n} \in \mathbb{N}\}$. If G is a countable group and $\{x_{\mathrm{n}} \mid \mathrm{n} \in \mathbb{N}\}$ an enumeration of G, let $U_{\mathrm{n}} := \langle x_1, x_2, \ldots, x_{\mathrm{n}}\rangle$ ($\mathrm{n} \in \mathbb{N}$); then $\{U_{\mathrm{n}} \mid \mathrm{n} \in \mathbb{N}\}$ is a nested local system for G. If the locally finite group G has such a nested local system, then G is countable. If an infinite group $G = \langle U \mid U \in \Sigma\rangle$ possesses a local system Σ consisting of simple subgroups, it is simple: suppose $N \neq \langle 1\rangle$ is a normal subgroup of G; if $N \cap U = \langle 1\rangle$ for all $U \in \Sigma$ then $N = \langle N \cap U \mid U \in \Sigma\rangle = \langle 1\rangle$; hence $N \cap U = U$ for some $U \in \Sigma$ and so $N \cap V = V$ for all $V \in \Sigma$ since $U, V \subseteq W$ for each $V \in \Sigma$ with some $W \in \Sigma$; thus $N = G$. An infinite simple group has, according to Philip Hall (see [46], p. 137, which introduces the **beautiful** term "bountiful"), some local system consisting of countably infinite simple subgroups (see [42], p. 18, [43], Theorem 4.4, [44], Theorem 2.5, and [45] O.H. Kegel: "Remarks on uncountable simple groups", in: Proceedings of Ischia Group Theory 2016, *Int. J. Group Theory* **7** (2018)). Thus simplicity is definitely a countably recognisable group theoretic property (see [2]). Periodic linear groups are locally finite (see [43], Theorem 1.L.1) and satisfy the Strong Sylow Theorem for *every* Prime p (see [54] and [44], 1.7). Simple periodic linear groups are countable (see [43], Theorem 1.L.2).

If G is a countably infinite locally finite simple group, then there will exist a nested local system $\{R_{\mathrm{n}} \mid \mathrm{n} \in \mathbb{N}\}$ for G of finite subgroups such that for each $\mathrm{n} \in \mathbb{N}$ the group R_{n} is perfect and there exists some maximal normal subgroup $M_{\mathrm{n}+1}$ of $R_{\mathrm{n}+1}$ satisfying $M_{\mathrm{n}+1} \cap R_{\mathrm{n}} = \langle 1\rangle$, so that $R_{\mathrm{n}+1} / M_{\mathrm{n}+1}$ is simple and $R_{\mathrm{n}} \precsim R_{\mathrm{n}+1} / M_{\mathrm{n}+1}$ (see **Chapter 3**); such a nested local system is called ***Kegel cover*** (or ***★-sequence***) ***for G***. We define the ***family*** $\mathcal{T}$ ***of types of known finite simple groups*** by using some assumed well-known symbols: $\mathcal{T} := \{\text{abelian}_p, \underline{\mathrm{A}}^{\mathrm{n}}, \mathrm{A} = \mathrm{PSL}_{\mathrm{n}}, \mathrm{B} = \mathrm{P\Omega}_{\text{odd n}}, \mathrm{C} = \mathrm{PSp}_{\mathrm{n}}, \mathrm{D} = \mathrm{P\Omega}^{+}_{\text{odd n}}, {}^2\mathrm{A} = \mathrm{PSU}_{\mathrm{n}}, {}^2\mathrm{D} = \mathrm{P\Omega}^{-}_{\text{even n}}, \mathrm{E}_6, \mathrm{E}_7, \mathrm{E}_8, \mathrm{F}_4, \mathrm{G}_2, {}^2\mathrm{B}_2, {}^3\mathrm{D}_4, {}^2\mathrm{E}_6, {}^2\mathrm{F}_4, {}^2\mathrm{G}, \text{sporadic} \bigstar\}$. If G is a known finite simple group of type $\Xi \in \mathcal{T}$, we call p resp. n resp. 2 resp. 4 resp. 6 resp. 7 resp. 8 resp. ★ (:= the order of G) the ***rank*** $\mathbf{r}(G)$ ***of G***. A countably infinite locally finite simple group is called to be ***of type*** $\Xi \in \mathcal{T}$, if it just has a Kegel cover $\Sigma = \{(R_{\mathrm{k}}, M_{\mathrm{k}}) \mid \mathrm{k} \in \mathbb{N}\}$ in such a way that infinitely many of the $R_{\mathrm{k}+1} / M_{\mathrm{k}+1}$'s belong to Ξ (wherefore we can replace Σ by these infinitely many $R_{\mathrm{k}+1}$'s), and is called to be ***of alternating type*** if it is of type $\underline{\mathrm{A}}^{\mathrm{n}}$. Note that such a group could a priori be of several types but we may placidly assume by the well-known pigeonhole principle (see https://en.wikipedia.org/wiki/Pigeonhole_principle) that in fact **all** $\boldsymbol{R_{\mathrm{k}+1} / M_{\mathrm{k}+1}}$**'s** belong to **the same** of the 19 known families.

The following figure (© 2012 by Iván Andrus [see https://irandrus.files.wordpress.com/2012/06/periodic-table-of-groups.pdf and https://irandrus.wordpress.com/2012/06/17/the-periodic-table-of-finite-simple-groups/]) depicts the **19 families** of known finite simple groups in a **beautiful** arrangement called "Periodic Table":



If the locally finite group G satisfies the Strong Sylow Theorem for the Prime p it contains a p-uniqueness subgroup (see [15], Theorem 3.9, and [44], Theorem 1.5, in conjunction with [15], Proposition 2.3). Thus, if for a countably infinite locally finite simple group G with Kegel cover $\{(R_{\mathrm{k}}, M_{\mathrm{k}}) \mid \mathrm{k} \in \mathbb{N}\}$ and p-uniqueness subgroup P we could prove that **the ranks of the** $\boldsymbol{R_{\mathrm{k}+1}/M_{\mathrm{k}+1}}$**'s are bounded in terms of** $\boldsymbol{P}$, then we could very straightforwardly deduce Prof. Otto H. Kegel's **Theorem 2.7** (see [44]: "*For the locally finite simple group G the following are equivalent:* ***(i)*** *Every countable simple subgroup of G contains a p-uniqueness subgroup;* ***(ii)*** *G satisfies the Strong Sylow Theorem for the Prime p;* ***(iii)*** *G is linear.*") and his central **Theorem 3.4** (see [44]: "*If* $\{F_i\}_{i\in\mathrm{N}}$ *is a smooth simple straight* *split sequence of finite p-perfect subgroups of the locally finite group G, then the countably infinite group* $U = \langle F_i\,;\ i \in \mathbb{N} \rangle$ *has* $2^{\aleph_0}$ *maximal p-subgroups.*").

Note – To study crucial configurations, Kegel developed in [44] the quite excogitated concept of "*(smooth simple straight) split sequences of finite p-perfect subgroups with their associated ascending sequences of subgroups*" which is related to his equally very fine concept of the "*Sylow-separated (ascending) sequences of p-subgroups with associated sequences of Sylow p-subgroups*" he had developed already nearly ten years earlier in "O.H. KEGEL: 'Chain conditions and Sylow's theorem in locally finite groups', in: Symposia Matematica, Volume **XVII**, Convegno sui Gruppi Infiniti, Istituto Nazionale di Alta Matematica (iNδAM) 'Francesco Severi', Roma, 11-14 Dicembre 1973, *Academic Press*, London-New York (1976), 251-259. ISBN 978-0-12612-217-6."

So, in his four workshop lectures on Sylow theory in locally finite groups at the famed and such eminent Singapore Group Theory Conference of June 1987, Kegel stated as a theorem and proved "by inspection" what is actually a **conjecture** (see [44], **Theorem 2.4**): "Let P be a p-uniqueness subgroup of the finite simple group S which belongs to one of the seven rank-unbounded families. Then the rank of S is bounded in terms of P." In this paper we prove the conjecture for the case that the finite simple group S is some $\underline{A}^n$ ($n \in \mathbb{N}$) thereby getting Kegel's Theorem 2.7 and Theorem 3.4 for the case that the countably infinite locally finite simple group is **of alternating type**.

If Σ is a local system of countably infinite simple subgroups of the simple locally finite group G with (G countable $\Rightarrow \Sigma = \{G\}$) and P^U for each Σ-group U is a p-uniqueness subgroup of U, which exists if G satisfies the Strong Sylow Theorem for the Prime p (see [15]), and $\{(R_k, M_k) \mid k \in \mathbb{N}\}$ is for each $U \in \Sigma$ a Kegel cover for U of alternating type, then for each $U \in \Sigma$ will exist a $k = k(U) \in \mathbb{N}$ with $P^U \subseteq R_k{}^U$, whence $P^U \bullet M_m{}^U / M_m{}^U \approx P^U / P^U \cap M_m{}^U$ is a p-uniqueness subgroup of $R_m{}^U / M_m{}^U$ for all $m \geq k(U)$, and we could deduce easily from the following **Theorem 1** that the ranks $\{r\,(R_m{}^U / M_m{}^U) \mid m \geq k(U)\}$ are bounded by $f_p(|P^U|)$ for all $U \in \Sigma$, so that all Σ-groups would be linear (see [47]) and so G would be linear, too, and so also countable.

Theorem 1 (see [14])**.** *Let* $n \in \mathbb{N}$ *and let p be a prime such that* $p \leq n$. *Let P be a finite p-group acting on* $\underline{A}^n$.
Let $\mathbf{g}_p(|P|)$ *be the number of conjugacy classes of subgroups of P and let* k *be the number of P-invariant Sylow p-subgroups of* $\underline{A}^n$. *Then* $g_p(|P|) \leq 2^{|P| - 1}$.

a) *If isomorphic subgroups of P are conjugate and* $b := \log_p|P|$ *(so that* $|P| =: p^b$*), then* $g_p(|P|) \leq$
$p^{((b-2)^4 + 2\,(b-2)^3 + (b-2)^2)/4 - ((b-2)^2 + b - 2)/2 - 90} + (|P| - 1)/(p - 1) + 25.$

b) *Let* $m := k + p + 1$. *Then* $n \leq m \bullet |P| \bullet g_p(|P|) - 1$.
If $k = 1$, *then* $n \leq f_p(|P|) := (p + 2) \bullet |P| \bullet 2^{|P| - 1} - 1$.

Having proved **Theorem 1** we state **a way 1)** and **a way 2)** how to optimise **Theorem 1**, make a couple of remarks and suggestions on **Planning future research** and state three conjectures.

A periodic linear group is locally finite (see [43], Theorem 1.L.1) and satisfies the Strong Sylow Theorem for *every* Prime p (see [54] and [44], 1.7). As the next undertaking we are proving **Conjecture 2** of **Page 8** regarding the **General Linear Groups** over locally finite fields (see [14]):

Theorem 2. *Let* $n \in \mathbb{N}$ *and let p be a prime.*
Let $\mathcal{F}$ *be a locally finite (commutative) field.*

a) *If* $\mathcal{F}$ *has characteristic p and* $a_p = a_p(GL(n, \mathcal{F}))$ *then* $n \leq (p + 2) \bullet p^{a_p} - 1$.

b) *If* $\mathcal{F}$ *has characteristic* $\neq p$ *and* $a_p = a_p(GL(n, \mathcal{F}))$ *then* $n \leq (p + 2) \bullet p^{2a_p} - 1$.

Afterwards we are breaking down **Theorem 2** to the **Special Linear Groups** over locally finite fields:

Theorem 3. *Let* $n \in \mathbb{N}$ *and let p be a prime.*
Let $\mathcal{F}$ *be a locally finite (commutative) field.*

a) *If* $\mathcal{F}$ *has characteristic p and* $a_p = a_p(SL(n, \mathcal{F}))$ *then* $n \leq (p + 2) \bullet p^{a_p} - 1$.

b) *If* $\mathcal{F}$ *has characteristic* $\neq p$ *and* $a_p = a_p(SL(n, \mathcal{F}))$ *then* $n \leq (p + 2) \bullet p^{2a_p} - 1$.

We continue with breaking down **Theorem 3** to the **Projective Special Linear (PSL) Groups** over locally finite fields:

Theorem 4. *Let* $n \in \mathbb{N}$ *and let p be a prime.*
Let $\mathcal{F}$ *be a locally finite (commutative) field and let P be a minimal p-unique subgroup of* $PSL(n, \mathcal{F})$.

a) *If* $\mathcal{F}$ *has characteristic p and* $a_p = a_p(PSL(n, \mathcal{F}))$ *then* $n \leq f_p(|P|) := (p + 2) \bullet p^{a_p} - 1$.

b) *If* $\mathcal{F}$ *has characteristic* $\neq p$ *and* $a_p = a_p(PSL(n, \mathcal{F}))$ *then* $n \leq f_p(|P|) := (p + 2) \bullet p^{2a_p} - 1$.

An infinite simple locally finite group G always has a local system Σ consisting of countably infinite simple locally finite subgroups and each Σ-group U has a Kegel cover $\{(R_k{}^U, M_k{}^U) \mid k \in \mathbb{N}\}$ (see **Page 3**). If all the factors $R_k{}^U / M_k{}^U$ of the Kegel covers for all Σ-groups U are of type Ξ = "$A = PSL_n$", then G is called to be ***of projective special linear type***. If G satisfies the Strong Sylow Theorem for the Prime p, then each Σ-group U has a p-uniqueness subgroup P^U (see [15]).

For each $U \in \Sigma$ exists some $k = k\,(U) \in \mathbb{N}$ with $P^U \subseteq R_k{}^U$, whence $P^U \bullet M_m{}^U / M_m{}^U \approx P^U / P^U \cap M_m{}^U$ is a great p-uniqueness subgroup of $R_m{}^U / M_m{}^U$ for all $m \geq k(U)$. If G is of projective special linear type, it follows from **Theorem 4** that the ranks $\{r\,(R_m / M_m) \mid m \geq k(U)\}$ will be bounded by $f_p(|P^U|)$ for all the Σ-groups U, which hence are linear and so G will be linear and therefore also countable (see [47]). Summarising we can see the consequences of the Strong Sylow Theorem for the Prime p according to **Theorem 1** and **Theorem 4**:

Theorem 5. *Let G be a simple locally finite group* ***of alternating type*** *or* ***of projective special linear type*** *satisfying the Strong Sylow Theorem for the even one Prime p. Then G is linear and countable.* □

Having proved **Theorems 1, 2, 3** and **4** we make a couple of further remarks and suggestions on **Planning future research** and announce **very beautifully The Second Trilogy**.

2. Proof of Theorem 1

Proof. We begin with some general remarks. For any group G we denote by $\mathrm{U}(G)$ the set of all its subgroups and for every $U \in \mathrm{U}(G)$ by $\mathrm{R}(G,U) := \{Ux \mid x \in G\}$ the set of all right cosets of U in G. Then G operates by multiplication from the right for every $\mathrm{U} \in \mathrm{U}(G)$ transitively on $\mathrm{R}(G,U)$ with $\mathrm{Cor}_G U := \{U^x \mid x \in P\}$ as the kernel. If G acts (from the right) on a set Ω, so that Ω is a *G-set*, and $\alpha \in \Omega$ is any point, then $G_\alpha := \{x \in G \mid \alpha^x = \alpha\} \subseteq G$ is the *stabiliser of* α. Another G-set Ψ is said to be *G-isomorphic* to Ω in case there exists a bijection $\xi : \Omega \to \Psi$ such that $\xi(\alpha^x) = \xi(\alpha)^x$ for all the $\alpha \in \Omega$ and $x \in G$. The ***classification of transitive G-sets*** reads as follows (see [50], Chapter 6): *Every transitive G-set* $\Omega \neq \emptyset$ *is G-isomorphic to* $\mathrm{R}(G,G_\alpha)$ *for all* $\alpha \in \Omega$*, and for any two* $U, V \in \mathrm{U}(G)$ *the two sets* $\mathrm{R}(G,U)$ *and* $\mathrm{R}(G,V)$ *are G-isomorphic if and only if U and V are conjugate in G.* Hence for the action of P we will have a bijection between the class $\mathcal{J}(\boldsymbol{P})$ of P-isomorphism types of transitive P-sets and the set of all conjugacy classes (in P) of subgroups of P, and so $|\mathcal{J}(P)| = \mathrm{g}_p(|P|)$. Thus for every P-set Ω the class $\mathcal{J}(\boldsymbol{P},\boldsymbol{\Omega})$ of P-isomorphism types of P-orbits on Ω has at most $\mathrm{g}_p(|P|)$ elements and since every subgroup of P is a subset containing 1, we can summarising deduce that $|\mathcal{J}(P,\Omega)| \leq \mathrm{g}_p(|P|) \leq |\mathrm{U}(P)| \leq 2^{|P|-1}$.

Consider now some $\mathrm{n} \in \mathbb{N}$ and a p-subgroup P of $\underline{\mathrm{S}}^{\mathrm{n}}$ for some prime p which is contained in exactly $\mathrm{k} \in \mathbb{N}$ Sylow p-subgroups of $\underline{\mathrm{S}}^{\mathrm{n}}$. Then $\mathbf{n \leq (k + \boldsymbol{p} + 1) \bullet |\boldsymbol{P}| \bullet g_{\boldsymbol{p}}(|\boldsymbol{P}|) - 1}$ ★.

RATIONALE – Suppose $\mathrm{n} \geq (\mathrm{k} + p + 1) \bullet |P| \bullet \mathrm{g}_p(|P|)$. Then $G := P$ is a finite group which operates on the set $\Omega := \{1, 2, \ldots, \mathrm{n}\}$ with $\mathbf{|\Omega| \geq (k + \boldsymbol{p} + 1) \bullet |\boldsymbol{G}| \bullet g_{\boldsymbol{p}}(|\boldsymbol{G}|)}$. We show that the number of G-isomorphic G-orbits on Ω must be at least $\mathrm{k} + p + 1$. The group G partitions Ω into r orbits $\Psi_1, \Psi_2, \ldots, \Psi_r$. Since the orbit lengths $|\Psi_1|, |\Psi_2|, \ldots, |\Psi_r|$ divide the group order $|G|$, it follows that $|\Omega| = \Sigma\{|\Psi_i| \mid 1 \leq \mathrm{i} \leq \mathrm{r}\} \leq \mathrm{r} \bullet |G|$; hence if $\mathrm{r} \geq (\mathrm{k} + p + 1) \bullet |\mathcal{J}(G,\Omega)|$, then by the pigeonhole principle there will be **at least k + *p* + 1 many *G*-isomorphic *G*-orbits on Ω**. ■ Therefore P has at least $\mathrm{k} + p + 1$ many P-isomorphic P-orbits on Ω. This implies, as we show below, that there are at least $|\mathrm{Syl}_p \underline{\mathrm{S}}^{\mathrm{k}+p+1}|$ many P-invariant Sylow p-subgroups of $\underline{\mathrm{S}}^{\Omega}$. Since also $|\mathrm{Syl}_p \underline{\mathrm{S}}^{\mathrm{n}}| \geq \mathrm{n} - 2$ for $\mathrm{n} \in \mathbb{N}$ (see **Lemma 1** below), $|\mathrm{Syl}_p \underline{\mathrm{S}}^{\mathrm{k}+p+1}| \geq (\mathrm{k} + p + 1) - 2 = (\mathrm{k} + 1) + (p - 2) \geq \mathrm{k} + 1$ follows. ■

a) For all $0 \leq \mathrm{k} \leq \mathrm{b}$ let $\mathbf{j_k}$ denote *the number of conjugacy classes of subgroups of index* $\mathrm{p}^{\mathrm{b-k}}$ *in P*. Then clearly $\mathrm{j}_0 = 1$, $\mathrm{j}_1 = 1$ and $\mathrm{j}_b = 1$, but also $\mathrm{j}_{\mathrm{b}-1} \leq (|P| - 1) / (p - 1)$: the Frattini subgroup $\Phi(P)$ of P has an elementary abelian factor group of rank $\leq \mathrm{b}$, since a maximal subgroup of a finite p-group is normal of index p, whence $\mathrm{j}_{\mathrm{b}-1}$ represents the number of the one-dimensional subspaces of the $\mathrm{GF}(p)$-vectorspace $P / \Phi(P)$. **Now suppose that the isomorphic subgroups of *P* are conjugate.** Then $\mathrm{j}_2 = 2$, since there are two isomorphism types of groups of order p^2, the cyclic group and the elementary abelian group, $\mathrm{j}_3 = 5$, since there are five isomorphism types of groups of order p^3, and $\mathrm{j}_4 \leq 15$, since there are 14 isomorphism types of groups of order 2^4 and 15 isomorphism types of groups of order p^4 for $p \neq 2$ (see [23]). It follows that $\mathrm{j}_0 + \mathrm{j}_1 + \mathrm{j}_2 + \mathrm{j}_3 + \mathrm{j}_4 + \mathrm{j}_{\mathrm{b}-1} + \mathrm{j}_\mathrm{b} \leq (|P| - 1) / (p - 1) + 25$. Considering a chief series for a group of order p^{k} ($\mathrm{k} \in \mathbb{N}$) one can determine the number of maximal possible multiplication tables of groups of order p^{k} and thus obtain rather simply the estimate $\mathrm{i}_p\mathrm{k} \leq p^{(\mathrm{k}^3 - \mathrm{k})/6}$ for the number $\mathbf{i_{\boldsymbol{p}}k}$ of isomorphism types of groups of order p^{k} (see [28], Theorem 3.1). Since we can calculate $\Sigma\{(\mathrm{k}^3 - \mathrm{k}) / 6 \mid 5 \leq \mathrm{k} \leq \mathrm{b} - 2\} =$ (see under https://www.numberempire.com/seriescalculator.php) $((\mathrm{b}-2)^4+2(\mathrm{b}-2)^3+(\mathrm{b}-2)^2)/4-((\mathrm{b}-2)^2+\mathrm{b}-2)/2-90$, it now follows the rather cool inequality $\Sigma\{\mathrm{j}_\mathrm{k} \mid 5 \leq \mathrm{k} \leq \mathrm{b} - 2\} \leq p^{((\mathrm{b}-2)^4 + 2(\mathrm{b}-2)^3 + (\mathrm{b}-2)^2)/4 - ((\mathrm{b}-2)^2 + \mathrm{b} - 2)/2 - 90}$. Summarising we get $\mathrm{g}_p(|P|) \leq$

$$p^{((\mathrm{b}-2)^4 + 2(\mathrm{b}-2)^3 + (\mathrm{b}-2)^2)/4 - ((\mathrm{b}-2)^2 + \mathrm{b} - 2)/2 - 90} + (|P| - 1)\,)/(p - 1) + 25. \quad \square$$

b) We may assume that the group P operates faithfully on $\underline{\mathrm{A}}^{\mathrm{n}}$ which is a normal subgroup of index 2 in $\underline{\mathrm{S}}^{\mathrm{n}}$. If $\mathrm{n} \leq 5$ or $\mathrm{n} \geq 7$ the automorphism group $\mathrm{Aut}(\underline{\mathrm{A}}^{\mathrm{n}})$ of $\underline{\mathrm{A}}^{\mathrm{n}}$ is known to be isomorphic to the group of inner automorphisms of $\underline{\mathrm{S}}^{\mathrm{n}}$ which is isomorphic to $\underline{\mathrm{S}}^{\mathrm{n}}$ (see [51], Satz 1.9). $\mathrm{Aut}(\underline{\mathrm{A}}^6)$ is the semidirect product of a group $\underline{\mathrm{C}}_2$ of order 2 with $\underline{\mathrm{S}}^6$ (see [32]). Thus P is (isomorphic to) a p-subgroup of $\underline{\mathrm{S}}^{\mathrm{n}}$ or of $\underline{\mathrm{C}}_2 \bullet \underline{\mathrm{S}}^6$ which normalises k Sylow p-subgroups of $\underline{\mathrm{A}}^{\mathrm{n}}$. Every Sylow 2-subgroup of $\underline{\mathrm{A}}^{\mathrm{n}}$ lies in only one Sylow 2-subgroup of $\underline{\mathrm{S}}^{\mathrm{n}}$, since $\underline{\mathrm{A}}^{\mathrm{n}}$ contains for $\mathrm{n} \geq 5$ just as many Sylow 2-subgroups as has $\underline{\mathrm{S}}^{\mathrm{n}}$, and a Sylow 2-subgroup of $\underline{\mathrm{A}}^{\mathrm{n}}$ is its own normaliser in $\underline{\mathrm{S}}^{\mathrm{n}}$ (see [59]). **Thus the *p*-subgroup *P* of** $\underline{\mathrm{S}}^{\mathrm{n}}$ (or of $\underline{\mathrm{C}}_2 \bullet \underline{\mathrm{S}}^6$, if $p = 2$) **lies in exactly k many Sylow *p*-subgroups of** $\underline{\mathrm{S}}^{\mathrm{n}}$**.** (If $\mathrm{k} \geq 2$ then even $\mathrm{k} \geq p + 1$ because the number of all Sylow p-subgroups of the semidirect product $P \bullet \underline{\mathrm{S}}^{\mathrm{n}}$ is congruent to 1 modulo p.) We digress now and permit a short **memory parenthesis**: When G is a finite group, P a p-subgroup of G and $S \in \mathrm{Syl}_p G$, then the operation of P by conjugation on $\mathrm{C}(G,S) := \{S^x \mid x \in G\}$ has at least one fixed point, that is $(\exists\, x \in G)(P^x \subseteq S)$, and for $P \in \mathrm{Syl}_p G$ exactly one, that is, $|\mathrm{Syl}_p G| = |G{:}\underline{\mathrm{N}}_G S| = |\mathrm{C}(G,S)| \equiv 1 \pmod{p}$; hence G satisfies the Strong Sylow Theorem for the Prime p, that is, every $U \in \mathrm{U}(G)$ conjugates *transitively* on $\mathrm{Syl}_p U$, and thus we have the **Frattini argument** for G (and p), that is, if N is a normal

★ If P is a p-uniqueness subgroup of $\underline{\mathrm{S}}^{\mathrm{n}}$, then $\mathrm{n} \leq (p + 2) \bullet |P| \bullet 2^{|P|-1}$. If the countable group $\underline{\mathrm{S}}^{(\mathrm{N})}$ would satisfy the Sylow Theorem for the prime p, then by Theorem 3.4 of [15] it would even satisfy the Strong Sylow p-Theorem, and thus it would by Theorem 3.9 of [15] contain a p-uniqueness subgroup P. Now $\underline{\mathrm{S}}^{(\mathrm{N})}$ has a nested local system $\{U_\mathrm{n} \mid \mathrm{n} \in \mathbb{N}\}$ with $U_\mathrm{n} \approx \underline{\mathrm{S}}^{\mathrm{n}}$ for all $\mathrm{n} \in \mathbb{N}$. Since P is finite, there exists an $\mathrm{m} \in \mathbb{N}$ with $P \subseteq U_\mathrm{m}$. Then P would be singular in U_n for all $\mathrm{n} \in \mathbb{N}$ with $\mathrm{n} \geq \mathrm{m}$ and we get the rubbish $\mathrm{n} \leq (p + 2) \bullet |P| \bullet 2^{|P|-1}$ for all $\mathrm{n} \geq \mathrm{m}$. Similarly, every finite p-subgroup of $\underline{\mathrm{S}}^{(\mathrm{N})}$ is contained in at least $\aleph_0$ Sylow p-subgroups of $\underline{\mathrm{S}}^{(\mathrm{N})}$ since $\underline{\mathrm{S}}^{(\mathrm{N})}$ does not satisfy the Sylow p-Theorem.

subgroup of G and $P \in \mathrm{Syl}_p N$, then $\underline{\mathrm{N}}_G P$ covers G/N, that is, $G = N \bullet \underline{\mathrm{N}}_G P$. ■ We now put $\mathrm{m} := \mathrm{k} + p + 1$ and are supposing $\mathrm{n} \geq \mathrm{m} \bullet |P| \bullet \mathrm{g}_p(|P|)$. Then according to the remarks made at the outset, when arguing for the RATIONALE, there will be at least m many P-isomorphic P-orbits on Ω.

In order to proceed we need a lower bound for $|\mathrm{Syl}_p \underline{S}^{\Omega}|$:

Lemma 1. *Let p be a prime and let* $\mathrm{n} \in \mathbb{N}$.
α) *If* $p > \mathrm{n}$, *then* $|\mathrm{Syl}_p \underline{S}^{\mathrm{n}}| = 1$.
β) *If* $((p, \mathrm{n}) = (p, 1), (2, 2), (2, 3), (3, 3), (2, 4), (3, 4))$, *then* $|\mathrm{Syl}_p \underline{S}^{\mathrm{n}}| = (\mathrm{n} = 1, \mathrm{n} - 1 = 1, \mathrm{n} = 3, \mathrm{n} - 2 = 1, \mathrm{n} - 1 = 3, \mathrm{n} = 4)$.
γ) *If* $p \leq \mathrm{n}$ *and* $\mathrm{n} \geq 5$, *then* $|\mathrm{Syl}_p \underline{S}^{\mathrm{n}}| \geq \mathrm{n}$.
δ) *If* $p \leq \mathrm{n}$, *then* $|\mathrm{Syl}_p \underline{S}^{\mathrm{n}}| \geq \mathrm{n} - 2$.

RATIONALE – **α)** $\underline{S}^{\mathrm{n}}$ is a p'-group for $p > \mathrm{n}$ since $\mathrm{n}! = |\underline{S}^{\mathrm{n}}|$.

β) $|\mathrm{Syl}_p \underline{S}^1| = 1$ for all p because of $\underline{S}^1 = \langle 1 \rangle$ and $\mathrm{Syl}_2 \underline{S}^2 = \{\underline{S}^2\}$ because of $|\underline{S}^2| = 2$. Since $|\underline{S}^3| = 2 \bullet 3$ and $|\underline{S}^4| = 2^3 \bullet 3$ we have $|\mathrm{Syl}_2 \underline{S}^3|, |\mathrm{Syl}_2 \underline{S}^4| \in \{1, 3\}$ and $|\mathrm{Syl}_3 \underline{S}^3|, |\mathrm{Syl}_3 \underline{S}^4| \in \{1, 4\}$ because of $|\mathrm{Syl}_p G| \equiv 1 \pmod p$. From $|\underline{S}^3 : \underline{A}^3| = 2$ follows that $\underline{A}^3$ is a normal subgroup of $\underline{S}^3$ whence $|\mathrm{Syl}_2 \underline{S}^3| = 3$ because $\underline{S}^3$ is non-abelian. We know that $\underline{S}^4$ has exactly two non-trivial proper normal subgroups, namely the Klein four-group and the $\underline{A}^4$, and therefore has neither a normal Sylow 2-subgroup nor a normal Sylow 3-subgroup, whence $|\mathrm{Syl}_2 \underline{S}^4| = 3$ and $|\mathrm{Syl}_2 \underline{S}^4| = 4$.

γ) We show first: (i) *If* $\mathrm{n} \geq 5$ *then* $\underline{S}^{\mathrm{n}}$ *contains just one non-trivial normal subgroup, namely the* $\underline{A}^{\mathrm{n}}$. RATIONALE – Let (if possible) $\langle 1 \rangle \neq N \subseteq \underline{S}^{\mathrm{n}}$ be normal in $\underline{S}^{\mathrm{n}}$ with $N \neq \underline{A}^{\mathrm{n}}$; then $N \cap \underline{A}^{\mathrm{n}} = \langle 1 \rangle$ since $\underline{A}^{\mathrm{n}}$ is simple, hence $|N| \bullet |\underline{A}^{\mathrm{n}}| = |N \bullet \underline{A}^{\mathrm{n}}|$ divides $|\underline{S}^{\mathrm{n}}|$, and so $|N| = 2$; as a 2-transitive group $\underline{S}^{\mathrm{n}}$ is primitive whence N operates trivially or transitively which is clearly impossible for $|N| = 2$. ■ Since $|\mathrm{Syl}_p \underline{S}^{\mathrm{n}}| \equiv 1 \pmod p$ it follows from (i), $|\underline{S}^{\mathrm{n}} : \underline{A}^{\mathrm{n}}| = 2$, and $|\underline{S}^{\mathrm{n}}| = n!$ that $|\mathrm{Syl}_p \underline{S}^{\mathrm{n}}| \geq 3$. Since $|\mathrm{Syl}_p \underline{S}^{\mathrm{n}}| = |\underline{S}^{\mathrm{n}} : \underline{\mathrm{N}}_{\underline{S}^{\mathrm{n}}} S|$ for $S \in \mathrm{Syl}_p \underline{S}^{\mathrm{n}}$ it now suffices to show the following: (ii) *Let* $\mathrm{n} \geq 5$ *and* $3 \leq \mathrm{k} \leq \mathrm{n} - 1$. *Then* $\underline{S}^{\mathrm{n}}$ *has not any subgroup of index* k *at all.* RATIONALE – Suppose there exists a subgroup U of $\underline{S}^{\mathrm{n}}$ with $|\underline{S}^{\mathrm{n}} : U| = k$. The transitive operation of $\underline{S}^{\mathrm{n}}$ on $\mathrm{R}(\underline{S}^{\mathrm{n}}, U)$ via right multiplication gives rise to some homomorphism $\varphi: \underline{S}^{\mathrm{n}} \rightarrow \underline{S}^{\mathrm{k}}$. Because of $\mathrm{k} \leq \mathrm{n} - 1$ we have $\langle 1 \rangle \neq \text{kernel } \varphi \subseteq U$ and since $\mathrm{k} \geq 3$ we have $|\text{kernel } \varphi| < |\underline{A}^{\mathrm{n}}|$. By (i) this is impossible. ■

δ) follows from point **β)** and point **γ)**. ■

We return to the group P operating on Ω with at least m many P-isomorphic P-orbits. Application of **Lemma 1** gives $|\mathrm{Syl}_p \underline{S}^{\mathrm{m}}| \geq \mathrm{m} - 2 = (\mathrm{k} + p + 1) - 2 = (\mathrm{k} + 1) + (p - 2) \geq \mathrm{k} + 1$. Therefore it remains to prove that if $\mathrm{Syl}_p(\underline{S}^{\Omega}, P) := \{S \in \mathrm{Syl}_p \underline{S}^{\Omega} \mid S \text{ is } P\text{-invariant}\}$ and there are at least m many P-isomorphic P-orbits on Ω, then $|\mathrm{Syl}_p(\underline{S}^{\Omega}, P)| \geq |\mathrm{Syl}_p \underline{S}^{\mathrm{m}}|$. For each $1 \leq \mathrm{i} \leq \mathrm{r}$ let V_{i} be the point stabiliser of $\Omega \setminus \Psi_{\mathrm{i}}$ in $\underline{S}^{\Omega}$; hence $V_{\mathrm{i}} \approx \underline{S}^{\Psi_{\mathrm{i}}}$. Then we truly have $P \subseteq D := \langle V_{\mathrm{i}} \mid 1 \leq \mathrm{i} \leq \mathrm{r} \rangle$. Let B be the set of permutations on Ω which interconvert in entire blocks the P-isomorphic Ψ_{i}'s and let the remaining Ψ_{i}'s pointwise fixed. Then $B \subseteq \underline{S}^{\Omega}$ with $B \approx \underline{S}^{\mathrm{m}}$ and wirh $B \cap D = \langle 1 \rangle$. Because B interchanges only P-isomorphic P-orbits, it is normalised by D. Hence $K := \langle B, D \rangle$ is the semidirect product $B \bullet D$, and so D is normal in K with $K/D \approx B$. Now let $Q \in \mathrm{Syl}_p K$ with $P \subseteq Q$. Since D is normal in K and the finite group K satisfies the Sylow Theorem for the Prime p, we have $P \subseteq D \cap Q \in \mathrm{Syl}_p D$ and by the Frattini argument (see above) $\underline{\mathrm{N}}_K(D \cap Q)/\underline{\mathrm{N}}_D(D \cap Q) \approx K/D$. It follows that $|\mathrm{Syl}_p(S,P)| \geq |\{R \in \mathrm{Syl}_p K \mid P \subseteq R\}| \geq |\{R \in \mathrm{Syl}_p K \mid D \cap R = D \cap Q\}| = |\mathrm{Syl}_p(\underline{\mathrm{N}}_K(D \cap Q)/\underline{\mathrm{N}}_D(D \cap Q))| = \mathrm{Syl}_p(K/D)| = |\mathrm{Syl}_p \mathrm{S}|$. ☐ **Q.E.D. (Quod Erat Demonstrandum)**

Corollary. *Let G be a simple locally finite group of alternating type with Kegel covers* $\{(R_{\mathrm{k}}{}^U, M_{\mathrm{k}}{}^U) \mid U \in \Sigma, \mathrm{k} \in \mathbb{N}\}$ *as described on* **Page 4** *satisfying the Strong Sylow Theorem for the Prime p and let* P^U *for each* Σ*-group U be a p-uniqueness subgroup of U* (see [15]). *Then we have the inequality* $\mathrm{r}(R_{\mathrm{m}}{}^U / M_{\mathrm{m}}{}^U) \leq \mathrm{f}_p(|P^U|) := (p + 2) \bullet |P^U| \bullet 2^{|P| - 1} - 1$ *for all* $\mathrm{m} \geq \mathrm{k}(U)$ *and for all* $U \in \Sigma$, *and G is a linear group and a countable group.*

Proof. Our **Theorem 1**, [47], and [43], Theorem 1.L.2. ☐

We keep the overall context of the **Corollary** and let G be a locally finite group satisfying the Strong Sylow Theorem for the Prime p and let P be a p-uniqueness subgroup of G. In view of **Theorem 1** it is of rather considerable interest *whether resp. when isomorphic (finite) subgroups of P are conjugate*. Therefore let Q and Q^* be isomorphic subgroups of P and also let r be their common index in P. The left regular representation $\lambda_g: h \rightarrow gh$ for all $h \in P$ $(g \in P)$ and the right regular representation $\rho_g: h \mapsto hg^{-1}$ for all the $h \in P$ $(g \in P)$ both embed P into the symmetric group $\underline{S}^P$ on P. Now a famous result by Philip Hall (see [26], Lemma 1) establishes that either regular representation maps isomorphic subgroups onto conjugate subgroups: let $x \mapsto x^*$ $(x \in Q)$ be an isomorphism of Q onto Q^*; let $\{y_1, y_2, \ldots, y_r\}$ be a complete set of left coset representatives of Q in P and $\{y_1^*, y_2^*, \ldots, y_r^*\}$ be such a set of left coset representatives of Q^* in P; the mapping $\xi: y_i x \mapsto y_i^* x^*$ $(x \in Q \mid \mathrm{i} = 1, 2, \ldots, \mathrm{r})$ is a permutation of P, so that $\xi \in \underline{S}^P$; if $\mathrm{t} \in Q$ and if ρ is any regular representation of P, we then have $y_i^* x^* \xi^{-1} \rho(t) \xi = y_i x \rho(t) \xi = y_i(xt)\xi = y_i^*(xt)^* = y_i^* x^* t^*$, since $*$ is a homomorphism, so that $\xi^{-1}\rho(t)\xi = \rho(t^*)$; hence ξ transforms $\rho(Q)$ into $\rho(Q^*)$. ■ However, we should in fact need conjugacy not only in $\underline{S}^P$ but in P itself. This is an **open problem**. Note that if this would be solved without restrictions then in a (locally finite) p-group, the simplest locally finite group satisfying the Strong Sylow Theorem for the Prime p … , isomorphic finite subgroups would be conjugate, a rather striking property. Hence the solution will probably need restrictions.

Sìsifo by **Tiziano** - Oil on Canvas, 1548 - 1549
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Les dieux avaient condamné **Sisyphe** à rouler **sans cesse** un rocher jusqu'au sommet d'une montagne d'ou la pierre retombait par son propre poids. … Il faut imaginer **Sisyphe** heureux.

Die Götter hatten **Sisyphos** dazu verurteilt, einen Felsblock **unablässig** den Berg hinaufzuwälzen, von dessen Gipfel der Stein kraft seines eigenen Gewichts wieder hinunterrollte. … Wir müssen uns **Sisyphos** als einen glücklichen Menschen vorstellen.

The gods had condemned **Sisyphus** to **ceaselessly** rolling a rock to the top of a mountain, whence the stone would fall back of its own weight. … One must imagine **Sisyphus** happy.

Gli dei avevano condannato **Sìsifo** a far rotolare **senza posa** un macigno sino alla cima di una montagna, dalla quale la pietra ricadeva per azione del suo stesso peso. … Bisogna immaginare **Sìsifo** felice.

3. About Kegel covers

Let G be a locally finite group. A set of pairs $\{(H_i,M_i) \mid i \in \mathfrak{I}\}$ is called a *Kegel cover for G* if, for all i in $\mathfrak{I}$, H_i is a finite subgroup of G and M_i is a maximal normal subgroup of H_i, and if for each finite subgroup H of G there exists an $i \in \mathfrak{I}$ with $H \subseteq H_i$ and $H \cap M_i = \langle 1 \rangle$; the groups H_i/M_i $(i \in \mathfrak{I})$ are called *the factors of the Kegel cover* (see [49]). In [14] we introduced the concept of the ⋆-sequence for G. Let G be a countably infinite simple locally finite group. We then are defining a *⋆-sequence for G* as a set of pairs $\{(R_n,M_n) \mid n \in \mathbb{N}\}$ where $\{R_n \mid n \in \mathbb{N}\}$ is a nested local system for G and for all $n \in \mathbb{N}$ the group R_n is perfect, $R_n \neq R_{n+1}$ and M_{n+1} is some maximal normal subgroup of R_{n+1} with $M_{n+1} \cap R_n = \langle 1 \rangle$, that is, *the factor* R_n/M_n, which is a non-abelian finite simple group, is (isomorphic to) a proper section of the non-abelian simple group R_{n+1}/M_{n+1}, and therefore $\{R_n \mid n \in \mathbb{N}\}$ is totally ordered by involvement. Such a group G has a nice ⋆-sequence (see [14], and [42], p. 20, and [43], Lemma 4.5, which tough Kegel calls an "approximation principle", and [44], Theorem 2.6, and the origin as the rather smart concept of a so-called "$\mathfrak{a}$-*Folge*" introduced in [41], Definition 2.1 and Hilfssatz 2.2 [but see the **Remark** on p. 116 of [43] regarding Hilfssatz 2.2]; see also [49], Lemma 3.4). Brian Hartley refers to a ⋆-sequence, where the R_n's need not to be perfect, as a *Kegel sequence* (see [27], Definition 2.2). He moreover states rather enlightening that the nomenclature of covers and sequences is more recent and even dedicates the entire Chapter 2 of [21] to Kegel sequences and to Kegel covers.

Proposition 1. *Let G be a countably infinite simple locally finite group. If* $\{(R_n,M_n) \mid n \in \mathbb{N}\}$ *is some ⋆-sequence for G, then it is a Kegel cover for G.*

Proof. If H is a finite subgroup of G, there exists an R_k of the nested local system $\{R_n \mid n \in \mathbb{N}\}$ for G with $H \subseteq R_k \subseteq R_{k+1}$ $(k \in \mathbb{N})$ and then $H \cap M_{k+1} = \langle 1 \rangle$. □

U. MEIERFRANKENFELD (see [49]) classified (with the help of S. DELCROIX) simple locally finite groups G according to their Kegel covers (see [10]): **finitary** (there exists a field $\mathcal{F}$ and a faithful $\mathcal{F}G$-module V such that $V(g - 1) = [V, g]$ is finite dimensional for all $g \in G$) (see [25]), **of 1-type** (where each Kegel cover has an alternating factor), **of *p*-type** for a unique prime p (where each Kegel cover has some classical group in characteristic p as some factor), and **of ∞-type** (which have a Kegel cover all of whose factors allow embedding of every finite group). He proved earlier pretty much surprisingly that a non-finitary such group is <u>either</u> ***of alternating type*** (hence of **1-type** or **of ∞-type**) <u>or</u> (**of *p*-type** <u>and</u> ***of projective special linear type***) (see [48] and the marvellous preprint at https://users.math.msu.edu/users/meierfra/Preprints/Nflfsg/nflfsg.html).

It had been inadvertently suggested that the results of this paper were a consequence of [25] and alternatively of "J.I. HALL – B. HARTLEY: 'A group theoretical characterization of simple, locally finite, finitary linear groups', Arch. Math. (Basel) **60**, Issue **2** (February1993), 108-114." since the groups considered were thought to be finitary. However, this thinking is not true. The joint paper by Hall and Hartley does not refer to Kegel covers and especially do both papers not refer to the *p*-uniqueness subgroups (Flemisch) resp. to the singular *p*-subgroups (Kegel). It had then been wrongly argued that the Kegel kernels M_i were not considered which in the given situation were claimed to be $\langle 1 \rangle$ for all $i \in \mathbb{N}$. But the Kegel factors R_i/M_i were considered and not only the kernels M_i nor were the kernels all $\langle 1 \rangle$. By rather vivid imagination it had then been quite wrongly concluded that the groups considered would become finitary linear locally finite simple groups which were classified by [25] (which is true). Even if all this would be true, [25] does not prove the results of this paper nor all the more so the paper by Hall and Hartley.

4. Planning future research – Part 1

We have seen that a simple locally finite group G can be covered by countable simple locally finite groups U each of which possesses a ⋆-sequence $\{(R^U, M^U) \mid n \in \mathbb{N}\}$ and so is in some sense a limit of the (approximating) sequences R^U/M^U $(n \in \mathbb{N})$ of finite non-abelian simple groups. If all the factors of the Kegel covers for all U, that is, all the R^U/M^U's, belong to the same family Ξ of the infinite families $\{\underline{A}^n$, $A = PSL_n$, $B = P\Omega_{odd\,n}$, $C = PSp_n$, $D = P\Omega^+_{odd\,n}$, ${}^2A = PSU_n$, ${}^2D = P\Omega^-_{even\,n}$, E_6, E_7, E_8, F_4, G_2, ${}^2B_2$, ${}^3D_4$, ${}^2E_6$, ${}^2F_4$, ${}^2G_2\}$, we call G to be **of type** Ξ. We propose to prove Kegel's conjecture for all these types seratim, that is, one type after another in the given succession, and started already with the first type Ξ = "$\underline{A}^n$ ".

Our **Theorem 1** could be optimised in two ways:

1) Extend it from type $\underline{A}^n$ **step-by-step** to further types Ξ with an appropriate (similar) function f_p, that is, the rank $r(G)$ of a finite group G of type Ξ is bounded by $f_p(|P|)$ whenever P is a given *p*-uniqueness subgroup of G.

2) Determine for the type $\underline{A}^n$ and **peu à peu** for further types Ξ all the minimal *p*-unique subgroups, that is, the *p*-uniqueness subgroups of the non-abelian simple groups of type $\underline{A}^n$ and of type Ξ, which are minimal with respect to order (see [15]).

Note that whilst **way 2)** is of great interest for all types and also for sporadic ⋆ (whereas it is trivial for abelian_p), **way 1)** is not of interest for the families $\{E_6, E_7, E_8, F_4, G_2, {}^2B_2, {}^3D_4, {}^2E_6, {}^2F_4, {}^2G_2\}$ because these families have a fixed rank (label) and so are infinite only through the underlying field.

We recall from [15] the **Theorem 4.1** and its consequences:

Theorem 4.1 (see [14]). *Let G be a locally finite group satisfying the Strong Sylow Theorem for the Prime p.*

a) *Each Sylow p-subgroup of G contains at least one (w.r.t. order) minimal p-unique subgroup of G.*

b) *Every two (w.r.t. order) minimal p-unique subgroups of G have the same order.* □

Let G be a **beautiful** locally finite group satisfying the Strong Sylow *p*-Theorem and let $S \in Syl_pG$. According to our Theorem 4.1 a), S contains some (w.r.t. S) minimal *p*-unique subgroup F. We define $a_p = a_p(G) \in \mathbb{N}_0$ by $|F| =: p^{a_p}$, that is, we let a_p be the composition length of F. Then according to our Theorem 4.1 b) this definition is independent of the special choice of the Sylow p-subgroup S of G, whereby in consequence a_p is a (numerical) Sylow p-invariant of G. We call a_p the ***p-uniqueness of G***.

Then the optimising **way 1)** can be stated as follows:

Conjecture 1. *Let* $\mathcal{T} := \{$ abelian$_p$, A = PSL$_n$, B = P$\Omega_{\text{odd } n}$, C = PSp$_n$, D = P$\Omega^+_{\text{odd } n}$, ^{2}A = PSU$_n$, ^{2}D = P$\Omega^-_{\text{even } n}$, E$_6$, E$_7$, E$_8$, F$_4$, G$_2$, ^{2}B$_2$, ^{3}D$_4$, ^{2}E$_6$, ^{2}F$_4$, ^{2}G$_2$, sporadic $\star\}$ *be the family of types of known finite simple groups and let G be a finite simple group of type* $\Xi \in \mathcal{T}$. *Then the rank* $r(G)$ *of G is bounded in terms of the p-uniqueness* $a_p(G)$.

Brian Hartley (15 May 1939 until 8 October 1994) in his Mathematical Review of [44] (see MR981832 [MR 90c #20037 (March 1990)]) stated the following: "*If the simple locally finite group G satisfies the Strong Sylow Theorem for the (even one) Prime p, then G is linear. This depends on the classification of finite simple groups and an* assertion *about singular p-subgroups of classical groups. Another proof of this result has since been given by the reviewer (not yet published).*" The assertion mentioned is Kegel's conjecture (see [44], Theorem 2.4). However, due to the so very tragic death of Prof. Hartley in 1994, aged 55 (see [14]), this certainly *highly insight gaining* proof was never prepared for publication. Hartley wrote 1994 a very eye-opening paper on simple locally finite groups (see [27]) which, however, did not refer to Kegel's work [44] and not even included it in its list of 56 references. The paper could appear only posthumously which most likely is the reason for the full ignorantness of Kegel's paper. Hartley's paper was meticulously completed and carefully prepared for publication by Richard E. Phillips (3 December 1936 until 9 November 1999). We consider it much rewarding, even after 30 years, to inspect Hartley's estate *In Search of* **not** *Lost Notes* (see Marcel Proust [10 July 1871 until 18 November 1922]: "À la recherche du temps perdu" / "In Search of Lost Time" / "Auf der Suche nach der verlorenen Zeit" / "Alla ricerca del tempo perduto" / "En busca del tiempo perdido" / "Em busca do tempo perdido").

Now as a very first step towards solving **Conjecture 1** for the second type Ξ = "A = PSL$_n$", we state another conjecture w.r.t. the general linear group over locally finite fields (see [14]):

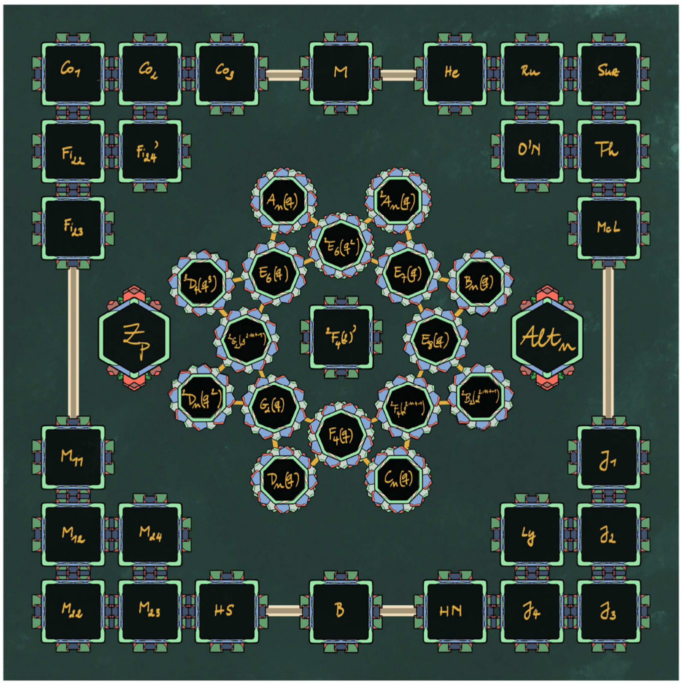
Conjecture 2. *Let* $n \in \mathbb{N}$ *and let p be a prime.*
Let $\mathcal{F}$ *be a locally finite (commutative) field.*

a) If $\mathcal{F}$ has characteristic p and $a_p = a_p(GL(n,\mathcal{F}))$ then $n \leq (p+2) \bullet p^{a_p} - 1$.

b) If $\mathcal{F}$ has characteristic $\neq p$ and $a_p = a_p(GL(n,\mathcal{F}))$ then $n \leq (p+2) \bullet p^{2a_p} - 1$.

In the entire paper we do not refer to the classification of finite simple groups (see [23], [61] and **Page 3**) but prefer to talk about the 19 families of "known" finite simple groups. Our efforts are directed towards knowing much better their Sylow subgroups. We hope to find useful insights about the Sylow subgroups of classical groups in the ATLAS of Finite Groups [8] and in the comprehensive literature about them.

The classification of the finite simple groups
(**13 sporadic groups** above **18 infinite families** around another "sporadic" group and further **13 sporadic groups** below)



(© 2022 by Mathsies – Own work, CC BY-SA 4.0, https://upload.wikimedia.org/wikipedia/commons/archive/a/a9/20220111205053%21Classification_of_the_finite_simple_groups.jpg, 28 December 2021, at 15:08 (UTC); [61])

Kegel's lectures [44] present the very basics of Sylow theory in locally finite groups, give an overview of the prodigious work of Brian Hartley and Andrew Rae on the Sylow theory in locally finite and p-soluble groups, and reveal in great detail the normal structure for groups satisfying the Strong Sylow Theorem for the Prime p in the general case (for $p \geq 5$). Chapters 2 and 4 of [12] give a rather good overview as well but alas without appreciating Kegel's very insight gaining work properly and avoiding all its **beautiful** details. We cite from the Preface of [12]: "*The condition that all the maximal p-subgroups of a locally finite group are conjugate is a very strong condition indeed; the structure of those groups has been obtained in the locally p-soluble case by Hartley and in the general case by Kegel. The Hartley-Kegel theorem is quite involved so I decided to simply state the results obtained.*" Also, simple groups are not in the scope of [12] and therefore [12] must be supplemented by [27].

Although this paper is about simple groups we cannot help to close with a brief attention to p-soluble groups since it is the joint study of the (locally) simple and the (locally) p-soluble groups which directs reliably the Sylow theory in (locally) finite groups.

In Chapter "2 Some length type inequalities" of his rather remarkable contribution [53], Alexandre Turull (see https://people.clas.ufl.edu/turull/) states a conjecture of Thomas R. Berger (which dates back to John G. Thompson in the 1970's):

Conjecture 2.3 (see [3])**.** *Let p be a prime. There exists a linear function f_p such that if G is a finite p-soluble group with p-length $\lambda_p(G)$ and P is a subgroup of G of order p^k ($k \in \mathbb{N}$) contained in precisely one Sylow p-subgroup then $\lambda_p(G) \leq f_p(k)$.*

Having studied the very most of the hereof related literature published by Brian Hartley, by Andrew Rae, and by Thomas R. Berger, we profess to have happily discovered such a linear function, namely our nice a_p. Therefore we can state Thomas R. Berger's conjecture more precisely (and best possible) as our

Conjecture 3. *Let p be a prime. Let G be a p-soluble finite group, $\lambda_p(G)$ be its p-length, and $a_p(G)$ be its p-uniqueness. Then $\lambda_p(G) \leq a_p(G) + 1$.*

It is much expected that the cases $p \geq 5$, $p = 3$ and $p = 2$ must be treated fairly separately and also that $p = 3$ and $p = 2$ will require fairly special methods as already indicated by the available literature. *A. Turull* gives in Section 2 of [53] a quite concise overview of **the classical Hall-Higman theory** created by *P. Hall*, *G. Higman*, *A.H.M. Hoare*, *T.R. Berger*, *F. Gross* and *E.G. Bryukhanova*, which introduces for finite p-soluble groups (best possible) inequalities between their p-length $\boldsymbol{\lambda_p}$ and the order $\boldsymbol{p^{b_p}}$ of a Sylow p-subgroup, its nilpotency class $\mathbf{c}_p$, its solubility length $\mathbf{d}_p$, its exponent $\boldsymbol{p^{e_p}}$, or the rank $\mathbf{r}_p$ of a maximal elementary abelian subgroup. Our aim is **to extend the Hall-Higman theory** to the **very beautiful** p-uniqueness $\boldsymbol{p^{a_p}}$

of a Sylow p-subgroup, an Herculean endeavour. It is in this context that *A. Turull* cites *T.R. Berger*'s conjecture and presents some results up to 1994 with regard to partly solving it but they are very gey far from being complete, in particular concerning the basic results of *B. Hartley* and *A. Rae*. But on the other hand *T.R. Berger* presents in [3] a, as he says, reasonably complete list of references up to 1979, including 15 of his own contributions, where eleven are related to p-length problems, and discusses his method of proof for p-length and other length type problems in a considerably detailed fashion.

5. Proof of Theorem 2

Proof. We begin with some general remarks (see [9], Chapter II, and [11], Chapters 1 and 2). Let $\mathcal{F}$ be a field, $V \neq \{0\}$ be a vector space over $\mathcal{F}$ with its automorphism group $GL(V)$, and let G be a group. V is called a *G-module over $\mathcal{F}$* and *G operates on V*, if a homomorphism of groups $\varphi: G \to GL(V)$ is declared. φ is then called a *linear representation of G on V over $\mathcal{F}$*. Every permutation representation of G on a set $\Omega \neq \emptyset$ now induces a G-module $\boldsymbol{V(\Omega)}$, called the *permutation module of (G,Ω) over $\mathcal{F}$*.

Therefore to every subgroup U of G belongs the G-module $\boldsymbol{V}(R(\boldsymbol{G,U}))$ (see **Page 5** and **Page 6**) with respect to **(w.r.t.)** multiplication from the right. A subspace W of V is called *G-invariant* or a *G-submodule*, if for all $x \in G$ we have $x^{\varphi}(W) \subseteq W$, that is, φ induces an operation of G on W. We say that G operates on V *irreducible*, if V contains exactly two G-submodules (namely $\{0\}$ and V), and *completely reducible*, if to every G-submodule W of V there exists a G-submodule X of V with $V = W \oplus X$, equivalently, if V is decomposable into a direct sum of minimal G-submodules. G operates on V *non-modular*, if $\operatorname{char}\mathcal{F} = 0$ or $\operatorname{char}\mathcal{F} \neq 0$ and G contains no $\operatorname{char}\mathcal{F}$-elements $\neq 1$; otherwise G operates *modular* on V. Now let V_1 be another G-module over $\mathcal{F}$ on which G operates via φ_1. Then V is called *G-isomorphic to V_1*, if there exists an isomorphism of vector spaces $\psi: V \to V_1$ such that the such **beautiful** diagram shown

commutes for all $x \in G$:

$$\begin{array}{ccc} V & \xrightarrow{x^{\varphi}} & V \\ \downarrow \psi & & \downarrow \psi \\ V_1 & \xrightarrow{x^{\varphi_1}} & V_1 \end{array}$$

Every irreducible G-module is G-isomorphic to a factor module of $V(R(G,<1>))$: the class $\mathcal{J}(G,\mathcal{F})$ of all G-isomorphism types of irreducible G-modules is a duly set of (finite-dimensional) vector spaces over $\mathcal{F}$ all of which have their dimension $\leq |G|$.

We now start the proof of **Theorem 2** by quoting two fairly well-known facts about non-modular linear representations (see [22], Chapter 3, Theorem 3.1, and [11], Theorem 10.8, for point **a)**, as well as [9], Theorem 27.22 with Remark 27.25, for point **b)**). We denote for point **b)** by $h(G)$ the *class number of G*, that is, the number $|\{x^G \mid x \in G\}|$ of conjugacy classes of G.

Proposition 2. *Let G be a finite group.*

a) (Heinrich Maschke, 1898) *Every non-trivial non-modular finite-dimensional G-module is completely reducible.*

b) *Let $\mathcal{F}$ be a field with $(\operatorname{char}\mathcal{F},|G|) = 1$. Then there are at most $h(G)$ many G-isomorphism types of irreducible G-modules over $\mathcal{F}$.* ■

We use **Proposition 2 b)** straight away to prove the following:

Proposition 3.

a) *There exists a function $\gamma: \mathbb{N} \to \mathbb{N}$ with the following property: If G is a finite group, $\mathcal{F}$ a field with $(\operatorname{char}\mathcal{F},|G|) = 1$ and $\mathcal{J}(G,\mathcal{F})$ the class of all G-isomorphism types of irreducible G-modules over $\mathcal{F}$, then $\mathcal{J}(G,\mathcal{F})$ is a genuine set with $|\mathcal{J}(G,\mathcal{F})| \leq \gamma(|G|)$.*

b) *Let G be a finite group, $\mathcal{F}$ a field with $(\operatorname{char}\mathcal{F},|G|) = 1$ and V a finite-dimensional G-module over $\mathcal{F}$. Let $\mathcal{J}(G,V)$ be the set of G-isomorphism types of irreducible G-submodules of V. Then $|\mathcal{J}(G,V)| \leq \gamma(|G|)$, where γ is the function from point* **a)**.

Rationale – **a)** We define $\gamma: \mathbb{N} \to \mathbb{N}$ simply by $\gamma(n) := n$. Then $h(G) \leq \gamma(|G|)$. Since by **Proposition 2 b)** there is an injective mapping of $\mathcal{J}(G,\mathcal{F})$ into $\{x^G \mid x \in G\}$ the assertion follows.

b) follows from point **a)**. ■

Up next we use **Proposition 2 a)** and **Proposition 3 b)** to prove

Proposition 4. *Let G be a finite group and $k \in \mathbb{N}$. Let V be a finite-dimensional non-modular G-module with $\dim(V) \geq |G| \cdot \gamma(|G|) \cdot k$, where γ is the function from* **Proposition 3 a)**. *Then there exist at least k many G-isomorphic irreducible G-submodules of V.*

RATIONALE – By **Proposition 2 a)** and a trivial induction on $\dim(V)$ there are m irreducible G-submodules U_i of V with $V = \bigoplus\{U_i \mid 1 \le i \le m\}$. If $0 \ne v_i \in U_i$ then $\langle v_i^x \mid x \in G\rangle$ is a G-submodule of V and thus $\dim(U_i) \le |G|$ $(1 \le i \le m)$. Now let $\mathcal{J}(G,V)$ be the set of G-isomorphism types of irreducible G-submodules of V. If $m \ge k \bullet |\mathcal{J}(G,V)|$ then by the pigeonhole principle there will be just k many G-isomorphic irreducible G-submodules of V. Because of $\dim(V) = \Sigma\{\dim(U_i) \mid 1 \le i \le m\}$ and $m \le \{\dim(U_i) \mid 1 \le i \le m\} \le m \bullet |G|$, we have $\dim(V)/|G| \le m \le \dim(V)$. Thus there are at least k many G-isomorphic irreducible G-submodules of V if only $\dim(V) \ge k \bullet |G| \bullet |\mathcal{J}(G,V)|$. So the assertion follows from **Proposition 3 b)**. ■

Recall that a finite group G operates *modular* on a G-module V if $G = \langle 1\rangle$ or G operates not non-modular on V. Therefore a finite p-group for the prime p operates modular on every vector space over the field $\mathcal{F}$ if and only if $\mathrm{char}\mathcal{F} = p$. We prove next two elementary facts (see [22], Chapter 2, Lemmata 6.2 and 6.3):

Proposition 5.

a) *Let G be a group, N be a normal subgroup of G and V be a G-module. Then* $C_V(N) := \{v \in V \mid v^x = v \text{ for all } x \in \mathbb{N}\}$ *is a G-submodule of V.*

b) *Let P be a finite p-group for the prime p and let V be a non-trivial modular P-module. Then* $C_V(P) \ne \{0\}$.

RATIONALE – **a)** Put $U := C_V(N)$. Then U is a subspace of V. Let $u \in U$ and $x \in G$. For $y \in N$ we have also $y^{x^{-1}} \in N$, since N is normal in G, and so $u^{y^{x^{-1}}} = u$. Therefore we have $u^{x^y} = u^x$ for all $y \in N$, that is, $u^x \in U$.

b) We carry out an induction on $|P|$. For $P = \langle 1\rangle$ we have $C_V(P) = V$ and nothing to prove. Let $|P| \ge p$ and M be a maximal subgroup of P. Then M is normal in P with $|P{:}M| = p$. Put $U := C_V(M)$. Then U is by point **a)** a P-submodule of V and by the induction hypothesis we have $U \ne \{0\}$. Let $y \in P \setminus M$ and $y' \in \mathrm{GL}(U)$ be the restriction of y to U. Then $y^p \in M$ and $\langle M,y\rangle = M \bullet \langle y\rangle = P$ and so $C_V(P) = U \cap C_V(y) = C_U(y')$. It remains for us to prove that $C_U(y') \ne \{0\}$. Let $\mathcal{F}$ be the field over which V is being a vector space and let $\mu(X)$ be the minimal polynomial of y' over $\mathcal{F}$. Then $\mu(X)$ divides the polynomial $X^p - 1$ of $\mathcal{F}[X]$, since y' has order 1 or p in $\mathrm{GL}(U)$, and $p = \mathrm{char}\mathcal{F} = \mathrm{char}\mathcal{F}[X]$ as well. Therefore $X^p - 1 = (X - 1)^p$. Hence $\mu(\kappa) = 0$ for $\kappa \in \mathcal{F}$ if and only if $\kappa = 1$, that is, 1 is the only eigenvalue of y' with $C_U(y') \ne \{0\}$ as its eigenspace. ■

We are in the very happy position to prove an intriguing toughening of **Proposition 5 b)** which is quite definitely not an elementary insight (see as well [43], p. 41, where, however, this core assertion is not proved properly and even only for an elementary abelian P, and [32], Chapter VIII, Lemma 10.17, where, however, only the very special example is considered that V is an abelian p-group and P has order p):

Proposition 6. *Let P be a finite p-group for the prime p, $\mathcal{F}$ be a field of characteristic p and V be a finite-dimensional P-module over $\mathcal{F}$. Then* $\dim(C_V(P)) \ge \dim(V)/|P|$.

RATIONALE – We refine the proof of **Proposition 5 b)** and carry out an induction on $|P|$. For $P = \langle 1\rangle$ we have nothing to prove. Let $|P| \ge p$ and M *be* a maximal subgroup of P. Then M is normal in P with $|P{:}M| = p$. Put $U := C_V(M)$. Let $y \in P\setminus M$ and $y' \in \mathrm{GL}(U)$ be the restriction of y to U. Then $y^p \in M$ and $\langle M,y\rangle = M \bullet \langle y\rangle = P$ and so $C_V(P) = U \cap C_V(y) = C_U(y')$. From **Proposition 5 a)** and the induction hypothesis follows that U is a P-submodule of V with $\dim(U)/p \ge \dim(V)/(|M| \bullet p) = \dim(V)/|P|$. It thus remains for us to prove the following:

(◇) $p \bullet \dim(C_U(y')) \ge \dim(U)$.

Put $n := \dim(U)$ and $d := \dim(C_U(y'))$. Let $\mu(X)$ be the minimal polynomial of y' over $\mathcal{F}$. Then $\mu(X)$ will divide the polynomial $X^p - 1$ of $\mathcal{F}[X]$, since y' has order 1 or p in $\mathrm{GL}(U)$. Because of $p = \mathrm{char}\mathcal{F} = \mathrm{char}\mathcal{F}[X]$ we have $X^p - 1 = (X - 1)^p$. Hence 1 is the unique eigenvalue of y' with $C_U(y')$ as related eigenspace. In particular $d \ge 1$. Let $\chi(X) := \det(y' - X\,\mathrm{id}_U)$ be the characteristic polynomial of y' over F. Then $\chi(X)$ has degree n and is divided by $\mu(X)$. In particular $U = \mathrm{kernel}(y' - \mathrm{id}_U)^n$ whence y' is unipotent. RECALL – Let G be a subgroup of $\mathrm{GL}(n,\mathcal{F})$. We call $x \in G$ *unipotent* if $(x - 1)^n = 0$, that is, if all eigenvalues of x are 1, and call G *unipotent* if each element of G is unipotent. Every unipotent subgroup of $\mathrm{GL}(n,\mathcal{F})$ is some conjugate of a subgroup of $\mathrm{UT}(n,\mathcal{F})$, the group of upper triangular matrices. If $\mathrm{char}\mathcal{F}$ is a prime p, then the unipotent elements of $\mathrm{GL}(n,\mathcal{F})$ are precisely the p-elements and $\mathrm{UT}(n,\mathcal{F})$ is a Sylow p-subgroup of $\mathrm{GL}(n,\mathcal{F})$. ■ Thus there is an $\mathcal{F}$-basis of U such that the matrix of U w.r.t. this $\mathcal{F}$-basis will lie in $\mathrm{UT}(n,\mathcal{F})$. This matrix can be decomposed in Jordan normal form as follows. Let $\tau := y' - \mathrm{id}_U$ and for each $m \in \mathbb{N}_0$ let $C_m := \mathrm{kernel}(\tau^m)$. The C_m's are $\mathcal{F}$-subspaces of U with $\{0\} = C_0 \subseteq C_1 \subseteq C_m \subseteq C_{m+1} \subseteq \ldots$. We have $C_1 = C_U(y')$ and $C_n = U$. Let $k \in \mathbb{N}$ be minimal w.r.t. $C_k = U$ and put $r := \dim(U/C_{k-1})$.

Then $u \mapsto u^{\tau^{k-1}}$ $(u \in U)$ induces an isomorphism of U/C_{k-1} onto an $\mathcal{F}$-subspace of C_1. It follows that $\mathbf{r \le d}$. We have $\tau^p = (y' - \mathrm{id}_U)^p = y'^p - \mathrm{id}_U = 0$ since y' has order 1 or p in $\mathrm{GL}(U)$ and $p = \mathrm{char}\mathcal{F} = \mathrm{char}\mathcal{F}[X]$ whence $\mathrm{image}(\tau^m) = \{0\}$ for all $m \in \mathbb{N}$ with $m \ge p$. It follows that $\mathbf{k \le \boldsymbol{p}}$. Now for each $u \in U\setminus C_{k-1}$ we define $W_u := \langle u, u^\tau, \ldots, u^{\tau^{k-1}}\rangle$ which will be a y'-invariant $\mathcal{F}$-subspace of U with $\dim(W_u) = k$. The k x k-matrix $A(y')$ of y' restricted to W_u w.r.t. $\{u^{\tau^{k-1}}, u^{\tau^{k-2}}, \ldots, u^\tau, u\}$ has the shape shown:

$$A(y') = \begin{pmatrix} 1 & 1 & . & . & 0 \\ . & 1 & 1 & . & . \\ . & . & . & . & . \\ . & . & . & . & 1 \\ 0 & . & . & . & 1 \end{pmatrix}.$$

There exist $u_1, u_2, \ldots, u_r \in U \setminus C_{k-1}$ with $U = \bigoplus\{W_{u_i} \mid 1 \le i \le r\}$. Then the n x n-matrix $A(y')$ of y' w.r.t. the basis $\{u_1^{\tau^{k-1}}, \ldots, u_1, u_2^{\tau^{k-1}}, \ldots, u_2, \ldots, u_r^{\tau^{k-1}}, \ldots, u_r\}$ of U has the above shape as well. It now follows that $\mathbf{n = k \bullet r}$ and hence $\mathbf{n \le \boldsymbol{p} \bullet d}$ by the previous inequalities. This is (◇) to be proved. ■

The inequality of **Proposition 6** is best-possible since for every prime p there exists a faithful finite-dimensional $\underline{C}_p$-module V over $\mathrm{GF}(p)$ with $\dim(C_V(\underline{C}_p)) = \dim(V)/p$: let q be a prime such that p divides q - 1; the $\underline{S}^4$ is a semidirect product of the $\underline{S}^3 = \underline{C}_2 \bullet \underline{C}_3$ with the four group $\underline{C}_2 \times \underline{C}_2$; this operation can be generalised to an operation of $\underline{C}_p \bullet \underline{C}_q$ on $V := \underline{C}_p^{(q-1)}$; if $p = 2$ one gets for every impair prime q the "generalised $\underline{S}^4$" of order $2^q q$; the (classical) Hall-Higman theory can now be used to show $\dim(C_V(\underline{C}_p)) = \dim(V)/p$ (see **Page 8** and **Page 9**).

We next apply some of the **beautiful new ideas** of the proof of **Theorem 1 b)** and of **Proposition 5** to prove for GL(V) a similar statement as for $\underline{S}^\Omega$ where $\Omega := \{1, 2, \ldots, n\}$:

Proposition 7. *Let V be a finite-dimensional vector space over the locally finite (commutative) field $\mathcal{F}$. The finite p-group P for the prime p shall operate on V.*
a) *Let char* $\mathcal{F} = p$ *and let* $V = \oplus\{U_i \mid 1 \le i \le m\}$ *be a direct decomposition of V into irreducible P-submodules according to* **Proposition 2 a)**. *Let k be the number of P-isomorphic* U_i*'s. Then there exist at least* $|\mathrm{Syl}_p\underline{S}^k|$ *many P-invariant Sylow p-subgroups of* GL(V).
b) *Let char* $\mathcal{F} = p$ *and* $k := \dim(C_V(P))$. *Then there are at least* $|\mathrm{Syl}_p\underline{S}^k|$ *many P-invariant Sylow p-subgroups of* GL(V).

RATIONALE – We may suppose without loss of generality **(w.l.o.g.)** that P is a subgroup of GL(V) and operates by conjugation on GL(V). If $S \in \mathrm{Syl}_p\mathrm{GL}(V)$ then $\mathrm{Syl}_p\underline{N}_{\mathrm{GL}(V)}(S) = \{S\}$ and hence P normalises S if and only if $P \subseteq S$. Therefore we have to prove the following:
$(\star)\ |\{S \in \mathrm{Syl}_p\mathrm{GL}(V) \mid P \subseteq S\}| \ge |\mathrm{Syl}_p\underline{S}^k|$.

a) We certainly may suppose w.l.o.g. that the first k of the U_i's are P-isomorphic. Let $H_i \subseteq \mathrm{GL}(V)$ be the point stabiliser of $\oplus\{U_j \mid 1 \le i \le m, j \ne i\}$; then $H_i \approx \mathrm{GL}(U_i)$ $(1 \le i \le m)$. Put $D := \langle H_i \mid 1 \le i \le m \rangle = \prod^0\{H_i \mid 1 \le i \le m\} \subseteq \mathrm{GL}(V)$. Then $P \subseteq D$. Let B be the set of automorphisms of V which interconvert in entire blocks the P-isomorphic U_i's and let the remaining U_i's pointwise fixed. Then $B \subseteq \mathrm{GL}(V)$ with $B \approx \underline{S}^k$ and $B \cap D = \langle 1 \rangle$. Since B interchanges only P-isomorphic U_i's, it is normalised by D. Hence $K := \langle B, D \rangle$ is the semidirect product $B \bullet D$, and hence D is normal in K with $K/D \approx B$. Now let $Q \in \mathrm{Syl}_p K$ with $P \subseteq Q$. Since D is normal in K, we have $P \subseteq D \cap Q \in \mathrm{Syl}_p D$ and by the Frattini argument, which follows from the (Strong) Sylow p-Theorem for the finite K, $\underline{N}_K(D \cap Q)/\underline{N}_D(D \cap Q) \approx K/D$. It follows that $|\{S \in \mathrm{Syl}_p\mathrm{GL}(V) \mid P \subseteq S\}| \ge |\{S \in \mathrm{Syl}_p\mathrm{GL}(V) \mid D \cap S = D \cap Q\}| \ge |\mathrm{Syl}_p(\underline{N}_K(D \cap Q)/\underline{N}_D(D \cap Q))| \ge |\mathrm{Syl}_p(K/D)| \ge |\mathrm{Syl}_p S|$. This is the inequality of $(\star)$ to be proved.

b) $C := C_V(P)$ is by **Proposition 5** a non-trivial P-submodule of V. Let $D := C_{\mathrm{GL}(V)}(C)$. Then $P \subseteq D$. Now let C_1 be a (not necessarily P-invariant) complement to C in V, that is, $V = C \oplus C_1$. Let B be the point stabiliser of C_1. Then $\mathrm{GL}(V) \approx B$ and $B \cap D = \langle 1 \rangle$. For all $b \in B$, $d \in D$ and $c \in C$ we have $c^{(d^b)} = (c^{b^{-1}})^{db} = (c^{b^{-1}})^b = c$. Hence B normalises D and so $K := \langle B, D \rangle = B \bullet D$ whence D is normal in K with $K/D \approx B$. Since $k = \dim(C)$, the group B contains a subgroup which is isomorphic to $\underline{S}^k$, namely the group of all permutation matrices of rank k over $\mathcal{F}$ (see [11], § 1.3). Therefore $|\mathrm{Syl}_p B| \ge |\mathrm{Syl}_p\underline{S}^k|$. Now $(\star)$ follows verbatim as in point **a)**. ■

Next we are notably very happy to be able to use the foregoing **Propositions 4** & **6** & **7** together with **Lemma 1** of **Page 9** to prove a core **Lemma** from which **Theorem 2** follows immediately. □

Lemma 2. *Let* $n \in \mathbb{N}$ *and let p be a prime. Let* $\mathcal{F}$ *be a locally finite (commutative) field and let P be a finite p-subgroup of* GL(n, $\mathcal{F}$) *which is contained in exactly* $k \in \mathbb{N}$ *Sylow p-subgroups of* GL(n, $\mathcal{F}$).
a) *If* $\mathcal{F}$ *has characteristic* $\ne p$ *then* $n \le (k + p + 1) \bullet |P|^2 - 1$.
b) *If* $\mathcal{F}$ *has characteristic p then* $n \le (k + p + 1) \bullet |P| - 1$.
c) *If P is a p-uniqueness subgroup of* GL(n, $\mathcal{F}$) *then* $n \le f_p(|P|) := (p + 2) \bullet |P|^2 - 1$.

RATIONALE – **a)** Let $n \ge (k + p + 1) \bullet |P|^2$. By **Proposition 4** and since $\gamma(|P|) = |P|$ by the proof of **Proposition 3 a)**, the space $\mathcal{F}^n$ then has at least $k + p + 1$ many irreducible P-isomorphic P-submodules. Thus P lies by **Proposition 7 a)** in at least $|\mathrm{Syl}_p\underline{S}^{k+p+1}|$ many Sylow p-subgroups of GL(n, $\mathcal{F}$). From **Lemma 1 δ)** of **Page 6** we can now conclude $|\mathrm{Syl}_p\underline{S}^{k+p+1}| \ge k + p + 1 - 2 \ge k + 1$.

b) Let $n \ge (k + p + 1) \bullet |P|$. We then have $\dim(C_{\mathcal{F}^n}(P)) \ge k + p + 1$ by **Proposition 6**. Therefore P lies by **Proposition 7 b)** in at least $|\mathrm{Syl}_p S|$ many Sylow p-subgroups of GL(n, $\mathcal{F}$). Now follows from **Lemma 1 δ)** of **Page 6** that $|\mathrm{Syl}_p S| \ge k + p + 1 - 2 \ge k + 1$.

c) follows from point **a)** and point **b)**. ■

6. Proof of Theorem 3

A subgroup of GL(n, $\mathcal{F}$) is locally finite if and only if $\mathcal{F}$ is locally finite, that is, if every finitely generated subfield of $\mathcal{F}$ is finite. $\mathcal{F}$ is locally finite if and only if it is isomorphic to a subfield of $\overline{\mathcal{F}_p}$, the algebraic closure of the nice prime field GF(p) = $\mathcal{F}_p$, for some prime p, and hence is countable. Since $\mathcal{F}_{p^m} \subseteq \mathcal{F}_{p^n}$ if and only if m divides n (m, n $\in \mathbb{N}$), we consider the chain $\mathcal{F}_p \subseteq \mathcal{F}_{p^{n!}} \subseteq \mathcal{F}_{p^{(n+1)!}} \subseteq \mathcal{F}_{p^{(n+2)!}} \subseteq \ldots$ of algebraic extensions, where $\mathcal{F}_{p^{(n+1)!}}$ is obtained by just adjoining some root α of an irreducible polynomial of degree n + 1 over $\mathcal{F}_{p^{n!}}$, that is, $\mathcal{F}_{p^{(n+1)!}} = \mathcal{F}_{p^{n!}}(\alpha)$ (n $\in \mathbb{N}$). Then $\overline{\mathcal{F}_p} = \bigcup\{\mathcal{F}_{p^{n!}} \mid n \in \mathbb{N}\}$ = [since $\mathcal{F}_{p^n} \subseteq \mathcal{F}_{p^{n!}}$] $\bigcup\{\mathcal{F}_{p^n} \mid n \in \mathbb{N}\}$ (see [4], Section 2.2). All the subfields of $\mathcal{F}_p$ (see [4], Section 2.3) correspond to all the locally finite fields in characteristic p.

Let $\mathcal{F}^* := \mathcal{F} \setminus \{0\}$ be the multiplicative group of $\mathcal{F}$ and let SL(n, $\mathcal{F}$) := {A $\in$ GL(n, $\mathcal{F}$) | det(A) = 1}.

Proof. GL(n, $\mathcal{F}$) = SL(n, $\mathcal{F}$) $\bullet$ $\mathcal{F}^*$ is the semidirect product of SL(n, $\mathcal{F}$) with $\mathcal{F}^*$ and the unique Sylow p-subgroup S_p of $\mathcal{F}^*$ is $\mathcal{F}^*$ if char $\mathcal{F} = p$ and $\langle 1 \rangle$ if char $\mathcal{F} \ne p$. Thus $\{S \mid S \in \mathrm{Syl}_p\mathrm{GL}(n, \mathcal{F})\} = \{T \bullet S_p \mid T \in \mathrm{Syl}_p\mathrm{SL}(n, \mathcal{F})\}$ whence every Sylow p-subgroup of SL(n, $\mathcal{F}$) lies in only one Sylow p-subgroup of GL(n, $\mathcal{F}$). Hence if P is a p-uniqueness subgroup of SL(n, $\mathcal{F}$) it is also a p-uniqueness subgroup of GL(n, $\mathcal{F}$). Therefore $n \le (p + 2) \bullet |P|^2 - 1$ if char $\mathcal{F} \ne p$ by **Lemma 2 a)** which is **Theorem 3 b)** and $n \le (p + 2) \bullet |P| - 1$ if char $\mathcal{F} = p$ by **Lemma 2 b)** which is **Theorem 3 a)**. □

7. Proof of Theorem 4

Let $D(SL(n,\mathcal{F})) := \{A \in SL(n,\mathcal{F}) \mid A$ is some scalar matrix$\}$ be the subgroup of $SL(n,\mathcal{F})$ of matrices in which all off-diagonal entries are zero and the diagonal entries are any scalars, that is, elements of $\mathcal{F}$, but not all zero. It is very well-known that $D(SL(n,\mathcal{F})$ is the centre of $SL(n,\mathcal{F})$ and that $PSL(n,\mathcal{F}) := SL(n,\mathcal{F}) / D(SL(n,\mathcal{F}))$.

Proof. If S is a **beautiful** Sylow p-subgroup of $SL(n,\mathcal{F})$, then $S\,D(SL(n,\mathcal{F}))/D(SL(n,\mathcal{F})) \approx S/Z(S)$ is a Sylow p-subgroup of $PSL(n,\mathcal{F})$ where $Z(S)$ denotes the centre ("Zentrum") of S. If Q is a p-uniqueness subgroup of $SL(n,\mathcal{F})$ then $P := Q\,D(SL(n,\mathcal{F}))/D(SL(n,\mathcal{F})) \approx Q/Z(Q)$ will be a p-uniqueness subgroup of $PSL(n,\mathcal{F})$ (see [44], 1.6, and [15], Proposition 2.3), and conversely, and $n \le f_p(|Q|)$ by **Theorem 3**. However, even $n \le f_p(|P|)$ since otherwise $n \ge (p+2) \bullet |P|$ resp. $n \ge (p+2) \bullet |P|^2$ if $\mathrm{char}\mathcal{F} = p$ resp. if $\mathrm{char}\mathcal{F} \ne p$. Since P operates on the underlying vector space $\mathcal{F}^n$, we have $\dim(C_{\mathcal{F}^n}(P)) \ge p+2$ by **Proposition 6** resp. the space $\mathcal{F}^n$ has at least $p+2$ many irreducible P-isomorphic P-submodules according to **Proposition 4**. Therefore P lies in at least $|\mathrm{Syl}_p \underline{S}^{p+2}|$ Sylow p-subgroups of $PSL(n,\mathcal{F})$ by **Proposition 7** which is at least 2 by **Lemma 1 *δ*)** of **Page 6**. □

8. Planning future research – Part 2

Our proofs of the **Conjecture 1** of **Page 8** for the types Ξ = "$\underline{A}^n$" and Ξ = "$A = PSL_n$", that is, to carve out the optimising **way 1)**, are characterised by the fact that *we need not at all know their Sylow p-subgroups*. There is no doubt that we can (easily) extend those proofs rather straightforwardly to the types $\Xi \in$ "$B = P\Omega_{\mathrm{odd}\,n}$, $C = PSp_n$, $D = P\Omega^+_{\mathrm{odd}\,n}$, ${}^2A = PSU_n$, ${}^2D = P\Omega^-_{\mathrm{even}\,n}$" by considering thoroughly the respective bilinear form defining these groups of Lie type, resp. the underlying vector spaces they act upon as isometries, and their resulting Sylow p-subgroups (without knowing them). They can well be considered proved which we shall confirm in a follow-up paper (see below: the **Part 1** of our **Second Trilogy**).

Optimising **Theorem 1**, **Theorem 2**, **Theorem 3** and **Theorem 4** along the **way 2)** of **Page 7** is much more challenging since it requires to determine the (minimal) p-uniqueness subgroups of $\underline{A}^n$ and of all the classical groups. Fortunately, a vast literature about these groups and their Sylow p-subgroups is available, even about the intersections of their Sylow p-subgroups. The starting point for future research into these hugely **beautiful** objects should be the papers by Léo A. Kaloujnine (see [32]-[40]) and by Alan J. Weir (see [55]-[58]) and Theorem 1.4 B of [11] together with [7]. The starting point for Sylow p-intersections could be [5] which has a sizeable list of references and all sorts of historical details.

A MATHEMATICIAN, like a painter or a poet, is a maker of patterns.
If his patterns are more permanent than theirs, it is because they are made with ***ideas***. … The mathematician's patterns, like the painter's or the poet's, must be ***beautiful***; the ***ideas***, like the colours or the words, must fit together in a harmonious way. ***Beauty*** is the first test: there is no permanent place in the world for ugly mathematics.

Godfrey Harold Hardy (7 February 1877 until 1 December 1947).
A Mathematician's Apology. § 10. July 18, 1940. ISBN 978-1-68422-185-1.
With a foreword by **Charles Percy Snow**. ISBN 978-1-107-60463-6.

The author is passionately curious about the future.
Der Autor ist sehr leidenschaftlich neugierig auf die Zukunft.
L'auteur est passionnément curieux de l'avenir.
L'autore è appassionatamente curioso del futuro.
O autor è muito apaixonadamente curioso sobre o futuro.

Felix Fortunatus Flemisch (17 May 1951 until today).
Firenze. April 11, 1992.

We now indicate how to continue the **way 1)** of **Page 7** for the remaining types $\Xi \in$ "$B = P\Omega_{\mathrm{odd}\,n}$, $C = PSp_n$, $D = P\Omega^+_{\mathrm{odd}\,n}$, ${}^2A = PSU_n$, ${}^2D = P\Omega^-_{\mathrm{even}\,n}$" and how to prove the **Conjecture 3** of **Page 9** by announcing the two follow-up papers "The Strong Sylow Theorem for the Prime p in the Locally Finite Classical Groups" and "The Strong Sylow Theorem for the Prime p in Locally Finite and p-Soluble Groups" which we hope to finalise in 2025. They are the first two parts of **The Second Trilogy about Sylow Theory in Locally Finite Groups** whose third part will be our forthcoming research paper "Augustin-Louis Cauchy's and Évariste Galois' Contributions to Sylow Theory in Finite Groups". **The First Trilogy** are [15] on p-uniqueness subgroups and [this paper] on $\underline{A}^n$ and $A = PSL_n$ (see the **Postscript** on **Page 15**).

Part 1 of **The Second Trilogy** considers the locally finite classical groups which are *the linear*, *symplectic*, *unitary* and *orthogonal groups* over locally finite fields. The linear groups are dealt with in this paper and the others are subgroups of the linear groups which are defined through a non-singular bilinear form (or a scalar product) which is either skew-symmetric (or alternate) or Hermitian or symmetric (defining a quadratic form) as *the group of isometries of the form*. They were introduced in the classical books [1] and [60] and are further studied in [6], [24] and [52]. We do not refer to the groups of Lie type resp. the Chevalley groups and the twisted Chevalley groups being defined through a Dynkin diagram automorphism followed by a field automorphism, which correspond to the classical groups (see [24], pp. 151-152) and whose fine introductory references are the "Lecture Notes on Chevalley Groups" by **Robert Steinberg** (1967 and 2016) together with the book "Simple Groups of Lie type" by **Roger W. Carter** (1972 and 1989). Thus we study $P\Omega_{\mathrm{odd}\,n}$, PSp_n, $P\Omega^+_{\mathrm{even}\,n}$, PSU_n and $P\Omega^-_{\mathrm{even}\,n}$ and not B, C, D, ${}^2A$ and ${}^2D$. Hence the proofs of **Part 1** for the further five types of Classical Groups can and will also eventually be based on our **very beautiful Theorem 2** about **the General Linear Groups**.

Part 2 of **The Second Trilogy** considers (locally) finite and p-soluble groups. It summarises the work by **B. Hartley** and **A. Rae** regarding $\boldsymbol{\lambda_p}$ and $\boldsymbol{p}^{a_p}$ (see **Page 38** of [15] and the **References** of [44]) and the foregoing work on the classical Hall-Higman theory regarding $\boldsymbol{\lambda_p}$ and $\boldsymbol{p}^{b_p}$, $\mathbf{c}_p$, $\mathbf{d}_p$, $\boldsymbol{p}^{e_p}$ and $\mathbf{r}_p$ by **P. Hall**, **G. Higman**, **A.H.M. Hoare**, **T.R. Berger**, **F. Gross**, **E.G. Bryukhanova** and last but not least by **A. Turull** [53] as indicated on **Page 8** and **Page 9**. It then proves **Conjecture 3** not only in **English** but partly in **Portuguese** for well-founded historical reasons.

Part 3 of **The Second Trilogy** pays tribute to **Augustin-Louis Cauchy**'s and **Évariste Galois**' contributions to Sylow theory in finite groups. It proves in a unified way **Lagrange's theorem** and **Cauchy's concealed second and third group theorems** by exploring and using the following three rectangles a.k.a. tableaux which we show here for the first time though with only minor comments in order to raise inquisitiveness:

complete right transversal for G in H	the **first row** consists of ***all*** elements z_k of G ($1 \le k \le \mathrm{M}$) acting on H in the **following rows** via multiplication from the left by their inverses				correspondence	set$_H$**Orbi(G)** $:= G \setminus H$ of ***all*** orbits of H under G acting by left translation
$t_1 := 1 =: z_1$	z_2	z_3	…	z_M	$\leftrightarrow$	$G = {}_1\mathrm{Orb}(G)$
t_2	$z_2\, t_2$	$z_3\, t_2$	…	$z_\mathrm{M}\, t_2$	$\leftrightarrow$	$Gt_2 = {}_{t_2}\mathrm{Orb}(G)$
t_3	$z_2\, t_3$	$z_3\, t_3$	…	$z_\mathrm{M}\, t_3$	$\leftrightarrow$	$Gt_3 = {}_{t_3}\mathrm{Orb}(G)$
…	…	…	…	…	…	…
t_R	$z_2\, t_\mathrm{R}$	$z_3\, t_\mathrm{R}$	…	$z_\mathrm{M}\, t_\mathrm{R}$	$\leftrightarrow$	$Gt_\mathrm{R} = {}_{t_\mathrm{R}}\mathrm{Orb}(G)$

rectangle $|G|$ x $[H{:}G]$ of elements

set of ***certain*** orbits of H under G acting by left translation	the **first row** consists of ***all*** right cosets Gx_1^k of G in H ($0 \le k \le p$-1) with the powers of some p-blank x_1 of G in H; the **following rows** consist of right cosets of G in H with the powers of left conjugates of x_1				correspondence	$X := \langle x_1 \rangle$; set of ***all*** orbits of H under $G \circlearrowleft \circlearrowright X$, the **simultaneous** actions of G by left translation and of X by right translation
$Gx_1^0\, t_1 = G$	Gx_1	Gx_1^2	…	$Gx_1^{p\text{-}1}$	$\leftrightarrow$	cosets $G\langle x_1\rangle = GX$ = double coset $G\,1\,X$
$Gx_2^0\, t_2 = Gt_2$	$Gx_2\, t_2$	$Gx_2^2\, t_2$	…	$Gx_2^{p\text{-}1}\, t_2$	$\leftrightarrow$	cosets $G\langle x_2\rangle t_2$ = double coset $G\,t_2\,X$
$Gx_3^0\, t_3 = Gt_3$	$Gx_3\, t_3$	$Gx_3^2\, t_3$	…	$Gx_3^{p\text{-}1}\, t_3$	$\leftrightarrow$	cosets $G\langle x_3\rangle t_3$ = double coset $G\,t_3\,X$
…	…	…	…	…	…	…
$Gx_\mathrm{S}^0\, t_\mathrm{S} = Gt_\mathrm{S}$	$Gx_\mathrm{S}\, t_\mathrm{S}$	$Gx_\mathrm{S}^2\, t_\mathrm{S}$	…	$Gx_\mathrm{S}^{p\text{-}1}\, t_\mathrm{S}$	$\leftrightarrow$	cosets $G\langle x_\mathrm{S}\rangle t_\mathrm{S}$ = double coset $G\,t_\mathrm{S}\,X$

tableau p x $[H{:}G]\,/\,p$ of cosets

set of ***certain*** orbits of H under G acting by left translation	the **first row** consists of ***all*** right cosets $G\,x_{1c}$ of G in H ($0 \le c \le |H|_p$-1) with the elements of some Sylow p-subgroup X of H, all of whose **elements of order** p are p-blanks of G in H; the **following rows** consist of right cosets of G in H with the elements of left conjugates of X				correspondence	$|X| = |H|_p = p^b$; set of ***all*** orbits of H under $G \circlearrowleft \circlearrowright X$, the **simultaneous** actions of G by left translation and of X by right translation
$G\,x_{10}\,t_1 = G$	$G\,x_{11}$	$G\,x_{12}$	…	$G\,x_{1\,p^b\text{-}1}$	$\leftrightarrow$	cosets $G\{x_{1c} \mid 0 \le c \le p^b\text{-}1\}$ $= GX =$ double coset $G\,1\,X$
$G\,x_{20}\,t_2 = G\,t_2$	$G\,x_{21}\,t_2$	$G\,x_{22}\,t_2$	…	$G\,x_{2\,p^b\text{-}1}\,t_2$	$\leftrightarrow$	cosets $G\{x_{2c} \mid 0 \le c \le p^b\text{-}1\}\,t_2$ = double coset $G\,t_2\,X$
$G\,x_{30}\,t_3 = G\,t_3$	$G\,x_{31}\,t_3$	$G\,x_{32}\,t_3$	…	$G\,x_{3\,p^b\text{-}1}\,t_3$	$\leftrightarrow$	cosets $G\{x_{3c} \mid 0 \le c \le p^b\text{-}1\}\,t_3$ = double coset $G\,t_3\,X$
…	…	…	…	…	…	…
$G\,x_{\mathrm{T}0}\,t_\mathrm{T} = G\,t_\mathrm{T}$	$G\,x_{\mathrm{T}1}\,t_\mathrm{T}$	$G\,x_{\mathrm{T}2}\,t_\mathrm{T}$	…	$G\,x_{\mathrm{T}\,p^b\text{-}1}\,t_\mathrm{T}$	$\leftrightarrow$	cosets $G\{x_{\mathrm{T}c} \mid 0 \le c \le p^b\text{-}1\}\,t_\mathrm{T}$ = double coset $G\,t_\mathrm{T}\,X$

rectangle $|H|_p$ x $[H{:}G]\,/\,|H|_p$ of cosets

Subsequently it first corrects a great misunderstanding of Cauchy's work of 1845/1846 in the quite renowned literature and then presents Cauchy's work of 1812/1815 in the sincere succession of the earlier work of **Joseph-Louis de Lagrange** (Giuseppe Luigi Lagrangia), of **Alexandre-Théophile Vandermonde** and of pioneer **Paolo Ruffini**, as indicated by Cauchy himself, thereby identifying and explaining the crucial parts of Cauchy's first publication of 1812/1815 on group theory.

It then presents what **Évariste Galois** surely knew about **Cauchy's group theorems** and even already about **Sylow's theorems** by referring to his published papers and with utmost care to his posthumously published papers and to his manuscripts.

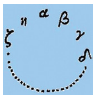
Afterwards it summarises a large number of papers on **Early group theory and early Sylow theory in finite groups** centred around both Cauchy's and Galois' work and completes this résumé with quite exciting own excavations. It then closes with grateful **Acknowledgements** and a sizeable list of **References** which is and must be chronologically ordered and not by the names of the authors or institutions as usual.

In the following we describe **Part 3** in more detail.

We are planning to revise thoroughly Sylow theory starting with **a really new proof** for **Cauchy**'s known as fundamental theorem in group theory (look at https://en.wikipedia.org/wiki/Cauchy%27s_theorem_(group_theory)) based on **beautiful** ideas by **Galois**. In the forthcoming (third) follow-up **Research Article** "Augustin-Louis Cauchy's and Évariste Galois' Contributions to Sylow Theory in Finite Groups" beyond our **First Trilogy** (look at **Page 15**) we first describe and then provide new but classical and rather unified proofs for the very fundamental theorems by **Lagrange** and by **Cauchy** on finite groups being of – in our modest opinion – considerable historical relevance.

We can describe consequences of **the absence of group elements of prime order p**, in spite of their ready availability in overgroups, thereby providing a considerably unified and also heretofore undiscovered approach to the theorems of Lagrange and of Cauchy and their implications for p-groups. Since this approach uses only ideas from a very well-known paper by **Augustin-Louis Cauchy** presented first in 1812 and then published later in 1815, this bears considerable historic relevance. While it is widely acknowledged that Cauchy had ***published*** his fundamental group theorem not until 1845/1846 and had there based it on double cosets of the finite permutation group and some Sylow p-subgroup of its symmetric overgroup, one could henceforth well argue that he had presented his theorem in a truly concealed way already a good thirty years earlier. Évariste Galois knew both Cauchy's paper of 1815 and – based on his own rather perceptive considerations – Cauchy's group theorem and even already Sylow's existence theorem. Cauchy's and Galois' ideas are particularly lucid in the embryonic case of permutation groups of prime degree p (≥ 5) when Sylow p-subgroups of the symmetric overgroup obviously exist. If $G \subseteq H$ with H being finite, then **the unified method of proof** consists in arranging the elements of H in **a rectangle** with $|G|$ columns and $[H{:}G]$ rows resp. the (right) cosets of G in H in **a rectangle** with p resp. with $|H|_p$ columns and $[H{:}G]/p$ resp. $[H{:}G]/|H|_p$ rows to obtain information about $[H{:}G]$ (see the three rectangles above).

Cauchy's theorem of 1812/1815 is a direct consequence of $[H{:}\langle x\rangle] \ge |G|$ if x is an element of H of order p with $x \notin G$ which we call a ***p-blank of G in H*** ☺. We find that Lagrange's theorem and Cauchy's theorem are just like two sides of a coin where "Lagrange" is representing the case $p^0 = 1$ and "Cauchy" represents the case $p^1 = p$ thereby offering a unified approach to both theorems. Therefore, "Cauchy" is not only a partial converse of "Lagrange" but it is in fact a smart **"swapping" of *p* for 1** as well: $p^0 = 1$ ⟲ $p = p^1$.

Cauchy depicts 1815 a *p*-cycle for some prime *p* as a regular *p*-gon  and studies *p*-cycles in considerable detail.

We present Cauchy's ***classical proof*** of **Lagrange's theorem** and supplement it with a **beautiful *modern proof***. Afterwards we present Cauchy's ***classical proofs*** of his **published first theorem**, of his **concealed second theorem** and of his **concealed third theorem**. Subsequently we introduce double cosets and show how they lead to a ***modern proof*** of Cauchy's second and third theorems what Cauchy did as well but not until 1845/1846 after very thoroughly reconsidering, sustainably impressed by a research paper of **Joseph Bertrand**, his work of 1812/1815, that is, after – believe it or not – 30 years.

We continue with **first correcting** a great misunderstanding of Cauchy's work of 1845/1846 in the literature and **then presenting** Cauchy's work of 1812/1815 in the very sincere succession of the earlier work of **Joseph-Louis de Lagrange** (Giuseppe Luigi Lagrangia), **Alexandre-Théophile Vandermonde** and **Paolo Ruffini**, as indicated by Cauchy himself, and identify, explain and comment the crucial parts of Cauchy's first publication on group theory. **Finally** we proudly present what **Évariste Galois** knew already about **Cauchy's group theorems** and about **Sylow's famous theorems** by referring to his published papers and also to his posthumously published papers. However, this will require quite considerable further (historical) research. We would be inestimably delighted if several group theory researcher would help us with this tedious but very suspenseful work and are ready to cöordinate all the work. We are then closing with fairly comprehensive **Acknowledgements** and a greatly sizeable list of **References**.

Augustin-Louis Cauchy
(21 August 1789 until 23 May 1857)

Évariste Galois
(25 October 1811 until 31 May 1832)

9. The First Trilogy and The Second Trilogy and their reviews

The First Trilogy are the papers

1a) Characterising Locally Finite Groups satisfying the Strong Sylow Theorem for the Prime *p* – Part 1 of a Trilogy (see [16]),

1b) Characterising Locally Finite Groups satisfying the Strong Sylow Theorem for the Prime *p* – Part 1 of a Trilogy. Second edition (see [17]),

2) About the Strong Sylow Theorem for the Prime *p* in Simple Locally Finite Groups – Part 2 of a Trilogy (see [18]), and

3) The Strong Sylow Theorem for the Prime *p* in Projective Special Linear Locally Finite Groups – Part 3 of a Trilogy (see [19]),

and **The Second Trilogy** are the papers

1) The Strong Sylow Theorem for the Prime *p* in the Locally Finite Classical Groups,

2) The Strong Sylow Theorem for the Prime *p* in Locally Finite and *p*-Soluble Groups, and

3) Augustin-Louis Cauchy's and Évariste Galois' Contributions to Sylow Theory in Finite Groups.

The mathematical subject matter of **The First Trilogy** is described in **its review** in **Contemporary Mathematics**, Volume **4**, Issue **3**, pp. 484-487 (see [20]). **1a)** and **1b)** of the Trilogy were subsequently submitted to **Advances in Group Theory and Applications (AGTA)** and peer reviewed and published there (see [15] and **Appendix 1**) and received **a review** by Mathematical Reviews (see MR4441631) and also **a review** by Zentralblatt für Mathematik (see Zbl 1496.20065). The **Postscript** on **Page 15** describes briefly the contents of **The First Trilogy**.

The review in Contemporary Mathematics was enlarged to **a much more detailed review** in the **Journal of Mathematical & Computer Applications (JMCA)** (see [21]).

The Second Trilogy is not yet published (and even not yet finally developed) and therefore cannot be reviewed, but **a review** along the pattern of [21] is planned and its contents is already summarised in great detail in **Chapter 8** above. This summary will be the basis of the planned review. It is well-expected that the published papers will receive **a review** by Mathematical Reviews and **a review** by Zentralblatt für Mathematik, at least when being published by **AGTA** or by **Contemporary Mathematics** or by **JMCA** including references to the previous publications.

However, with these two trilogies the development of Sylow theory in (locally) finite groups cannot be finished. In particular, it is a major challenge to determine **all (minimal) *p*-uniqueness subgroups** for the known finite simple groups and their natural overgroups, the symmetric and the linear groups, and for the (locally) *p*-soluble groups, distinguishing $p \geq 5$, $p = 3$ and $p = 2$.

Acknowledgments

The author is sincerely very grateful to the known and unknown **referees** for her/his corrections, suggestions and such friendly adjuvant advice which improved the manuscript quite considerably. He wishes to thank also so very heartfeltly his truly most fabulous wife **Helga** . Without her tenderest and unconditional support and her love and greatest patience over so many years, this publication would never have been born. Most importantly, he is forever and ever grateful to **Prof. Brian Hartley** and to his teacher **Prof. Otto H. Kegel** (see [15] and [44], p. 25) for their **beautiful** papers which provide really incredible insights and give marvellous pleasure in reading and understanding the magnificent Sylow theory of both (locally) *p*-soluble (locally) finite groups and simple (locally) finite groups.

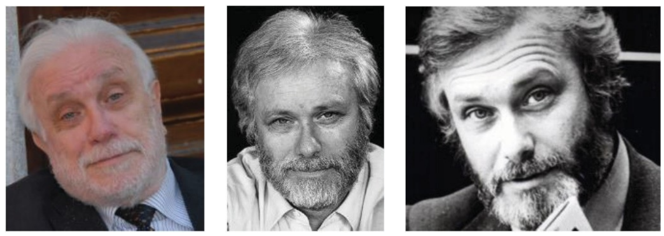
Postscript

The research paper [15] (see MR4441631 and Zbl 1496. 20065) has as many "actual" pages as there are "known" **sporadic finite simple groups**. As the overwhelming majority of group theorists (including the author) believe, these **26 groups** are now really all and *never* in the future further "sporadics" will appear (not counting the Tits group ${}^2F_4(2)'$ [as some do] because it did in fact not appear *sporadically* at the stage). A central **question** of Sylow theory in locally finite groups is, as pointed out by **Prof. Otto H. Kegel** (see [44]), how the rank of **these altogether seven rank-unbounded families of finite simple groups** $\{\underline{A}^n$, $A = PSL_n$, $B = P\Omega_{odd\ n}$, $C = PSp_n$, $D = P\Omega^+_{even\ n}$, ${}^2A = PSU_n$, ${}^2D = P\Omega^-_{even\ n}\}$ is bounded, say, *someway* ("in terms of") by any *p*-uniqueness subgroup *P*. More precisely, let us discover a nice function f_p of the order of *P* or (much more challenging) of the *p*-uniqueness of each of these (classical) groups *G*, that bounds the rank: $n \le f_p(|P|)$ or $n \le f_p(a_p(G))$. The author answered **Kegel's question** in the affirmative already for all the **beautiful** alternating groups $\underline{A}^n$ in his **Diplomarbeit** [14] and he is now publishing the answer as **Theorem 1 b)**: $n \le f_p(|P|) := (p + 2) \cdot |P| \cdot 2^{|P| - 1} - 1$. This is, although it is similar, much worse then the result obtained for all the **beautiful** linear groups $GL(n, \mathcal{F})$ (see **Lemma 2 c)** on **Page 11**). We could optimise our answer if we would come to know $a_p(\underline{A}^n)$, that is, the minimal *p*-unique subgroups of the alternating groups. Let us look for them!

In the paper at hand we answered the **question** as **Theorem 4** for the PSL groups $A = PSL_n$ thereby completing for the time being our (in our modest opinion) **beautiful (First) Trilogy** – [15] on *p*-uniqueness subgroups and [this paper] on $\underline{A}^n$ and $A = PSL_n$ – **about Sylow Theory in Locally Finite Groups** which provides a number of good suggestions to stimulate and encourage future research. All of these should become rather very challenging **beautiful** open problems for the international community of (locally finite) group theory researchers. We are ready to cöordinate related research work (see also **Page 14**). ☺ A detailed overview of the **19 families** of "known" finite simple groups is given by the figure "The Periodic Table Of Finite Simple Groups" (© 2012 by the great Iván Andrus [see https://irandrus.files.wordpress.com/2012/06/periodic-table-of-groups.pdf and https://irandrus.wordpress.com/2012/06/17/the-periodic-table-of-finite-simple-groups/]) on **Page 3** and by the **beautiful** figure on **Page 8** which shows the **19 families** of finite simple groups as **13 sporadic groups** above **18 infinite families** around another "sporadic" group (the Tits group ${}^2F_4(2)'$) and **13 sporadic groups** below ☺.

Siamo angeli con un'ala soltanto
e possiamo volare solo restando abbracciati.
We are angels who have but a single wing
and we can only fly if we cling to one another.
Wir sind Engel mit nur einem Flügel,
um fliegen zu können müssen wir uns umarmen.
Nous sommes des anges à une seule aile,
nous ne pouvons voler qu'en restant enlacés.
Somos ángeles con una única ala y sólo podemos volar abrazados.
Nós somos anjos com apenas uma asa
e só podemos voar quando nos abraçamos.

Luciano De Crescenzo
(✶ 18 August 1928 in Naples until ✝ 18 July 2019 in Rome).
Così parlò Bellavista. Napoli, amore e libertà. XXIII Piedigrotta. 1977 e settembre 2019. ISBN 978-88-04-71491-0.



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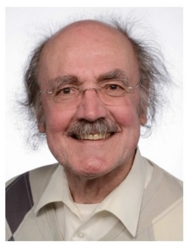
Felix F. Flemisch



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Long live Group Theory!

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Felix F. Flemisch received his first degree Bacc.Math. in 1974 from the Albert-Ludwigs-Universität at Freiburg im Breisgau, his postgraduate degree M.Sc. in 1975 from the University of London, UK, and finally his degree Dipl.-Math. at Freiburg i.Br. in 1985. Since May 1985 he was working for the telecom industry. On April 11, 1992, he married beloved **Helga** in Florence in Tuscany in Italy. Since October 2016 he is retired and is still resp. is again loving to work on mathematics, in particular on the very beautiful Group Theory ☺.

Conflict of Interest

The author declares gently that there are no competing personal or organisational or financial conflicts of interest with this original work or other conflicts of interest regarding the publication of this meticulous **Research Article**.

Pablo Picasso's *La Joie de vivre*

Pablo Picasso – *La Joie de vivre* which shepherds the **Research Article** as a flock along all abysses (see https://www.pablopicasso.org/joie-de-vivre.jsp)

About the author

Felix F. Flemisch was born on 17 May 1951 in **Munich** in Bavaria in Germany. In **June 1971** he received his **Abitur** ☺ whose subject Mathematics was taught in a pioneering spirit by **Dr. Helmut Bergold**. Afterwards he received his first-ever degree **Baccalaureus der Mathematik** (**Bacc.Math.**) in **July 1974** with the alas unpublished **beautiful bachelor's thesis** "Über einfache Punkte affiner Varietäten" from the venerable Albert-Ludwigs-Universität at **beautiful Freiburg im Breisgau** in green Baden-Württemberg in Germany under the such thorough supervision of esteemed **Akadem. Rat Dr. Herbert Götz**, and then his degree **Master of Science** (**M.Sc.**) from the Faculty of Science of the highly recognised University of London, United Kingdom, in **August 1975** at its grand Bedford College under the supervision of greatly adored **Prof. Paul Moritz Cohn** (8 January 1924 until 20 April 2006). From October 1975 until – very regrettably – only July 1976 he was employed as ***a fairly diligent Teaching Assistant with two graduations*** by the hoar Mathematische Fakultät of **Freiburg im Breisgau**'s Albert-Ludwigs-Universität. Subsequently he quite enthusiastically continued his postgraduate mathematical studies in such marvellous and such fabulous **Freiburg i.Br.** – with decent interruptions as ***a teacher*** and as ***a tutor*** – and then received his degree **Diplom-Mathematiker** (**Dipl.-Math.**) in **April 1985** under the impressive supervision of adored **Prof. Otto Helmut Kegel** (20 July 1934 until today). The Research Paper [15] publishes the most essential and partly well corrected portions of his German **Diplomarbeit** [14] of **October 1984** and a said scattered "sprinkling" of fairly new considerations and results which truly try to propose coming directions of research for the Sylow theory in (locally) finite groups. The publication at hand continues [15] with theorems about simple locally finite groups "of alternating type" and "of projective special linear type" and makes quite a number of suggestions for future research ☺. From February 1981 until April 1985 the author was enormous happily affiliated to the **Institut für Medizinische Biometrie und Statistik (IMBI)** at lovely **Freiburg im Breisgau** as ***a considered Wissenschaftlicher Mitarbeiter***. Since **May 1985** he was based dahoam in **Munich** and devotedly working with greatest joy for the telecom industry first as ***an eager System Software Developer***, then as ***a fastidious Systems Engineer***, and finally as ***a Director for International Standardisation of telecom software and concepts***. On the very **11 April 1992** (see also **Page 2**) he so blissful happily married the most fabulous and wonderful-ever woman **Helga** in **beautiful Florence** in Tuscany in Italy, which was a memorable marriage

celebrated along with about twenty friends and uniting the most venerable city **Weiden** in ***Upper Palatinate*** (i.d.OPf.) (Helga) with the huge cosmopolitan city **Munich** in ***Upper Bavaria*** (Felix). That was built really for eternity: Helga and Felix were meant to last forever ♡. Since **October 2016** the author is retired and is still resp. is again loving to work for mathematics, in particular for the **very beautiful** Group Theory ☺.

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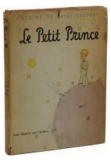
15 May 1939 – 8 October 1994

Submitted on **October 8, 1984**, exactly ten years before the very tragic death of Brian Hartley – whose splendid contributions to locally finite group theory ([21] and many about locally finite and *p*-soluble groups [see [15]]) the author had studied in great detail and with the deepest admiration and adoration – while mountain hiking. **Brian Hartley** was very well known to be such a keen and so passionate hill walker, and it happened freakishly while descending from rather steep Helvellyn (see https://en.wikipedia.org/wiki/Helvellyn) in the known as **beautiful** English Lake District's fells (see https://en.wikipedia.org/wiki/Lake_District), gey nearby his homeland, on **October 8, 1994**, that he collapsed with a grim heart attack and died (lack of any help) very very tragically (see https://en.wikipedia.org/wiki/Brian_Hartley and the references cited there, in particular MacTutor [see https://mathshistory.st-andrews.ac.uk/Biographies/Hartley/]). *It is only with the heart that one can see rightly. What is essential is invisible to the eyes. On ne voit bien qu'avec le cœur. L'essentiel est invisible pour les yeux. Gut sehen kann man nur mit dem Herzen. Worauf es wirklich* ankommt, *das sehen die Augen nicht.* **Antoine de Saint-Exupéry** (29 June 1900 until 31 July 1944). **Le Petit Prince** (April 6, 1943) (see https://en.wikipedia.org/wiki/The_Little_Prince).



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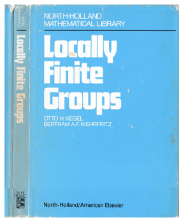

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61.14 Here are further interesting hyperlinks regarding CFSG:
https://math.mit.edu/research/highschool/primes/circle/documents/2022/Gracie.pdf;
https://mathworld.wolfram.com/ClassificationTheoremofFiniteGroups.html;
https://e.math.cornell.edu/people/mann/classes/chicago/Classification.pdf;
https://encyclopediaofmath.org/wiki/Simple_finite_group;
https://mathshistory.st-andrews.ac.uk/Extras/Simple_groups_classification/

MR – AMS Mathematical Reviews® (AMS American Mathematical Society) considered together with **MathSciNet**® (MathSciNet Mathematical Reviews) and with **MR Lookup** (American Mathematical Society MR Lookup A Reference Tool for Linking) (see https://www.ams.org/mr-database and https://mathscinet.ams.org/mathscinet and https://mathscinet.ams.org/mrlookup)

Zbl – Zentralblatt MATH (zbMATH Open® The first resource for mathematics) (see https://www.zbmath.org/)

Note – The MR number in brackets refers to the print edition of **Mathematical Reviews**®, which was printed very regrettably only until 2012 ☹, and includes the reviewer and the month of publication. Since 2013 references are to the online edition of **MathSciNet**® and the **electronic Mathematical Reviews**® **(eMR) Sections** (see https://www.ams.org/publications/mrsections/mrsections).

Appendix 1

Reference [15] with MR Review and Zbl Review

15. F.F. Flemisch: "Characterising Locally Finite Groups Satisfying the Strong Sylow Theorem for the Prime p", *Adv. Group Theory Appl.* **13** (June 2022), 13-39 (see MR4441631 and Zbl 1496.20065). https://www.advgrouptheory.com/journal/index.php#vol13 and https://www.advgrouptheory.com/journal/Volumes/13/Flemisch.pdf

While the **MR Review** is very disgracefully simply stating only the main result and is telling nothing at all about the ***new ideas***, the **Zbl Review** states at least the Abstract as a Summary and all References but also very regretfully states nothing about the **beautiful** ***new ideas*** ☹.

For example, we are rightly a little very proud of two discoveries: **1) Theorem 3.6** on **Page 28** of [15], which found a symmetry between non-conjugated Sylow p-subgroups, and then also **2)** that the **minimal** members of the set **Unique**$_p U$ from **Page 35** of [15] should play for a finite U a very similar important rôle as its **maximal** members which are the Sylow p-subgroups. It then becomes a challenge to determine the **minimal** members for sufficiently "known" (locally) finite groups, in particular for all the "known" finite simple groups and the finite p-soluble groups, and their core properties, in particular conjugacy and minimal w.r.t. order vs. minimal w.r.t. inclusion. These are mathematical ideas which propose exciting new directions for (timeless and eternal) Sylow theory in (locally) finite groups during the coming years where we intend to join in, to support, to coördinate and to try to shape. They could not have been included in The First Trilogy and are as well because of their complexity not scheduled to become part ot The Second Trilogy. Hence, they will be facinating topics of very hopefully joint research for the time after publication of The Second Trilogy.

The **MR Review** is available at **MR Lookup** under https://mathscinet.ams.org/mathscinet/relay-station?mr=4441631 and in detail on **Page 16** of the **eMR Section 1F for January 2023** at https://www.ams.org/mrslisting/2023/1F/2023-1F-01.pdf.

The **Zbl Review** is available at **Zentralblatt MATH** under https://zbmath.org/1496.20065 and its PDF at https://zbmath.org/pdf/07554056.pdf.

● For the complete **Appendix 1**, having 33 pages, see **Page 21** to **Page 53**.

Appendix 2

Introduction to the Talk by Felix F. Flemisch at IGT 2024 on April 11th, the 120th birthday of Philip Hall

My name is **Felix Flemisch**. I come from **Munich** in Bavaria in Germany. In the 1970ties and 1980ties I was a considerably busy and faithful student of **Prof. Otto H. Kegel** ♡ in such **beautiful** Freiburg i.Br. in Germany. In 2021 I luckily came again in contact with my adored teacher and met him in person and in good shape during June and July of 2022 in Freiburg. I present at IGT 2024 a **POSTER** about a new paper on **Sylow theory in simple locally finite groups** which is based on the famous **Kegel covers** and a **beautiful** paper of mine about rounding off **the general Sylow theory in locally finite groups**, friendly published by AGTA, under the rigid supervision of esteemed **Prof. Francesco de Giovanni** ✝. **Prof. Kegel** gave me kindly the hint to submit the paper to AGTA whose review process improved the paper substantially so that it now can be the basis for further work on the subject.

Both papers have a quite strong relationship to **Prof. Kegel's work on Sylow theory, each one proving a conjecture of him** and centred around the gay concept of a ***p*-uniqueness subgroup** which is a finite p-subgroup being friendly contained in such a unique Sylow p-subgroup. The **POSTER** shows the twelve slides of my talk as a PowerPoint presentation which include as well tough suggestions to stimulate and encourage future research. I much hope to enthuse group theorists with them and I am ready to coördinate related research work. This is my main interest why I present the **POSTER**. However, I am sadly aware that locally finite groups, and their Sylow theory in particular, seem not (yet) to be current topics of group theory research except some special questions presented on Tuesday. A limited number of nicely printed copies of the paper's **abstract**, its **POSTER** in DIN A3, and its **preprint** are available. I will deposit them tomorrow morning in SALA CARTAROMANA. An underlying **research paper** to this Talk will be published.

● For the complete **Appendix 2**, having 18 pages and including the **beautiful** twelve slides of the presentation, some **beautiful** photographs of Freiburg i.Br., two **beautiful** photographs of Prof. Otto H. Kegel and four photographs of the wonderfully **beautiful** Lake Ammersee in Bavaria, see **Page 54** to **Page 71**.

Appendix 1

Reference [15] with MR Review and Zbl Review

15. F.F. FLEMISCH: "Characterising Locally Finite Groups Satisfying the Strong Sylow Theorem for the Prime p", *Adv. Group Theory Appl.* **13** (June 2022), 13-39 (see MR4441631 and Zbl 1496.20065).
https://www.advgrouptheory.com/journal/index.php#vol13 and
https://www.advgrouptheory.com/journal/Volumes/13/Flemisch.pdf

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The **MR Review** is available at **MR Lookup** under https://mathscinet.ams.org/mathscinet/relay-station?mr=4441631 and in detail on **Page 16** of the **eMR Section 1F for January 2023** at https://www.ams.org/mrslisting/2023/1F/2023-1F-01.pdf.

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MR4441631 20D20 20D15 20F50

Flemisch, Felix F.

Characterising locally finite groups satisfying the strong Sylow theorem for the prime p. (English summary)

Adv. Group Theory Appl. **13** (2022), 13-39.

Let p be a prime and let G be a locally finite group. Then, G is said to satisfy the *Sylow Theorem for the prime p* if all maximal p-subgroups of G are conjugate. The group G is said to satisfy the *strong Sylow Theorem for the prime p* if every subgroup of G satisfies the Sylow Theorem for the prime p. Further, a finite p-subgroup P of G is said to be *singular* in G if for every finite subgroup F of G containing P there is a unique Sylow p-subgroup of F containing P. In this paper, it is shown that G satisfies the strong Sylow Theorem for the prime p if and only if every subgroup S of G contains a finite p-subgroup which is singular in S. This answers a question posed by Otto H. Kegel in 1987. The paper is based on the author's thesis from the year 1984 [*Lokal endliche Gruppen mit Sylow p-Satz oder mit min-p. I: Grundbegriffe, ein Charakterisierungssatz und lokale Prinzipien*, Diplomarbeit, Univ. Freiburg, 1984; per bibliography]. *Stefan Kohl* (see Theorem 2). *Marco Trombetti*

MR4482310 20D15 08A35
Ghumashyan, Heghine (AR-EU; Yerevan); **Guričan, Jaroslav** (SK-KMSK-NDM; Bratislava)
Endomorphism kernel property for finite groups. (English summary)
Math. Bohem. **147** (2022), *no.* 3, 347–358.

Summary: "A group G has the endomorphism kernel property (EKP) if every congruence relation θ on G is the kernel of an endomorphism on G. In this note we show that all finite abelian groups have EKP and we show infinite series of finite non-abelian groups which have EKP."

MR4440439 20D15 20J99
Kalteh, O. (IR-IAUMS-M; Mashhad); **Jafari, S. Hadi** (IR-IAUMS-M; Mashhad)
Capable groups of order p^3q. (English summary)
Algebra Discrete Math. **33** (2022), *no.* 1, 104–115.

A group G is called capable if there exists a group E such that $G \cong E/Z(E)$. The epicenter $Z^*(G)$ of G is the smallest central subgroup such that $G/Z^*(G)$ is capable. Obviously, G is capable if and only if $Z^*(G) = 1$.

This paper studies the capability of groups of order p^3q, where p and q are distinct prime numbers and $p > 2$. More specifically, by calculating the non-abelian exterior square $G \wedge G$, the authors determine the epicenter for groups of order p^3q in Theorem 2. As a corollary, they identify the capability of groups of order p^3q. *Junqiang Zhang*

MR4441631 20D20 20D15 20F50
Flemisch, Felix F.
Characterising locally finite groups satisfying the strong Sylow theorem for the prime p. (English summary)
Adv. Group Theory Appl. **13** (2022), 13–39.

Let p be a prime and let G be a locally finite group. Then, G is said to satisfy the *Sylow Theorem for the prime* p if all maximal p-subgroups of G are conjugate. The group G is said to satisfy the *strong Sylow Theorem for the prime* p if every subgroup of G satisfies the Sylow Theorem for the prime p. Further, a finite p-subgroup P of G is said to be *singular* in G if for every finite subgroup F of G containing P there is a unique Sylow p-subgroup of F containing P. In this paper, it is shown that G satisfies the strong Sylow Theorem for the prime p if and only if every subgroup S of G contains a finite p-subgroup which is singular in S. This answers a question posed by Otto H. Kegel in 1987. The paper is based on the author's thesis from the year 1984 [*Lokal endliche Gruppen mit Sylow p-Satz oder mit min-p. I: Grundbegriffe, ein Charakterisierungssatz und lokale Prinzipien*, Diplomarbeit, Univ. Freiburg, 1984; per bibliography]. *Stefan Kohl*

MR4346240 20D25 20D15 20D20
Allcock, Daniel (1-TX; Austin, TX)
Variations on Glauberman's ZJ theorem. (English summary)
Int. J. Group Theory **11** (2022), *no.* 2, 43–52.

It is well known that the classical ZJ theorem by G. Glauberman has been proved in various versions, depending on the various possible definitions of the Thompson subgroup. In this paper the author presents an "axiomatic" version of the ZJ theorem, and proposes new choices for the family of abelian subgroups of the Sylow p-subgroup S of the finite group G that can generate a sort of generalized Thompson subgroup for which a ZJ-type theorem holds. Furthermore, I believe that the paper could be very

The **Zbl Review** is available at **Zentralblatt MATH** under https://zbmath.org/1496.20065 and its PDF at https://zbmath.org/pdf/07554056.pdf.

Flemisch, F. F.

Characterising locally finite groups satisfying the strong Sylow theorem for the prime *p*. (English) Zbl 1496.20065

Adv. Group Theory Appl. 13, 13-39 (2022).

Summary:

During his lectures to the 1987 Singapore Group Theory Conference [10] Otto H. Kegel proposed the following question: "*If every subgroup S of the locally finite group G contains a finite p-subgroup which is singular in S, does G then satisfy the strong Sylow Theorem for the prime p?*" In this paper we answer the question in the affirmative. The paper formed an essential part of the author's German Diplomarbeit of 1984 (the "Charakterisierungssatz") written before he left academia [4]. We present the *Charakterisierungssatz* as Theorem 3.9, and summarise then the result as Theorem 3.10, stating that if *G* is a locally finite group and *p* is a prime, then *G* satisfies the strong Sylow theorem for the prime *p* if and only if every subgroup *S* of *G* contains a finite *p*-subgroup which is singular in *S*. Subsequently we present a few novel concepts for Sylow theory in (locally) finite groups to encourage future research. The paper is divided in four sections: Introduction; Good Sylow *p*-subgroups and *p*-uniqueness subgroups; Basic theorems of Sylow theory in locally finite groups and our *Charakterisierungssatz*; Novel concepts for Sylow theory in (locally) finite groups.

MSC:

20D20	Sylow subgroups, Sylow properties, π-groups, π-structure
20F50	Periodic groups, locally finite groups
20D15	Finite nilpotent groups, *p*-groups
20E25	Local properties of groups
20E34	General structure theorems for groups

Keywords:

singular (Sylow) *p*-subgroup; (very) good Sylow *p*-subgroup; *p*-uniqueness subgroup; minimal *p*-unique subgroup; (numerical) Sylow *p*-invariant a_p

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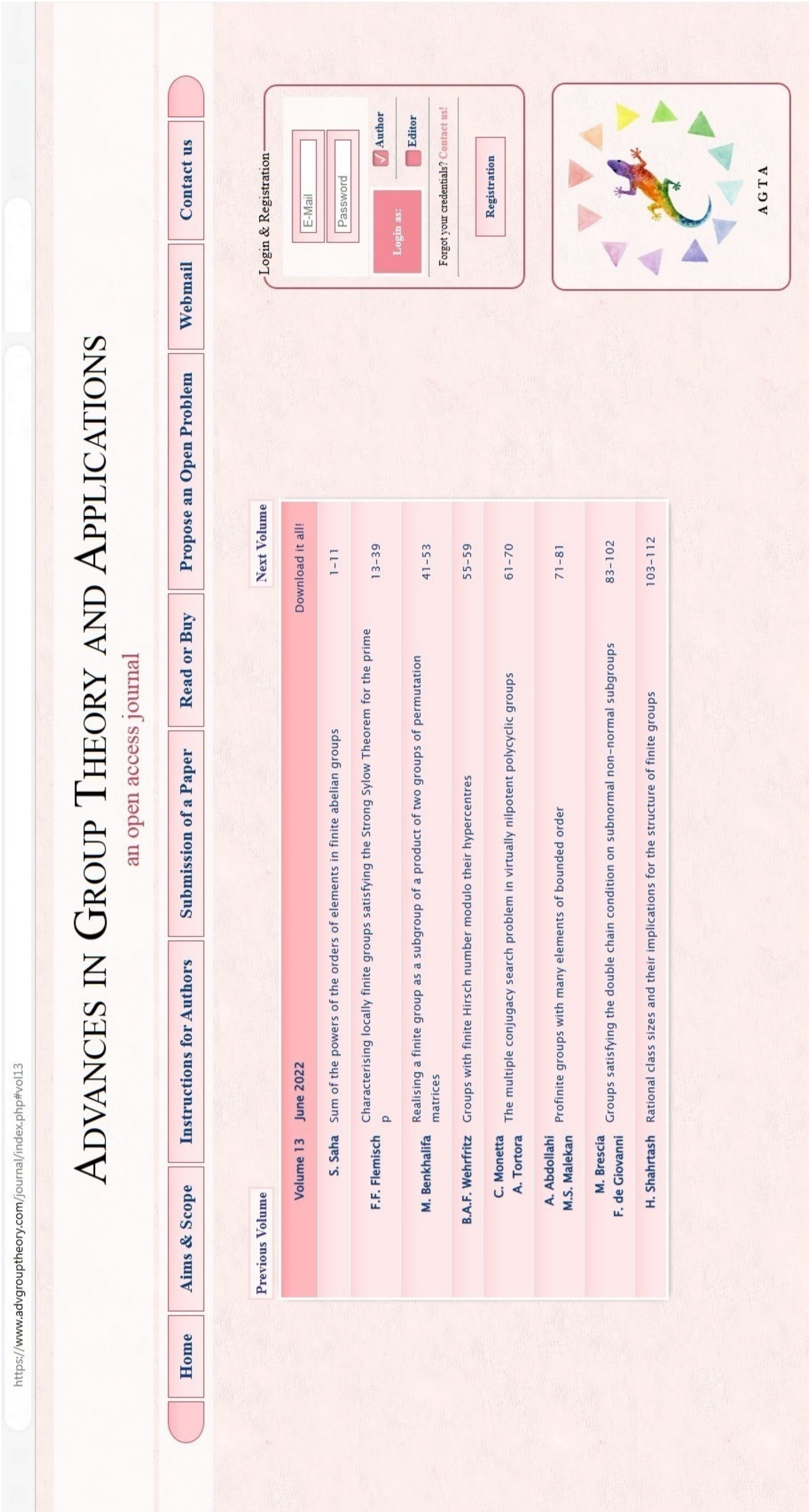
Summary: During his lectures to the 1987 Singapore Group Theory Conference Otto H. Kegel proposed the following question: *"If every subgroup S of the locally finite group G contains a finite p-subgroup which is singular in S, does G then satisfy the strong Sylow Theorem for the prime p?"* In this paper we answer the question in the affirmative. The paper formed an essential part of the author's German Diplomarbeit of 1984 (the *"Charakterisierungssatz"*) written before he left academia [*F. F. Flemisch*, "Lokal endliche Gruppen mit Sylow p-Satz oder mit min-p. I: Grundbegriffe, ein Charakterisierungssatz und lokale Prinzipien", Diplomarbeit, University of Freiburg, Germany (1984)]. We present the *Charakterisierungssatz* as Theorem 3.9, and summarise then the result as Theorem 3.10, stating that if G is a locally finite group and p is a prime, then G satisfies the strong Sylow theorem for the prime p if and only if every subgroup S of G contains a finite p-subgroup which is singular in S. Subsequently we present a few novel concepts for Sylow theory in (locally) finite groups to encourage future research. The paper is divided in four sections: Introduction; Good Sylow p-subgroups and p-uniqueness subgroups; Basic theorems of Sylow theory in locally finite groups and our *Charakterisierungssatz*; Novel concepts for Sylow theory in (locally) finite groups.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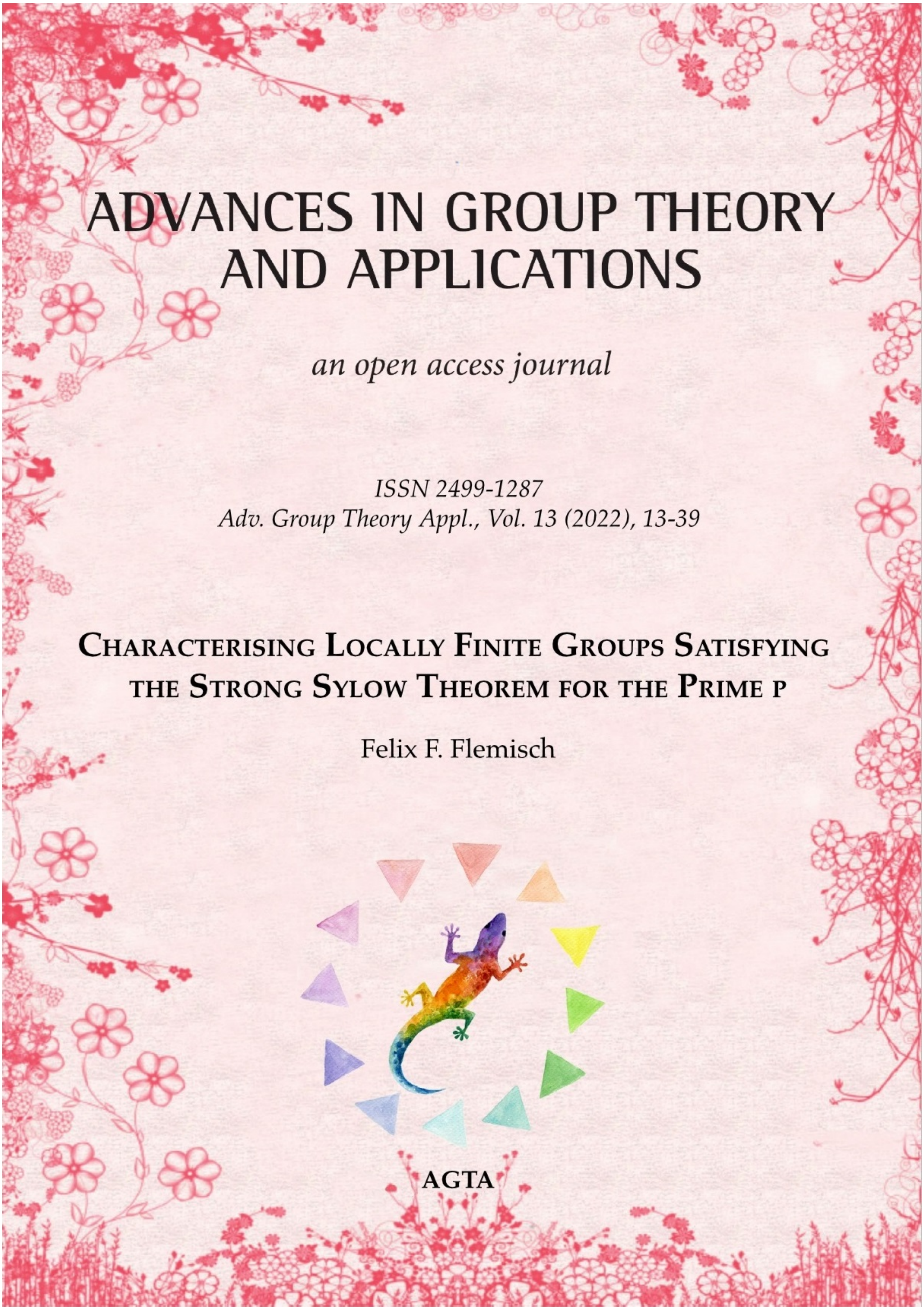
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CHARACTERISING LOCALLY FINITE GROUPS SATISFYING
THE STRONG SYLOW THEOREM FOR THE PRIME P
Felix F. Flemisch
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Characterising Locally Finite Groups Satisfying the Strong Sylow Theorem for the Prime p

Felix F. Flemisch

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Abstract

During his lectures to the 1987 Singapore Group Theory Conference Otto H. Kegel proposed the following question: *"If every subgroup* S *of the locally finite group* G *contains a finite* p*-subgroup which is singular in* S*, does* G *then satisfy the strong Sylow Theorem for the prime* p*?"* In this paper we answer the question in the affirmative. The paper formed an essential part of the author's German Diplomarbeit of 1984 (the "*Charakterisierungssatz*") written before he left academia [4]. We present the *Charakterisierungssatz* as Theorem 3.9, and summarise then the result as Theorem 3.10, stating that if G is a locally finite group and p is a prime, then G satisfies the strong Sylow theorem for the prime p if and only if every subgroup S of G contains a finite p-subgroup which is singular in S. Subsequently we present a few novel concepts for Sylow theory in (locally) finite groups to encourage future research. The paper is divided in four sections: Introduction; Good Sylow p-subgroups and p-uniqueness subgroups; Basic theorems of Sylow theory in locally finite groups and our *Charakterisierungssatz*; Novel concepts for Sylow theory in (locally) finite groups.

Mathematics Subject Classification (2020): 20D20, 20F50, 20D15

Keywords: singular p-subgroup; good Sylow p-subgroup; minimal p-unique subgroup

1 Introduction

In his four workshop lectures on Sylow theory in locally finite groups at the famed Singapore Group Theory Conference of June 1987 [10], Otto H. Kegel stated that he could not settle the following question: *if* *every subgroup* S *of the locally finite group* G *contains a finite* p*-subgroup which is singular in* S*, does* G *then satisfy the strong Sylow Theorem for the prime* p*?* Recall that the group G of arbitrary cardinality is defined to be *locally finite* if every finite subset of G is contained in a finite subgroup of G and the finite p-subgroup P of the locally finite group G is said to be *singular in* G if for every finite subgroup F of G containing P there is just a unique Sylow p-subgroup of F containing P. Here a p-group for the prime p is a group of arbitrary cardinality each of whose elements has order a finite power of p. Then a p-group is finite if and only if its order is a finite power of p. The locally finite group G is said to satisfy the *Sylow Theorem for the prime* p (or the *Sylow* p*-Theorem*) if the maximal p-subgroups of G are all conjugate in G and G satisfies the *strong Sylow Theorem for the prime* p if every subgroup of G satisfies the Sylow Theorem for the prime p. Kegel's lectures present the basics of Sylow theory in locally finite groups, give an overview of the work of Brian Hartley and Andrew Rae on Sylow theory in locally p-soluble groups, and reveal in great detail the normal structure for groups satisfying the strong Sylow Theorem for the prime p in the general case (for $p \geqslant 5$). Chapters 2 and 4 of [3] give a good overview as well but without appreciating Hartley's, Rae's and Kegel's fundamental papers properly and avoiding all their beautiful details.

In this publication we turn Kegel's question into a theorem: *If every subgroup* S *of the locally finite group* G *contains a finite* p*-subgroup which is singular in* S*, then* G *satisfies the strong Sylow Theorem for the prime* p*.* Since the converse is also true (see [4] and [10]), this characterises the locally finite groups which satisfy the strong Sylow Theorem for the prime p. The proof of our *Charakterisierungssatz* is not presented in its original form since it was written in German as the main result of the author's Diplomarbeit during 1978–1984 (see [4]). We decided against a presentation (for historical reasons) as an amalgam of English and German and translated all employed parts into English, thereby introducing a large number of corrections and embellishments, in particular Theorem 3.6.

The central discovery that enabled in those days the proof was the relationship of p-subgroups which are singular to the *good* p*-subgroups* (see [12]) and the *strongly local* p*-subgroups* (see [13]) of Andrew Rae. Let G be any locally finite group and let P be a p-subgroup of G. A *local system for* G is a family Σ of finite subgroups such that every element of G lies in a Σ-group and for every two Σ-groups there exists another Σ-group which contains both, for example, the local

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system of all finite subgroups of G. The p-group P is said to *reduce into a local system* Σ *for* G if for every Σ-group U we have that $P \cap U$ is a Sylow p-subgroup of U, and then P is a maximal p-subgroup of G (see below), P is said to be *good* if there exists a local system for G into which P reduces, and P is said to be *strongly local* or, as we prefer to say, *very good* if given any local system Σ for G there exists a subsystem of Σ into which P reduces. A very good p-subgroup is of course good, and, as we show below, any singular p-subgroup P of a locally finite group G is contained in a unique maximal p-subgroup of G which is very good and the existence of P enforces the conjugacy of the good Sylow p-subgroups in countable locally finite groups

We have the ambition to present not only our own results but also important known results to offer some context and a unified depiction. So when we refer to [4] it does not always mean (although it almost always means) that we present research results of ourselves.

2 Good Sylow p-subgroups and p-uniqueness subgroups

A maximal p-subgroup of a locally finite group G is called here a *Sylow* p*-subgroup* of G and we denote the set of all Sylow p-subgroups of G by $\mathrm{Syl}_p G$. If a p-subgroup of a locally finite group G reduces into a local system for G, it is a maximal p-subgroup.

Lemma 2.1 (see [4]) *Let* p *be a prime and let* P *be a* p*-subgroup of a locally finite group* G*. If there exists a local system* Σ *for* G *into which* P *reduces, then* P *is a Sylow* p*-subgroup of* G*.*

Proof — Let $S \in \mathrm{Syl}_p G$ with $P \leqslant S$. Suppose, $P \neq S$. Then there exists an element $x \in S \backslash P$. Let $U \in \Sigma$ with $x \in U$. It follows that $\langle P \cap U, x \rangle$ is a p-subgroup of U with $P \cap U < \langle P \cap U, x \rangle \leqslant S$. This contradicts the prerequisite $P \cap U \in \mathrm{Syl}_p U$. □

Notice that the above result is proved in [3], Lemma 2.2.10, only for nested local systems and in a more complicated way. The local system Σ for the locally finite group G is said to be *nested* (in German *geschachtelt*) if there is a sequence $\{U_n \mid n \in \mathbb{N}\}$ of finite subgroups of G such that $U_n \leqslant U_{n+1}$ for all $n \in \mathbb{N}$ and $\Sigma = \{U_n \mid n \in \mathbb{N}\}$. If G is a countable locally finite group and $\{x_n \mid n \in \mathbb{N}\}$ an enumeration of G, let $U_n := \langle x_1, x_2, \dots, x_n \rangle$ $(n \in \mathbb{N})$. Then $\{U_n \mid n \in \mathbb{N}\}$ is a nested

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local system for G. If the locally finite group G has a nested local system, then G is countable. We can identify all the good Sylow p-subgroups of countable locally finite groups by means of nested local systems for them.

Lemma 2.2 (see [4]) *Let* G *be a countable locally finite group.*

a) *If* Σ *is a local system for* G*, then* Σ *contains a local subsystem* Σ_1 *which is nested.*

b) *Let* $\Sigma = \{U_n \mid n \in \mathbb{N}\}$ *be a nested local system for* G*. Then there exist with respect to (w.r.t.)* Σ *good Sylow* p*-subgroups of* G*. In particular,* G *contains at least one good Sylow* p*-subgroup.*

Proof — a) Let Σ be a local system for G and $\{x_n \mid n \in \mathbb{N}\}$ an enumeration of G. For $x, y \in G$, we define $U_x \in \Sigma$ with $x \in U_x$ and $\langle U_x, U_y \rangle \leqslant U_{xy}$ as follows: let $U_{x_1} \in \Sigma$ with $x_1 \in U_{x_1}$; if subgroups $U_{x_1\,x_2\,x_3\,\dots\,x_n} \in \Sigma$ are already defined with

$$x_1, x_2, x_3, \dots, x_n \in U_{x_1\,x_2\,x_3\,\dots\,x_n} \ (n \in \mathbb{N}),$$

let $U_{x_{n+1}} \in \Sigma$ with $x_{n+1} \in U_{x_{n+1}}$ and $U_{x_1\,x_2\,x_3\,\dots\,x_n\,x_{n+1}} \in \Sigma$ with

$$\langle U_{x_1\,x_2\,x_3\,\dots\,x_n}, U_{x_{n+1}} \rangle \leqslant U_{x_1\,x_2\,x_3\,\dots\,x_n\,x_{n+1}} \ (n \in \mathbb{N}).$$

Then the countable subset $\Sigma_1 := \{U_{x_1\,x_2\,x_3\,\dots\,x_n} \mid n \in \mathbb{N}\}$ of Σ is a nested local system for G.

b) Let $P_1 \in \mathrm{Syl}_p U_1$. If

$$P_1 \leqslant P_2 \leqslant \dots \leqslant P_n$$

are already finite p-subgroups of G with $P_i \in \mathrm{Syl}_p U_i$ $(1 \leqslant i \leqslant n)$, let $P_{n+1} \in \mathrm{Syl}_p U_{n+1}$ with $P_n \leqslant P_{n+1}$ $(n \in \mathbb{N})$. Define $S := \bigcup_n P_n$. Then S is a p-subgroup of G, which reduces into Σ, and so is good with $S \in \mathrm{Syl}_p G$ by Lemma 2.1. □

Another argument for proving Lemma 2.2 b) comes from Kegel's Lemma 1.1 of [10] and is very similar to that of Lemma 2.1. Note also that Lemmata 2.1 and 2.2 a) are (and were) well-known but we presented slick improved proofs and did not find Lemma 2.2 a) in the literature. For Lemma 2.2 b) see also [12], 1.11.

We can now introduce the p-uniqueness subgroups and present the close relationship between them and the good Sylow p-subgroups.

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In [4] we call p-*dominant* a p-subgroup of the locally finite group G if it is finite and is contained in a unique Sylow p-subgroup S of G, and call then S *singular* (in German *einzigartig* or *einmalig* or *singulär*, in a double sense). Although "dominant" in German is "dominant" in English we now find it smarter to define such a p-subgroup of G as a p-*uniqueness subgroup* (in German, quite a bit unwieldy, p-*Einzigartigkeitsuntergruppe* or p-*Einmaligkeitsunterguppe*) *of* G *for* S or *w.r.t.* S. We observe that there is no danger of confounding our p-uniqueness subgroups with the p-uniqueness subgroups which play a major role in the classification of the finite simple groups (see page 82 of [5]).

Proposition 2.3 *Let* G *be a locally finite group and let* p *be a prime. Let* P *be a finite* p*-subgroup of* G*. The following properties are equivalent:*

1) P *is a* p*-uniqueness subgroup of* G*.*

2) P *is singular in* G*.*

3) *Whenever* P_1 *and* P_2 *are finite* p*-subgroups of* G *with* $P \leqslant P_1 \cap P_2$ *then* $\langle P_1, P_2 \rangle$ *is a* p*-group.*

Proof — 1) ⇒ 2) Suppose P is not singular in G. Then we have a finite subgroup F of G such that P is contained in at least two Sylow p-subgroups P_1 and P_2 of F. Let S_i be a Sylow p-subgroup of G which contains P_i $(i = 1, 2)$. If $S_1 = S_2$ then $\langle P_1, P_2 \rangle \leqslant \langle S_1, S_2 \rangle \cap F$ is a p-group which contradicts $P_1 \in \mathrm{Syl}_p F$ and $P_2 \in \mathrm{Syl}_p F$. Thus $S_1 \neq S_2$. Therefore P is not a p-uniqueness subgroup of G.

2) ⇒ 3) Let $P \leqslant P_1 \cap P_2$ where P_1 and P_2 are finite p-subgroups of G and suppose that $F := \langle P_1, P_2 \rangle$ is not a p-group. Then $P \leqslant F$ and since $\langle P_1, P_2 \rangle$ is not a p-group there are two distinct Sylow p-subgroups Q_1 and Q_2 of F containing P_1 and P_2, respectively. But then $P \leqslant Q_1 \cap Q_2$ and so P is not singular in G.

3) ⇒ 1) Suppose that 3) holds and that P is not a p-uniqueness subgroup of G. Then there are distinct Sylow p-subgroups Q_1 and Q_2 of G such that $P \leqslant Q_1 \cap Q_2$. Let $x \in Q_1 \setminus Q_2$ and $y \in Q_2 \setminus Q_1$. It follows that $P_1 := \langle P, x \rangle$ and $P_2 := \langle P, y \rangle$ are finite p-groups and that $\langle P_1, P_2 \rangle$ is not a p-group, contradicting 3). □

Kegel discovered insight gaining equivalent conditions for the conjugacy of good Sylow p-subgroups in countable locally finite groups. We expandedly restate and improvedly reprove his result in our terminology thereby adding the property of the existence of a p-uniqueness subgroup. We also notice hat Kegel's argument for 2) ⇒ 4) on page 6 and following of [10] is really not fully convincing.

Theorem 2.4 (see [10], Theorem 1.2) *For the countable locally finite group* G *and the prime* p *the following properties are equivalent:*

1) *There exists a nested local system* $\{G_i \mid i \in \mathbb{N}\}$ *for* G *and an index* i_0 *such that for every pair* $j \geqslant i \geqslant i_0$ *of indices every Sylow* p*-subgroup* P_i *of* G_i *lies in a unique Sylow* p*-subgroup* P_j *of* G_j*.*

2) *There exists a finite* p*-subgroup* P_0 *of* G *which is singular in* G*.*

3) *There exists a* p*-uniqueness subgroup* P_0 *of* G*.*

4) *Any two good Sylow* p*-subgroups of* G *are conjugate in* G*.*

Proof — 1) ⇒ 2) Choose $P_{i_0} \in \mathrm{Syl}_p G_{i_0}$ and put $P_0 := P_{i_0}$. Let F be any finite subgroup of G containing P_0. For every index j such that $F \leqslant G_j$, the unique Sylow p-subgroup of G_j containing P_0 must contain a Sylow p-subgroup of F, and no other Sylow p-subgroup of F can contain P_0. Clearly 2) ⇒ 1). From Proposition 2.3 follow 2) ⇒ 3) and 3) ⇒ 2). To show 4) ⇒ 1) assume that for any nested local system $\{G_i \mid i \in \mathbb{N}\}$ for G and any index i_0, there are infinitely many pairs $j \geqslant i \geqslant i_0$ of indices for which some (and hence any by conjugation) Sylow p-subgroup of G_i is contained in at least two Sylow p-subgroups of G_j. We then can construct, similar to Theorem 3.2 or Theorem 3.8 below, $2^{\aleph_0}$ maximal p-subgroups of G which are good by Lemma 2.2 and cannot all be conjugate in G. Thus 4) entails 1), and hence 2). It remains to show 3) ⇒ 4). Let P and Q be good Sylow p-subgroups of G obtained as two unions of Sylow p-subgroups of nested local systems $\{G_i \mid i \in \mathbb{N}\}$ and $\{H_i \mid i \in \mathbb{N}\}$ for G (see Lemma 2.2) and let S_0 be the unique Sylow p-subgroup of G containing P_0; we show that P is conjugate to S_0 and S_0 is conjugate to Q, and therefore P is conjugate to Q; if P and S_0 are not conjugate then one of them must have property $(\star)$ of Theorem 3.1 (see below) which means in particular that it is not singular; so P has property $(\star)$; now P reduces into $\{G_i \mid i \in \mathbb{N}\}$, that is, $P \cap G_i \in \mathrm{Syl}_p G_i$ for all $i \in \mathbb{N}$; there exists an index i_0 such that $P_0 \leqslant G_{i_0}$; then $P_0 \leqslant P_{i_0}$ for some unique $P_{i_0} \in \mathrm{Syl}_p G_{i_0}$; now, by Sylow's classical theorem, let x be an element of G_{i_0} such that $P_{i_0}^x = P \cap G_{i_0}$; then $P_{i_0}^x$ is a finite p-subgroup of P which is contained in just only one Sylow p-subgroup of G thereby contradicting property $(\star)$ of P; for exactly the same reasons S_0 is conjugate to Q; therefore P must be conjugate to Q. □

Let S be a Sylow p-subgroup of the locally finite group G. A finite subgroup F of G is called S-*dominant* if S reduces into every

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subgroup U of G which contains F, that is, $S \cap U \in \mathrm{Syl}_p U$ for all subgroups U of G such that $F \leqslant U$.

Lemma 2.5 (see [4]) *Let G be a locally finite group, p a prime, $S \in \mathrm{Syl}_p G$ and F a finite subgroup of G. The following properties are equivalent:*

1) *F is S-dominant.*

2) *For each finite subgroup U of G with $F \leqslant U$ we have $S \cap U \in \mathrm{Syl}_p U$.*

Proof — 1) ⇒ 2) is clear, so we only need to prove that 2) implies 1). Since F is finite, there exists a local system Σ for G such that for each Σ-group U we have $F \leqslant U$. Let V be a subgroup of G with $F \leqslant V$. Then $\Sigma_1 := \{V \cap U \mid U \in \Sigma\}$ is a local system for V into which $S \cap V$ reduces. Therefore from Lemma 2.1 follows $S \cap V \in \mathrm{Syl}_p V$. □

Lemma 2.6 (see [4]) *Let G be a locally finite group and $S \in \mathrm{Syl}_p G$. The following properties are equivalent:*

1) *S is very good.*

2) *There exists an S-dominant subgroup of G.*

Proof — 1) ⇒ 2) Suppose no S-dominant subgroup of G exists. Then, according to Lemma 2.5, to every finite subgroup F of G there exists one finite subgroup U_F of G with $F \leqslant U_F$ and $S \cap U_F \notin \mathrm{Syl}_p U_F$. Then $\Sigma := \{U_F \mid F \text{ finite subgroup of } G\}$ is a local system for G that possesses no local subsystem into which S reduces.

2) ⇒ 1) Let F be an S-dominant subgroup of G and Σ a local system for G. Let $\Sigma_1 := \{U \mid U \in \Sigma \text{ and } F \leqslant U\}$. Then Σ_1 is, because of the S-dominance of F, a local subsystem of Σ into which S reduces. □

Lemma 2.7 (see [4]) *Let G be a locally finite group and let p be a prime.*

a) *If F is a p-uniqueness subgroup of G and S is the singular Sylow p-subgroup of G with $F \leqslant S$, then F is an S-dominant subgroup of G.*

b) *Every singular Sylow p-subgroup of G is very good.*

Proof — Since b) follows from a) and Lemma 2.6 we only need to prove a). Let U be a subgroup of G with $F \leqslant U$. Let $P \in \mathrm{Syl}_p U$ and $T \in \mathrm{Syl}_p G$ with $F \leqslant S \cap U \leqslant P \leqslant T$. From $F \leqslant S$ and the p-uniqueness of F follows $T = S$. Therefore $S \cap U \geqslant S \cap P = P$. □

The following consequence of this lemma is a relevant insight.

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Theorem 2.8 (see [4]) *Let* p *be a prime and* P *be a* p*-uniqueness subgroup of the locally finite group* G *(or, equivalently by Proposition 2.3, let* P *be a singular* p*-subgroup of* G*). Then the singular Sylow* p*-subgroup* S *of* G *containing* P *is very good.*

We can now summarise the relationship between good Sylow p-subgroups and p-uniqueness subgroups together with the Sylow p-subgroups containing them as follows:

- singular Sylow p-subgroups are very good;
- p-uniqueness subgroups are singular, and conversely;
- in countable locally finite groups good Sylow p-subgroups are identified by nested local systems;
- in countable locally finite groups the existence of a p-uniqueness subgroup compels the conjugacy of all good Sylow p-subgroups.

We end the discussion of good Sylow p-subgroups by pointing out that there exist 1) countable locally finite groups with Sylow p-subgroups which are not good (see the note at page 5 of [10]: "It may be worthwhile to point out that a countable infinite locally finite group may have maximal p-subgroups which" are not good) and 2) locally finite groups of cardinality $2^{\aleph_0}$ without good Sylow p-subgroups.

First, we let G be a finite group with $|\mathrm{Syl}_p G| \geqslant 2$, e.g. the symmetric group $\underline{S}^{2p}$ of degree 2p for the prime p for which we know surely that

$$|\mathrm{Syl}_p \underline{S}^{2p}| \geqslant 2p - 2 \geqslant 2.$$

Consider the $\mathbb{N}$*-fold cartesian power*

$$\begin{aligned} G^{[\mathbb{N}]} &:= \prod\{G_i \mid G_i := G \text{ for all } i \in \mathbb{N}\} \\ &= \{(x_1, x_2, \dots) \mid x_i \in G_i \text{ for all } i \in \mathbb{N}\} \end{aligned}$$

of G and notice that *it satisfies the Sylow* p*-Theorem*.

Proof — For $S, T \in \mathrm{Syl}_p G^{[\mathbb{N}]}$ there are $S_i, T_i \in \mathrm{Syl}_p G_i = \mathrm{Syl}_p G$ $(i \in \mathbb{N})$ such that S, resp. T, is the cartesian product of the S_i's, resp. the T_i's. If $x_i \in G_i = G$ with $S_i^{x_i} = T_i$ $(i \in \mathbb{N})$ and $x := (x_i)_{i \in \mathbb{N}}$, then $S^x = T$. □

The group $G^{[\mathbb{N}]}$ contains the $\mathbb{N}$*-fold direct power*

$$G^{(\mathbb{N})} := \prod{}^0 \{(x_i)_{i \in \mathbb{N}} \in G^{[\mathbb{N}]} \mid x_i = 1 \text{ for almost all } i \in \mathbb{N}\},$$

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which *does not satisfy the Sylow* p*-Theorem*.

Proof — Let $S, T \in \mathrm{Syl}_p G^{(\mathbb{N})}$. If there is an $x \in G^{(\mathbb{N})}$ with $S^x = T$, then $S^{x\pi_i} = T^{\pi_i}$ for almost all $i \in \mathbb{N}$. Thus for $P, Q \in \mathrm{Syl}_p G$ with $P \neq Q$, the groups $P^{(\mathbb{N})}$ and $Q^{(\mathbb{N})}$ are not in $G^{(\mathbb{N})}$ — but in $G^{[\mathbb{N}]}$ — conjugate Sylow p-subgroups of $G^{(\mathbb{N})}$. Alternatively, it follows from $|G^{(\mathbb{N})}| = \aleph_0$ and $|\mathrm{Syl}_p G^{(\mathbb{N})}| = 2^{\aleph_0}$ — since $|\mathrm{Syl}_p G| \geqslant 2$ we can refer to Theorems 3.1 and 3.2 (see below) — that not all Sylow p-subgroups of $G^{(\mathbb{N})}$ can be conjugate. □

The example $G^{(\mathbb{N})} \leqslant G^{[\mathbb{N}]}$ shows that in uncountable locally finite groups the Sylow p-Theorem is not inherited by normal subgroups.

Moreover, $G^{[\mathbb{N}]}$ contains *the diagonal subgroup*

$$D := \left\{ (x_i)_{i\in\mathbb{N}} \in G^{[\mathbb{N}]} \mid (\exists\, x \in G)(\forall\, i \in \mathbb{N})\, x_i = x \right\} \simeq G$$

via the isomorphism

$$\delta : D \longrightarrow G, \quad \left((x_i)_{i\in\mathbb{N}}\right)^{\delta} := x,$$

from D onto G with $D \cap G^{(\mathbb{N})} = \langle 1 \rangle$. Since $G^{(\mathbb{N})}$ is a normal subgroup of $G^{[\mathbb{N}]}$, we have $\langle G^{(\mathbb{N})}, D \rangle = DG^{(\mathbb{N})}$; this is a countable subgroup of $G^{[\mathbb{N}]}$. The Sylow p-subgroups of $G^{[\mathbb{N}]}$ (resp. of $G^{(\mathbb{N})}$) are cartesian (resp. direct) products of the Sylow p-subgroups of the G_i's ($i \in \mathbb{N}$), namely $\prod\{S_n^{\pi_i} \mid i \in \mathbb{N}\}$ (resp. $\prod^0\{S_n^{\pi_i} \mid i \in \mathbb{N}\}$) for $S_n \in \mathrm{Syl}_p G$ ($n \in \mathbb{N}$), where π_i: $G^{[\mathbb{N}]} \to G_i$ is the projection $\pi_i\left((x_k)_{k\in\mathbb{N}}\right) := x_i$ on the factor G_i ($i \in \mathbb{N}$). Any $P \in \mathrm{Syl}_p D$ normalises exactly one Sylow p-subgroup $S(P)$ of $G^{[\mathbb{N}]}$ (resp. exactly one Sylow p-subgroup $S^0(P)$ of $G^{(\mathbb{N})}$), namely $S(P) = \prod\{P^{\pi_i} \mid i \in \mathbb{N}\}$ (resp. $S^0(P) = \prod^0\{P^{\pi_i} \mid i \in \mathbb{N}\}$). Therefore every Sylow p-subgroup of D is a p-uniqueness subgroup of $DG^{(\mathbb{N})}$, and $P\,S^0(P)$, for $P \in \mathrm{Syl}_p D \simeq \mathrm{Syl}_p G$, is a singular Sylow p-subgroup of $DG^{(\mathbb{N})}$ and so is good, even very good, by Theorem 2.8; these Sylow p-subgroups are conjugate: if $P_1, P_2 \in \mathrm{Syl}_p D$ and $P_1^x = P_2$ with $x \in D$, then

$$\left(P_1\, S^0(P_1)\right)^x = \left(P_1 \textstyle\prod^0\{P_1^{\pi_i} \mid i \in \mathbb{N}\}\right)^x$$
$$= P_2 \textstyle\prod^0\{P_2^{\pi_i} \mid i \in \mathbb{N}\} = P_2\, S^0(P_2)$$

(see also Theorem 2.4). The countable group $G^{(\mathbb{N})}$ also has by Lemma 2.2 good Sylow p-subgroups, which are not conjugate, and we

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are able to designate some distinguished of them explicitly: let

$$U_i := G_1 \times G_2 \times \ldots \times G_i \quad (i \in \mathbb{N});$$

then $\Sigma := \{U_i \mid i \in \mathbb{N}\} \cap G^{(\mathbb{N})}$ is a nested local system for $G^{(\mathbb{N})}$; if $P_i \in \mathrm{Syl}_p G_i = \mathrm{Syl}_p G$ $(i \in \mathbb{N})$, then

$$P^0 := \big(P_1 \times P_2 \times \ldots\big) \cap G^{(\mathbb{N})}$$

is a p-subgroup of $G^{(\mathbb{N})}$ which reduces into Σ and thus is a good Sylow p-subgroup of $G^{(\mathbb{N})}$ by Lemma 2.1.

The group $DG^{(\mathbb{N})}$ has indeed also (many) Sylow p-subgroups, which are not good: since $|\mathrm{Syl}_p G| \geqslant 2$ we can construct using the method employed in the proof of Theorem 3.2 or that employed in the proof of Theorem 3.8 an infinitely ($\aleph_0$) high tree of finite p-subgroups of $DG^{(\mathbb{N})}$ with $\langle 1 \rangle$ as the root which branches properly at each location with proper inclusions and where two immediate successors of each point do not generate a p-group; this tree has $2^{\aleph_0}$ branches which constitute $2^{\aleph_0}$ many ascending unions of finite p-subgroups and thus $2^{\aleph_0}$ many p-subgroups P_ι where any two of them do not generate a p-group; choosing for each P_ι a Sylow p-subgroup S_ι of $DG^{(\mathbb{N})}$ containing P_ι now gives $2^{\aleph_0}$ Sylow p-subgroups of $DG^{(\mathbb{N})}$ $(1 \leqslant \iota \leqslant 2^{\aleph_0})$ on the treetop; since the good Sylow p-subgroups of the countable group $DG^{(\mathbb{N})}$ are conjugate (Theorem 2.4), at most $\aleph_0$ of these $2^{\aleph_0}$ Sylow p-subgroups can be good; there remain (with or without the continuum hypothesis) at least $2^{\aleph_0} - \aleph_0$ many Sylow p-subgroups in the treetop which are not good and too many to be conjugate in $DG^{(\mathbb{N})}$. We note that Rae [12] constructs, by introducing the unwieldy concept of "weakly goodness" and by referring to another group he constructed (see [12], 5.11), a countable locally soluble group possessing a Sylow p-subgroup which is not good (see [12], 5.31). This example is much more complicated than ours.

Second, let p and q be primes with $q \equiv 1 \pmod p$ and

$$A := \langle a, b \mid a^p = b^q = (ab)^p = 1 \rangle.$$

Then $|A| = pq$ and A has q Sylow p-subgroups and a normal Sylow q-subgroup, so is metabelian. If $(p, q) = (2, 3)$, then $A = \underline{S}^3$ is the symmetric group of degree 3. The group A contains the elements a and $a' := ab$ of order p which are not p-*consonant*, that is, they do

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not generate a p-group. The $\mathbb{N}$-fold cartesian power $A^{[\mathbb{N}]}$ of A is locally finite and metabelian of exponent pq. László G. Kovács, Bernhard H. Neumann and Hugo de Vries constructed, based on the elements a and a' (and exemplarily for $(p, q) = (2, 3)$), an $\mathbb{N}$-fold interdirect power H of A, that is, $A^{(\mathbb{N})} \leqslant H \leqslant A^{[\mathbb{N}]}$, with the following properties (see [11], Theorem 3.7): H is metabelian of exponent q and order $2^{\aleph_0}$ with a countable Sylow p-subgroup and a Sylow p-subgroup of order $2^{\aleph_0}$ (hence without Sylow Theorem for the prime p). They also constructed, using again a and a', an $\mathbb{N}$-fold interdirect power H of A with the following amazing properties (see [11], Theorem 4.4, and also [12], 1.13): H has order $2^{\aleph_0}$, each Sylow p-subgroup of H is countable, H has a countable normal (hence unique) Sylow q-subgroup, which has no complement in H, and each Sylow p-subgroup has a complement in H, which is normal in H and contains elements of order p. No Sylow p-subgroup of H can be good: suppose a Sylow p-subgroup S of H reduces into a local system Σ for H; we then choose a Σ-group U containing an element x of order p of a complement of S, and a $P \in \mathrm{Syl}_p U$ containing x; since $S \cap U \in \mathrm{Syl}_p U$ there is a $y \in U$ with $P^y = S \cap U$; then $\langle x \rangle^y \leqslant S$ whereas $\langle x \rangle^y$ belongs to the normal complement of S, which is a contradiction.

In the following section we shall point out that there exist countable locally finite groups 3) without singular Sylow p-subgroups, 4) with good Sylow p-subgroups which are not very good, and 5) with very good Sylow p-subgroups which are not singular.

3 Basic theorems of Sylow theory in locally finite groups and our "Charakterisierungssatz"

In this section we first present — with quite considerably improved proofs — the basics of Sylow theory in locally finite groups (Theorem 3.1 to Theorem 3.5) and subsequently prepare and carry out the proof of our *Charakterisierungssatz* (Theorem 3.6 to Theorem 3.9) which, in turns, allows us to prove very easily our main theorem (Theorem 3.10).

In the following statement, the property $(\star)$ means that S is not singular; see the same property $(*)$ on page 8 of [10]. This property was for the first time discovered by Ali O. Asar [1].

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Theorem 3.1 (see [4], and Theorem 3.6 below for a generalisation) *Any locally finite group* G *which does not satisfy the Sylow Theorem for the prime* p *contains a Sylow* p*-subgroup* S *with the following property:*

($\star$) *Every finite subgroup of* S *lies in at least two Sylow* p*-subgroups of* G.

Proof — Let S and T be two Sylow p-subgroups of G which are not conjugate (in G). If T is not singular, that is, T does have property ($\star$), the result is immediate, so suppose that T is singular and let Y be a p-uniqueness subgroup for T. We show that then S has property ($\star$), that is, S is not singular. To this end let X be an arbitrary finite subgroup of S. Then $\langle X, Y\rangle$ is a finite group. According to the Sylow p-Theorem for finite groups there is an $x \in G$ such that X and Y^x lie in the same Sylow p-subgroup of $\langle X, Y\rangle$. Then $\langle X, Y^x\rangle$ is a p-group. From the assumption on Y it now follows that $\langle Y^x, X\rangle \leqslant T^x$. Hence X lies in at least the two Sylow p-subgroups S and T^x of G. Therefore X is not a p-uniqueness subgroup for S. □

We now prepare an alternative proof of the basic theorem of Sylow theory known as the "Asar-Hartley theorem" (see [1] and [3], Theorem 2.3.11, for the original proof). Our proofs of Theorem 3.2 a) and b) with reference to a) are much clearer and more detailed than the original proof by Asar, which may be considered rather cumbersome. Note also that in [10], Theorem 1.3, Kegel sagely combines Theorem 3.1 with Theorem 3.2 c).

Theorem 3.2 (see [4]) *Let* G *be a locally finite group and let* P *be a* p*-subgroup of* G *for the prime* p.

a) *Suppose* P *has the following property:* (†) *To every finite subgroup* F *of* P *there exists an* $x = x\,(F) \in G$ *with* $F^x \leqslant P$ *such that* $\langle P, P^x\rangle$ *is not a* p*-group. Then there are* $2^{\aleph_0}$ *infinite ascending chains*

$$X_{i_1} < X_{i_1 i_2} < \ldots < X_{i_1 i_2 \ldots i_n} < \ldots$$

of finite p*-subgroups of* G *with indices* $i_k \in \{0, 1\}$ $(k \in \mathbb{N})$ *such that for all* $n \in \mathbb{N}$ *and each choice of indices* i_k $(1 \leqslant k \leqslant n)$, *the group* $\langle X_{i_1 i_2 \ldots i_n 0}, X_{i_1 i_2 \ldots i_n 1}\rangle$ *is not a* p*-group.*

b) *Let* $P \in \mathrm{Syl}_p G$ *with the property* ($\star$). *Then* P *has property* (†).

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c) *Let* $P \in \mathrm{Syl}_p G$ *with the property* $(\star)$ *and let* X *be a finite subgroup of* P*. Then there are* $2^{\aleph_0}$ *many infinite ascending chains*

$$X < X_{i_1} < X_{i_1 i_2} < \ldots < X_{i_1 i_2 \ldots i_n} < \ldots$$

with the properties from point a)*.*

Proof — a) Let X be a finite subgroup of P and y an element of G such that: 1) $\langle P, P^y \rangle$ is not a p-group, and 2) $X^y \leqslant P$. Because of the first property there exists a finite subgroup X_0 of P with $X \leqslant X_0$ such that $\langle X_0, X_0^y \rangle$ is not a p-group, and because of the second property we have $\langle X, X^y \rangle \leqslant P$, hence $X_0 \neq X \neq X_0^y$. If we substitute in the last two sentences X by X_0, we get two finite p-subgroups X_{00} and X_{01} of G with $X_0 < X_{00}$ and $X_0 < X_{01}$ such that $\langle X_{00}, X_{01} \rangle$ is not a p-group. Since P^y has the property $(\dagger)$, too, we can quite analogously substitute X by $X_1 := X_0^y$ and so get two finite p-subgroups X_{10} and X_{11} of G with $X_1 < X_{10}$ and $X_1 < X_{11}$ such that the subgroup $\langle X_{10}, X_{11} \rangle$ is not a p-group. We now have constructed four ascending chains

$$X < X_0 < X_{00}, \quad X < X_0 < X_{01}, \quad X_1 < X_{10} \quad \text{and} \quad X_1 < X_{11}$$

of finite p-subgroups of G such that the subgroups $\langle X_0, X_1 \rangle, \langle X_{00}, X_{01} \rangle$ and $\langle X_{10}, X_{11} \rangle$ are not p-groups. Now let $n \in \mathbb{N}$ with $n \geqslant 2$ and let already be constructed 2^n ascending chains

$$X_{i_1} < X_{i_1 i_2} < \ldots < X_{i_1 i_2 \ldots i_n}$$

of finite p-subgroups of G with indices $i_k \in \{0, 1\}$ $(1 \leqslant k \leqslant n)$ such that for each $m \in \mathbb{N}$ with $m \leqslant n-1$ and each choice of indices i_k $(1 \leqslant k \leqslant m)$ the subgroup $\langle X_{i_1 i_2 \ldots i_m 0}, X_{i_1 i_2 \ldots i_m 1} \rangle$ of G is not a p-group. Whilst repeating the construction of the first two sentences successively with the 2^n groups $X_{i_1 i_2 \ldots i_n}$ in place of X, we get, because each conjugate of P possesses the property $(\dagger)$, in each case two p-subgroups $X_{i_1 i_2 \ldots i_n 0}$ and $X_{i_1 i_2 \ldots i_n 1}$ of G such that

$$X_{i_1 i_2 \ldots i_n} < X_{i_1 i_2 \ldots i_n 0} \cap X_{i_1 i_2 \ldots i_n 1}$$

and

$$\langle X_{i_1 i_2 \ldots i_n 0}, X_{i_1 i_2 \ldots i_n 1} \rangle$$

is not a p-group. Therewith we now have constructed 2^{n+1} ascend-

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ing chains

$$X_{i_1} < X_{i_1 i_2} < \ldots < X_{i_1 i_2 \ldots i_n} < X_{i_1 i_2 \ldots i_n i_{n+1}}$$

having the requested properties. Therefore we can w.r.t. inclusion recursively construct a tree of height $\aleph_0$ of finite p-subgroups of G, which branches properly at each location with proper inclusions, hence must contain $2^{\aleph_0}$ infinite branches. Also any two immediate successors of an arbitrary point do not generate a p-group. These branches are just the required chains.

b) Let F be a finite subgroup of P and R be a Sylow p-subgroup of G with $F \leqslant R \neq P$. Then there is an element x in R with $x \notin P$ and the group $\langle F, x \rangle$ is a finite p-group. Let $Y := \langle F, x \rangle \cap P$. Then we have $Y \neq \langle F, x \rangle$. It is well-known that as a finite p-group $\langle F, x \rangle$ satisfies the normaliser condition. Therefore Y is a proper subgroup of $N_{\langle F, x \rangle}(Y)$. Let y be an element in $\langle F, x \rangle$, but not in Y, which normalises Y. Then $y \notin P$. Since y is a p-element and P by assumption a Sylow p-subgroup of G, it follows that $y \notin N_G(P)$ and that $\langle P, P^y \rangle$ is not a p-group. This is the property (†) from point a) for P.*

c) We combine the proofs of point a) and point b). Let $R \in \mathrm{Syl}_p G$ with $X \leqslant R \neq P$, $x \in R \backslash P$ and $T := P \cap \langle X, x \rangle$. Being a finite p-group, $\langle X, x \rangle$ satisfies the normaliser condition. Hence there exists a $t \in \langle X, x \rangle \backslash T$ with $t \in N_{\langle X, x \rangle}(T)$. Then $\langle P, P^t \rangle$ is not a p-group, since else $t \in P$, and so there exists a finite subgroup X_0 of P with $X \leqslant X_0$ such that with $X_1 := X_0^t$ the group $\langle X_0, X_1 \rangle$ is not a p-group. Thus, we have $X_0 \neq X \neq X_1$ since $\langle X, X^t \rangle \leqslant T$ is a p-group. Of course, $X \leqslant X_0$, but also $X \leqslant X_1$ because of $t \in X$. We can repeat this construction whilst replacing X by X_0 and also by its conjugate X_1. Thereby we construct subgroups $X_{00}, X_{01}, X_{10}, X_{11}$ and four ascending chains

$$X < X_0 < X_{00},\ X < X_0 < X_{01},\ X < X_1 < X_{10} \text{ and } X < X_1 < X_{11}$$

of finite p-subgroups of G. We subsequently repeat this construction with each of the $X_{i_1 i_2}$'s and whilst doing this infinitely often we construct $2^{\aleph_0}$ many chains

$$X < X_{i_1} < X_{i_1 i_2} < \ldots < X_{i_1 i_2 \ldots i_n} < \ldots$$

* Asar [1, Lemma 1] (unwieldy) considers instead of R a p-subgroup Y of G such that $Y < U$ $(= P)$, chooses $y \in Y \backslash U$, defines $F^* := U \cap \langle F, y \rangle$ with $F \leqslant U \cap Y$, finds $F \leqslant F^*$ and $N_{\langle F,\, y \rangle}(F^*) > F^*$, and finally concludes $N_G(F^*) < N_G(U)$, since U is the unique maximal p-subgroup of $N_G(U)$ and $N_{\langle F,\, y \rangle}(F^*) < U$.

of finite p-subgroups of G with the properties from point a). So we can, starting from an arbitrary subgroup X of P as a "minimal point" or a "root", recursively w.r.t. inclusion construct a tree of height $\aleph_0$ of finite p-subgroups of G, which branches properly at each location, hence must contain $2^{\aleph_0}$ infinite branches. Also any two immediate successors of an arbitrary point do not generate a p-group. These branches are just the required chains. □

Theorem 3.2 enables us to prove very easily the "Asar-Hartley theorem" which characterises locally finite groups satisfying the strong Sylow Theorem for the prime p by a cardinality result without the need to endeavour the continuum hypothesis (for a proof closer to the original one of Asar, the reader can consult [10], pp. 8–9).

Theorem 3.3 (see Asar [1], Hartley [6],[8][*]) *Let* G *be a locally finite group and* p *be a prime. Suppose that for every countable subgroup* H *of* G *we have* $|\mathrm{Syl}_p H| < 2^{\aleph_0}$. *Then* G *satisfies the strong Sylow* p*-Theorem.*

Proof — Suppose G does not satisfy the strong Sylow Theorem for the prime p. Then there is a subgroup U of G which does not satisfy the Sylow Theorem for the prime p. Thus according to Theorems 3.1 and 3.2 there are $2^{\aleph_0}$ many infinite ascending chains

$$X_{i_1} < X_{i_1 i_2} < \ldots < X_{i_1 i_2 \ldots i_n} < \ldots$$

of finite p-subgroups of U with the properties from point a) of Theorem 3.2. Let $\mathcal{M}$ be the set of all p-subgroups of U which are an ascending union of one of these chains. Then it follows $|\mathcal{M}| = 2^{\aleph_0}$ and that any two $\mathcal{M}$-groups cannot generate a p-group. Now let

$$H_n := \langle X_{i_1 i_2 \ldots i_n} \mid i_k \in \{0,1\},\ 1 \leqslant k \leqslant n \rangle \quad (n \in \mathbb{N})$$

and

$$H := \bigcup_{n \in \mathbb{N}} H_n.$$

Then H is a countable subgroup of U and so of G. Since H contains every $\mathcal{M}$-group it follows that $|\mathrm{Syl}_p H| = 2^{\aleph_0}$. This contradicts the assumption on the countable subgroups of G. □

[*] The result for countable locally finite groups was obtained independently by Brian Hartley using a quite different method which allowed him to generalise it from the prime p to a set of primes π when the finite groups of a nested local system have each a nilpotent Hall π-subgroup (see [6]). However, Hartley has extended his proof in [8] to uncountable locally finite groups by another beautiful method.

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The cardinality statement of Theorem 3.3 has an immediate first corollary for countable locally finite groups.

Theorem 3.4 *Let* G *be a countable locally finite group. The following properties are equivalent:*

1) *For every (countable) subgroup* H *of* G *we have* $|\mathrm{Syl}_p H| < 2^{\aleph_0}$.
2) G *satisfies the strong Sylow Theorem for the prime* p.
3) G *satisfies the Sylow Theorem for the prime* p.
4) $|\mathrm{Syl}_p G| < 2^{\aleph_0}$.
5) *Every (countable) subset of* G *is contained in a subgroup* U *of* G *with* $|\mathrm{Syl}_p U| < 2^{\aleph_0}$.

The second corollary of Theorem 3.3 would certainly as a conjugacy assertion be very difficult to be proved but is as a cardinality statement trivial. Recall first that a class of groups $\mathfrak{X}$ is *countably recognisable* if, whenever all countable subgroups of a group G belong to $\mathfrak{X}$, then G itself is an $\mathfrak{X}$-group (see Baer [2]).

Theorem 3.5 *The locally finite group* G *satisfies the strong Sylow Theorem for the prime* p *if and only if every countable subgroup of* G *satisfies the strong Sylow Theorem for the prime* p*. In particular, the class* Syl-p *of all locally finite groups satisfying the strong Sylow Theorem for the prime* p *is countably recognisable.*

We now can prove our key discovery whenever the Sylow Theorem for the prime p is not valid in a countable locally finite group which shows a symmetry between not conjugate Sylow p-subgroups.

Theorem 3.6 *Let* G *be a countable locally finite group and* p *be a prime. If two Sylow* p*-subgroups of* G *are not conjugate, then neither is singular.*

Proof — Let S and T be Sylow p-subgroups of G which are not conjugate. We saw in Theorem 3.1 that one of S or T is not singular. Without loss of generality (w.l.o.g.) we may suppose that S is not singular. To prove the result we must show that T is not singular either. If T is not good, it cannot be singular, since by Theorem 2.8 singular Sylow p-subgroups are very good. So let T be good w.r.t. the nested local system $\{G_n \mid n \in \mathbb{N}\}$ for G and let F be an arbitrary finite

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subgroup of T. We show that F cannot be a p-uniqueness subgroup for T and so T is not singular since F is chosen arbitrarily. Since S and T are not conjugate, we have $S \neq T$.

There exists an $m = m(F) \in \mathbb{N}$ with $F \leqslant G_m$. After the renumeration $\{n \mapsto n+m-1 \mid n \in \mathbb{N}\}$, it is possible to assume $F \leqslant G_1$. Then $F \leqslant T \cap G_1 \in \mathrm{Syl}_p G_1$. If $T \cap G_n$ is the unique Sylow p-subgroup of G_n for all $n \in \mathbb{N}$ then T is the unique Sylow p-subgroup of G and we obtain the contradiction that $S = T$. Hence there is an $n \in \mathbb{N}$ such that G_n has a Sylow p-subgroup R with $R \neq T \cap G_n$. Renumbering again if needed we may assume that $R \in \mathrm{Syl}_p G_1$ with $R \neq T \cap G_1$. Choose $y \in R \backslash (T \cap G_1)$, so in particular $y \notin T$. By the Sylow p-Theorem for finite groups there is an $x \in G_1$ such that $(T \cap G_1)^x = R$ and so $F^x \leqslant R$ since $F \leqslant T \cap G_1$. From $\langle F^x, y\rangle \leqslant R$ follows that $\langle F^x, y\rangle$ is a finite p-group. Let $Y := \langle F^x, y\rangle \cap T$. Then $Y \neq \langle F^x, y\rangle$ since $y \notin T$.

But Y satisfies, as is well-known, the normaliser condition and so we can choose $z \in N_{\langle F^x, y\rangle}(Y) \backslash Y$. Then $z \notin T$ since otherwise z belongs to $T \cap \langle F^x, y\rangle = Y$. But z is a p-element outside of T and $T \in \mathrm{Syl}_p G$, and so $z \notin N_G(T)$. Therefore $\langle T, T^z\rangle$ is not a p-group. In particular, $T \neq T^z$ and $F \leqslant T \cap T^z$. Therefore the arbitrarily chosen F is not a p-uniqueness subgroup for T. □

Whenever a countable locally finite group contains a singular Sylow p-subgroup then all good Sylow p-subgroups will be conjugate by Theorem 2.4. Whenever every countable subgroup of a (countable) locally finite group contains a singular Sylow p-subgroup then all Sylow p-subgroups are conjugate. This core insight is spelled out by the following theorem.

Theorem 3.7 (see [4]) *Let* G *be a locally finite group and let* p *be a prime. Suppose that every countable subgroup of* G *contains a singular Sylow* p*-subgroup. Then* G *satisfies the strong Sylow Theorem for the prime* p.

Proof — According to Theorem 3.5 we can assume that G is countable, and according to Theorem 3.4 it suffices to show that G satisfies the Sylow Theorem for the prime p. However, this is now immediate since by assumption G has a singular Sylow p-subgroup S. Let T be any Sylow p-subgroup of G. If S and T are not conjugate, then by Theorem 3.6 neither is singular. With this contradiction S and T are conjugate and the result follows. □

Since the above result is very significant, we provide an alternative proof by proving the contrapositive.

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Proof — Suppose G does not satisfy the Sylow Theorem for the prime p. Then, according to Theorem 3.1, Theorem 3.2 b), and Theorem 3.2 a), there are $2^{\aleph_0}$ infinite ascending chains

$$X_{i_1} < X_{i_1 i_2} < \ldots < X_{i_1 i_2 \ldots i_n} < \ldots$$

of finite p-subgroups of G with the properties from Theorem 3.2 a). Let

$$U_n := \langle X_{i_1 i_2 \ldots i_n} \mid i_k \in \{0, 1\}, 1 \leqslant k \leqslant n \rangle \quad (n \in \mathbb{N})$$

and

$$U := \bigcup_{n \in \mathbb{N}} U_n = \langle X_{i_1 i_2 \ldots i_n} \mid i_k \in \{0, 1\}, 1 \leqslant k \leqslant n \in \mathbb{N} \rangle.$$

Then U is a (countable) subgroup of G and $\{U_n \mid n \in \mathbb{N}\}$ is a nested local system for U. We show that U does not contain any singular Sylow p-subgroup. Let F^* be a finite p-subgroup of U. There exists an $m = m(F^*) \in \mathbb{N}$ with $F^* \leqslant U_m$. By definition of U_m there are indices $j_1, j_2, \ldots, j_m, \ldots, k_1, k_2, \ldots, k_m, \ldots, l_1, l_2, \ldots, l_m$ with

$$F^* \leqslant \langle X_{j_1 j_2 \ldots j_m}, X_{k_1 k_2, \ldots, k_m}, \ldots, X_{l_1 l_2 \ldots l_m} \rangle.$$

Then

$$P_1 := \langle X_{j_1 j_2 \ldots j_m 0}, X_{k_1 k_2 \ldots k_m 0}, \ldots, X_{l_1 l_2 \ldots l_m 0} \rangle$$

and

$$P_2 := \langle X_{j_1 j_2 \ldots j_m 1}, X_{k_1 k_2 \ldots k_m 1}, \ldots, X_{l_1 l_2 \ldots l_m 1} \rangle$$

are finite p-subgroups of U with $F^* \leqslant P_1 \cap P_2$ such that $\langle P_1, P_2 \rangle$ is not a p-group. We now choose $Q_{1,0}, Q_{2,0} \in \mathrm{Syl}_p U_m$ with $P_1 \leqslant Q_{1,0}$ and $P_2 \leqslant Q_{2,0}$. If

$$Q_{1,0} \leqslant Q_{1,1,} \leqslant \ldots \leqslant Q_{1,n} \quad \text{and} \quad Q_{2,0} \leqslant Q_{2,1} \leqslant \ldots \leqslant Q_{2,n}$$

are already p-subgroups of U with $Q_{1,i}, Q_{2,i} \in \mathrm{Syl}_p U_{m+i}$ $(0 \leqslant i \leqslant n)$, let $Q_{1,n+1}, Q_{2,n+1} \in \mathrm{Syl}_p U_{m+n+1}$ such that $Q_{1,n} \leqslant Q_{1,n+1}$ and $Q_{2,n} \leqslant Q_{2,n+1}$ $(n \in \mathbb{N}_0)$. Let

$$Q_1 := \bigcup_{n \in \mathbb{N}_0} Q_{1,n} \quad \text{and} \quad Q_2 := \bigcup_{n \in \mathbb{N}_0} Q_{2,n}.$$

Then Q_1 and Q_2 are both p-subgroups of U with $F^* \leqslant Q_1 \cap Q_2$ such

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that $\langle Q_1, Q_2\rangle$ is not a p-group. Per construction, Q_1 and Q_2 reduce into the nested local system $\{U_{m+n} \mid n \in \mathbb{N}_0\}$ for U. By Lemma 2.1, the groups Q_1 and Q_2 are two good Sylow p-subgroups of U containing F^*, that is, F^* is not a p-uniqueness subgroup of U. Thus U does not contain any p-uniqueness subgroup. □

Third, we supplement Theorem 3.7 with an example of a countable locally finite group H without the (strong) Sylow Theorem for the prime p but with a (countable) subgroup U without singular Sylow p-subgroups. Let $H := DG^{(\mathbb{N})}$ be the group from p. 21, $V := G^{(\mathbb{N})}$ and F be a finite subgroup of the good Sylow p-subgroup P^0 of V from p. 21. We show that F cannot be a p-uniqueness subgroup of V. Since F is finite, there is an $m = m(F) \in \mathbb{N}$ with $F \leqslant U_m$. Because of $|\mathrm{Syl}_p G| \geqslant 2$ there is a $Q_{m+1} \in \mathrm{Syl}_p G_{m+1}$ with $Q_{m+1} \neq P_{m+1}$. Then

$$Q^0 := \big(P_1 \times P_2 \times \ldots \times P_m \times Q_{m+1} \times P_{m+2} \times \ldots\big) \cap G^{(\mathbb{N})}$$

contains the group F and we have $Q^0 \neq P^0$. So V has the distinguished good Sylow p-subgroup P^0 which is not singular (notice that by Theorem 3.1 there must be such a Sylow subgroup since V does not satisfy the Sylow p-Theorem). By the second part of the proof of Theorem 3.7, there is a (countable) subgroup U of V which does not contain any singular Sylow p-subgroup.

Fourth, let $G = \underline{S}^{(\mathbb{N})}$ be the countable locally finite group of finitary permutations on a countably infinite set (that is, which move only finitely many elements), p a prime, and $\{n_i \mid i \in \mathbb{N}\}$ a sequence in $\mathbb{N}$ with $n_i + 2p \leqslant n_i + 1$ $(i \in \mathbb{N})$. Then $\Sigma := \{\underline{S}^{n_i} \mid i \in \mathbb{N}\}$ is a nested local system for G. By Lemma 2.2 b) there exists an $S \in \mathrm{Syl}_p G$ which is good w.r.t. Σ. We know that $|\mathrm{Syl}_p \underline{S}^{2p}| \geqslant 2p-2 \geqslant 2$. Let $T_1, T_2 \in \mathrm{Syl}_p \underline{S}^{2p}$ with $T_1 \neq T_2$. Let $i \in \mathbb{N}$. Then

$$\underline{S}^{n_i} \leqslant \underline{S}^{n_i} \times \underline{S}^{2p} \leqslant \underline{S}^{n_{i+1}}.$$

We put $F_i := \underline{S}^{n_i} \times T_2$, if $S \cap \underline{S}^{2p} = T_1$, and $F_i := \underline{S}^{n_i} \times T_1$ otherwise. Then we have $S \cap F_i \notin \mathrm{Syl}_p F_i$ and $\underline{S}^{n_i} \leqslant F_i \leqslant \underline{S}^{n_{i+1}}$. Hence $\{F_i \mid i \in \mathbb{N}\}$ is a (nested) local system for G containing no local subsystem of Σ into which S reduces. Thus S is a good Sylow p-subgroup of G which is not very good.

Fifth, the good Sylow p-subgroup P^0 of $V := G^{(\mathbb{N})}$ provides an example of a Sylow p-subgroup which is very good but not singular.

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Let Σ^* be a local system for V; by Lemma 2.2 a) there exists a nested local subsystem $\Sigma_1 = \{V_n \mid n \in \mathbb{N}\}$ of Σ and by Lemma 2.2 b) there is a Sylow p-subgroup Q of V which is good w.r.t Σ_1. Since P^0 is good w.r.t. $\Sigma = \{U_i \mid i \in \mathbb{N}\}$, it will contain a conjugate of every finite p-subgroup P of V: there is a Σ-group $U = U(P)$ with $P \leqslant U$; let $R \in \mathrm{Syl}_p U$ with $P \leqslant R$; by Sylow Theorem there is a $y \in U$ with $R^y = P^0 \cap U$; hence $P^y \leqslant P^0$. Therefore

$$(Q \cap V_n)^{x_n} \leqslant V_n^{x_n} \cap P^0$$

for some $x_n \in V$ ($n \in \mathbb{N}$). Thus $V_n^{x_n} \cap P^0$ is a Sylow p-subgroup of V_n and therefore $|P^0 \cap V_n| = |Q \cap V_n|$. It follows that $P^0 \cap V_n$ has the size of a Sylow p-subgroup of V_n ($n \in \mathbb{N}$), and consequently P^0 reduces into the subsystem Σ_1 of the given local system Σ^*.

The following core result may be very well-known but we can present a novel and shorter proof.

Theorem 3.8 (see [4]) *Let* G *be a locally finite group and let* p *be a prime. To any finite* p*-subgroup* P *of* G *shall pertain two finite* p*-subgroups* P_1 *and* P_2 *of* G *with* $P \leqslant P_1 \cap P_2$ *such that* $\langle P_1, P_2 \rangle$ *is not a* p*-group. Then there will exist a countable subgroup* H *of* G *with* $|\mathrm{Syl}_p H| = 2^{\aleph_0}$.

Proof — We construct recursively an infinite ascending chain

$$F_0 < F_1 < \ldots < F_n < \ldots$$

of finite subgroups of G and for every $n \in \mathbb{N}_0$ a set Σ_n of p-subgroups of F_n such that for every $n \in \mathbb{N}_0$ we have: (i) $|\Sigma_n| = 2^n$; (ii) every two Σ_n-groups do not generate a p-group; (iii) for $n \geqslant 1$ every Σ_{n-1}-group lies in at least two Σ_n-groups.

Let $F_0 := \langle 1 \rangle$ and $\Sigma_0 := \{\langle 1 \rangle\}$. Let $n \in \mathbb{N}$ and suppose

$$F_0 < F_1 < \ldots < F_{n-1} \quad \text{and} \quad \{\Sigma_i \mid i < n\}$$

have already been constructed. We let Σ_n be the set of all finite p-subgroups P_1, P_2 of G such that $\langle P_1, P_2 \rangle$ is not a p-group and there exists exactly one Σ_{n-1}-group P with $P \leqslant P_1 \cap P_2$. From the properties (i)–(iii) of Σ_{n-1} and from the prerequisite on G then follow (i)–(iii) for Σ_n. Let F_n be the span of all Σ_n-groups. Hereafter F_n is a finite

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subgroup of G with $F_{n-1} < F_n$. Let

$$H := \bigcup_{i \in \mathbb{N}_0} F_i.$$

Then H is a countable subgroup of G. Let $\mathcal{M}$ be the set of all p-subgroups of G which are an ascending union of a chain

$$S_0 < S_1 < \ldots < S_n < \ldots$$

of finite p-subgroups $S_i \in \Sigma_i$ $(i \in \mathbb{N}_0)$. According to (i) and (iii) we have $|\mathcal{M}| = 2^{\aleph_0}$ and according to (ii) any two $\mathcal{M}$-groups cannot generate a p-group. H contains every $\mathcal{M}$-group, so from the properties of $\mathcal{M}$ (and the countability of H) it follows that $|\mathrm{Syl}_p H| = 2^{\aleph_0}$. We have constructed an infinitely high ($\aleph_0$) tree of finite p-subgroups of G which branches properly at each location with proper inclusions and in which any two immediate successors of an arbitrary point do not generate a p-group. This tree has $2^{\aleph_0}$ many infinite branches. □

We are ready to state and prove our *Charakterisierungssatz*.

Theorem 3.9 (see [4]) *Let* G *be a locally finite group and let* p *be a prime. The following properties are equivalent:*

1) G *satisfies the strong Sylow Theorem for the prime* p.

2) *In every subgroup* U *of* G *every Sylow* p*-subgroup of* U *is singular.*

3) *Every countable subgroup* H *of* G *contains a* p*-uniqueness subgroup of* H.

4) *Every countable subgroup* H *of* G *contains a singular Sylow* p*-subgroup of* H.

5) *Every countable subgroup of* G *satisfies the Sylow Theorem for the prime* p.

6) *If* H *is a countable subgroup of* G, *then* $|\mathrm{Syl}_p H| < 2^{\aleph_0}$.

Proof — 2) ⇒ 3) and 3) ⇒ 4) are clear. 4) ⇒ 5) is valid by Theorem 3.7, 5) ⇒ 6) is valid by Theorem 3.4, and 6) ⇒ 1) is valid by Theorem 3.3. It remains to show 1) $\Rightarrow$ 2).* Assume 1) holds and let $U \leqslant G$. Then U satisfies the strong Sylow Theorem for the prime p. By Theorems 3.5 and 3.4 we have that $|\mathrm{Syl}_p H| < 2^{\aleph_0}$ for any countable subgroup H of U. By Theorem 3.8 there is a finite p-subgroup P of U such that for all finite p-subgroups P_1 and P_2 of U with $P \leqslant P_1 \cap P_2$ the group $\langle P_1, P_2 \rangle$ is a p-group. By Proposition 2.3 it follows that P is a p-uniqueness subgroup of U. Let $S \in \mathrm{Syl}_p U$ with $P \leqslant S$. Moreover, let $T \in \mathrm{Syl}_p U$ and $x = x(T) \in U$ with $S = T^{x^{-1}}$. Then P^x is a p-uniqueness subgroup of U with $P^x \leqslant T$, and hence T is singular by means of P^x. □

It would have been easier to show that Theorem 3.9 1) implies that every Sylow p-subgroup S of an arbitrary subgroup U of G is very good. In fact, let Σ be a local system for U. By Lemma 2.2 a) there exists a nested local system Σ_1 of Σ, and by Lemma 2.2 b) there is a $T \in \mathrm{Syl}_p U$ which reduces into Σ_1. Since G satisfies the strong Sylow Theorem for the prime p, we find an $x \in U$ such that $S = T^x$. Let $\Sigma_2 := \{Y \mid Y \in \Sigma_1, x \in Y\}$. Then Σ_2 is a local subsystem of Σ into which S reduces: for $S \cap Y = T^x \cap Y = (T \cap Y)^x \in \mathrm{Syl}_p Y$ when $Y \in \Sigma_2$.

Having proved our *Charakterisierungssatz*, we are now ready to prove the announced main theorem characterising the locally finite groups which satisfy the strong Sylow p-Theorem.

Theorem 3.10 *Let* G *be a locally finite group and let* p *be a prime. The following properties are equivalent:*

1) G *satisfies the strong Sylow Theorem for the prime* p.
2) *Every subgroup* S *of* G *contains a finite* p*-subgroup which is singular in* S.

Proof — The result follows from a combination of Proposition 2.3 and Theorem 3.9. □

* In Theorem 1.5 of [10] (*If the locally finite group* G *satisfies the strong Sylow Theorem for the prime* p *there exists a finite* p*-subgroup* P *which is singular in* G), Kegel ingeniously constructs, by contradiction, an infinite ($\aleph_0$) tower of countable subgroups of G, such that none of the finite p-subgroups of a member can be singular in the upper next, whose union has $2^{\aleph_0}$ maximal p-subgroups and therefore contradicts Theorem 3.4.

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4 Novel concepts for Sylow theory in (locally) finite groups

We end this paper with some further thoughts, a result, and some questions that could be quite useful for future researchers into Sylow theory in (locally) finite groups. The status quo of Sylow theory in locally finite groups has been beautifully summarised in [3] and [10]; here, a special place is occupied by the contributions of Brian Hartley (see [6],[7],[8]), who also contributed prodigiously to simple locally finite groups (see [9]). Concerning [9], which appeared posthumously, we notice that it does not cite [10] (not even in its list of 56 references). This is regrettable since Hartley states in his 1990 Mathematical Review of [10] the following: "If the simple locally finite group G satisfies the strong Sylow Theorem for the (even one) prime p, then G is linear. This depends on the classification of finite simple groups and an *assertion* about singular p-subgroups of classical groups. Another proof of this result has since been given by the reviewer (not yet published)." However, due to the tragic death of Brian Hartley on October 8, 1994, aged 55, this certainly very interesting proof was never prepared for publication. With someone of Hartley's stature, there is no question that his word is good enough and that in any case he supplied a new proof with probably quite a number of new insights. It might therefore be worthwhile and even most desirable to inspect Hartley's estate.

In every locally finite group G, for all subgroups U of G, the set $\mathrm{Unique}_p U$ of finite p-subgroups which are p-uniqueness subgroups of U is non-empty if G satisfies the strong Sylow Theorem for the prime p, that is, if G belongs to the class Syl-p of locally finite groups satisfying the strong Sylow Theorem for the prime p, and should this set be non-empty for a countable U then all the good Sylow p-subgroups of U are conjugate. Let U be finite. Then we have already $\mathrm{Unique}_p U \neq \emptyset$ because we have $\mathrm{Syl}_p U \leqslant \mathrm{Unique}_p U$. The Sylow p-subgroups of U are of course the maximal members of $\mathrm{Unique}_p U$, with respect to inclusion and order. It is a very very considerable challenge to try to determine the minimal members of $\mathrm{Unique}_p U$, with respect to either inclusion or order, in case that U and $\mathrm{Syl}_p U$ are sufficiently "known", in particular if U is a "known" finite simple group or a p-soluble group. Note that whenever $P < Q < R$ are p-subgroups of U where Q is a minimal p-uniqueness subgroup, or will be minimal singular in U, then P is contained in at least two, in fact in at least $p+1$, Sylow p-subgroups of U and R will be another p-uniqueness subgroup of U. The author is much hoping that some progress be made to this challenge in the future. For example, the question of whether (resp. when) the minimal p-uniqueness subgroups are conjugate, quite similar to the maximal ones, is surely of some interest, or, whether minimal w.r.t. inclusion implies minimal w.r.t. order, the converse being clearly obvious. We would then also come to better know the p-uniqueness subgroups of locally finite groups, in particular the simple and the locally p-soluble ones, and, many thanks to Kegel's Theorem 4.4, of locally finite groups in general belonging to the lovely class Syl-p. A good starting point would be to study minimal p-uniqueness subgroups of the finite symmetric and alternating groups where a Sylow 2-subgroup of an alternating group is a next to maximal 2-uniqueness subgroup of the symmetric overgroup so that we have to study only the symmetric groups and to show at least that their ranks are "somehow" bounded in terms of a p-uniqueness subgroup and in ideal circumstances to determine all the minimal ones (see what follows).

Let G be a locally finite group, $S \in \mathrm{Syl}_p G$ and $F \leqslant G$. We call F *minimal* p-*unique w.r.t.* S, if F is a minimal p-uniqueness subgroup of G w.r.t. order such that $F \leqslant S$, that is, F is p-unique with $F \leqslant S$ and each (finite) subgroup P of S with $|P| < |F|$ lies in at least two Sylow p-subgroups of G. If there exists an $S \in \mathrm{Syl}_p G$, such that F is, w.r.t. S, minimal p-unique, then F is called *minimal* p-*unique* (in G). Obviously, G is p-closed if and only if $\langle 1 \rangle$ is minimal p-unique (in G).

Theorem 4.1 (see [4]) *Let G be a locally finite group satisfying the strong Sylow Theorem for the prime p.*

a) *Each Sylow p-subgroup of G contains at least one minimal p-unique subgroup of G.*

b) *Each two minimal p-unique subgroups of G have the same order.*

Proof — a) Let $S \in \mathrm{Syl}_p G$ and let $U(G,S)$ be the set of all p-uniqueness subgroups F of G such that $F \leqslant S$. According to Theorem 3.9 we have $U(G,S) \neq \emptyset$ and of course each $U(G,S)$-group has finite order. Thus $U(G,S)$ contains (w.r.t. S) a minimal p-unique subgroup due to the well ordering of $\mathbb{N}$.

b) Let F_1 and F_2 be two minimal p-unique subgroups of G. For symmetry reasons it suffices to show $|F_1| \leqslant |F_2|$. Let S_1, $S_2 \in \mathrm{Syl}_p G$

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with $F_1 \leqslant S_1$ and $F_2 \leqslant S_2$. Since $G \in$ Syl-p there is an $x \in G$ such that $S_1 = S_2^x$. Then F_2^x is a p-uniqueness subgroup of G with $F_2^x \leqslant S_1$. Thus $|F_1| \leqslant |F_2^x| = |F_2|$ since F_1 is minimal p-unique w.r.t. S_1. □

Let G be a locally finite group satisfying the strong Sylow p-Theorem and let $S \in \mathrm{Syl}_p G$. According to Theorem 4.1 a) S contains (w.r.t. S) a minimal p-unique subgroup F. We define $a_p = a_p(G) \in \mathbb{N}_0$ by $|F| =: p^{a_p}$, that is, we let a_p be the composition length of F. According to Theorem 4.1 b) this definition is independent of the special choice of the Sylow p-subgroup S of G. Whereby consequently a_p is a (numeric) Sylow p-invariant of G. We call a_p the p-*uniqueness* of G. This Sylow p-invariant enqueues into the list — even is in the vanguard — of other Sylow p-invariants which play a major role in (locally) finite group theory, e.g. the order p^{b_p} of a Sylow p-subgroup, its nilpotency class c_p, its solubility length d_p, its exponent p^{e_p}, the composition length $i_p - 1$ of a proper maximal (w.r.t. order) Sylow p-intersection and further. The real peculiarity of a_p is that it is not determined by a Sylow p-subgroup as abstract p-group alone but depends on its embedding into the whole group and the respective relationships to the other Sylow p-subgroups. Then (w.r.t. inclusion or order maximal) intersections of two or several Sylow p-subgroups are of interest and deserve further study. For example, two core questions for Sylow theory in (locally) finite groups are how the p-length of a finite p-soluble group and the rank of a (known) finite simple group are bounded in terms of a p-uniqueness subgroup.

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ISCHIA GROUP THEORY 2016 (see [45])

Otto H. Kegel & Andrea Caranti (Ischia, 2016) • courtesy of F. de Giovanni
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Appendix 2

Talk by Felix F. Flemisch at Ischia Group Theory 2024

- **Introduction to the Talk by Felix F. Flemisch at IGT 2024 on April 11th, the 120th birthday of Philip Hall**
- **The 12 Slides of the Talk**
- **Professor Otto H. Kegel that's him all over**
- **The Mathematical Institute in Freiburg im Breisgau**
- **The Ancient University City Freiburg im Breisgau**
- **Thank you very much for your Patient Attention**

Introduction to the Talk by Felix F. Flemisch at IGT 2024 on April 11th, the 120th birthday of Philip Hall

This Talk of **only three minutes** was originally scheduled to take place in the first session at IGT 2024 by chairman **Prof. Dieter Kilsch** on Tuesday from 9.20 to 11.15. It should in no way detract from his thorough Talk about **Prof. Kegel** (9.20-9.25) and from the Talks by **Mahmut Kuzucuoğlu** (9.25-10.10) [who honoured Prof. Kegel brilliantly – **I thank Mahmut for the excellent work**], by Luise-Charlotte Kappe (10.10-10.50) and by Elena Bunina resp. Viji Z. Thomas (10.50-11.15). The "Lecture" therefore should take place at the very end of the session just before the coffee break (11.15-11.18) thereby steeling three minutes from the break ... ☺.
However, in the meantime this schedule was cancelled and I was kindly given the time slot from 18.05 to 18.30 on Thursday in the session by chairman **Alessio Rosso**, since **Dimitry Malinin** cannot come to our conference and **Natalia Maslova** moved to Wednesday. I present **first** this Introduction and **then** the **12 slides** of my **POSTER**.

My name is **Felix Flemisch**. I come from **Munich** in Bavaria in Germany. In the 1970ties and 1980ties I was a considerably busy and faithful student of **Prof. Otto H. Kegel** ♡ in such **beautiful** Freiburg i.Br. in Germany. In 2021 I luckily came again in contact with my adored teacher and met him in person and in good shape during June and July of 2022 in Freiburg. I present at IGT 2024 a **POSTER** about a new paper on **Sylow theory in simple locally finite groups** which is based on the very famous **Kegel covers** and on a **beautiful** paper of mine about rounding off **the general Sylow theory in locally finite groups**, friendly published by AGTA, under the rigid supervision of esteemed **Prof. Francesco de Giovanni** ✞. **Prof. Kegel** gave me kindly the hint to submit the paper to AGTA whose review process improved the paper very substantially so that it now can be the sound basis for further work on the subject.

Both papers have a quite strong relationship to **Prof. Kegel's work on Sylow theory, each one proving a conjecture of him** and centred around the quite gay concept of a ***p*-uniqueness subgroup** which is a finite *p*-subgroup being friendly contained in a **unique** Sylow *p*-subgroup. The **POSTER** shows the **twelve slides** of my talk as a PowerPoint presentation which include as well rather tough suggestions to stimulate and encourage future research. I much hope to enthuse group theorists with them and I am ready to support and coördinate related research work. This is my main interest why I present the **POSTER**. However, I am sadly aware that locally finite groups, and their Sylow theory in particular, seem not (yet) to be current topics of group theory research except some special questions presented on Tuesday. A limited number of nicely printed copies of the paper's **abstract**, its **POSTER** in DIN A3, and its **preprint** are available. I will deposit them tomorrow morning in SALA CARTAROMANA. An underlying **research paper** of this Talk will be published.

The Mathematical Institute in Freiburg im Breisgau

This is the **Mathematical Institute** at Albert-Ludwigs-University in **Freiburg im Breisgau** in Germany where from 1975 until 1999 **Prof. Kegel** occupied his chair, gave **beautiful** lectures and seminars, invited researchers over researchers, and hosted students in the morning **offering a cup of coffee** (or two) thereby doing careful supervision work and suggesting fascinating research topics.

Slide 1


The Strong Sylow Theorem for the Prime p in Simple Locally Finite Groups
Dipl.-Math. Felix F. Flemisch, M.Sc., Bacc.Math.
Dedicated to Prof. Otto H. Kegel
on the occasion of his 90th birthday on July 20
Ischia Group Theory 2024 from April 8 to April 13
Talk on Thursday, April 11, the 120th birthday of Prof. Philip Hall

Ischia Group Theory 2024 *The 11th edition* **Slide 2**

The Strong Sylow Theorem for the Prime p in Simple Locally Finite Groups

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Dedicated to **Prof. Otto H. Kegel** on the occasion of his 90th birthday
http://www.advgrouptheory.com/GTArchivum/Pictures/gtphotos/OttoKegel.jpg

This **Research Article** continues [15]. We begin with giving a profound overview of the structure of arbitrary simple groups and in particular of the simple locally finite groups and reducing their Sylow theory for the prime p to a quite famous conjecture by Prof. Otto H. Kegel (see [44], Theorem 2.4: "Let the p-subgroup P be a p-uniqueness subgroup in the finite simple group S which belongs to one of the seven rank-unbounded families. Then the rank of S is bounded in terms of P.") about the rank-unbounded ones of the 19 known families of finite simple groups. We introduce a new scheme to describe the 19 families, the family $\mathcal{T}$ of types, define the rank of each type, and emphasise the rôle of Kegel covers: Prof. Kegel rediscovered from Prof. Philip Hall (see [46]) that an infinite simple group has a local system consisting of countably infinite simple subgroups (see [45], [46] and [44], Theorem 2.5) (and conversely) and if they are locally finite he discovered groundbreakingly that they have a Kegel cover (see [44], Theorem 2.6), that is, a nested local system $\{G_n\}$ with maximal normal subgroups $M_n \le G_n$ such that $G_n \cap M_{n+1} = <1>$ so that G_n embeds into G_{n+1}/M_{n+1}. This part of the Research Article presents a unified rather complete picture of known results all of whose proofs are by reference.

We then apply **new ideas** to prove the conjecture for **the Alternating Groups**.

Thereupon we are remembering Kegel covers and ★-sequences and the classification of simple locally finite groups according to their Kegel covers. Next we suggest a way 1) and a way 2) how to prove and even how to optimise Kegel's conjecture step-by-step or peu à peu which leads to Conjecture 1, Conjecture 2 and Conjecture 3 thereby unifying *Sylow theory in locally finite simple groups* with *Sylow theory in locally finite and p-soluble groups* whose joint study directs very reliably Sylow theory in (locally) finite groups. For any unexplained terminology we refer to [15].

We then continue the program begun above to optimise along the way 1) the theorem about the first type Ξ = "$\underline{A}^n$" of infinite families of finite simple groups step-by-step to further types by proving it for the second type Ξ = "$A = PSL_n$". We apply **new ideas** to prove Conjecture 2 about **the General Linear Groups** over locally finite fields, stating that their rank is bounded in terms of their p-uniqueness, and then break down this insight to **the Special Linear Groups** and **the Projective Special Linear (PSL) Groups** over locally finite fields. We close with good suggestions for future research ► regarding the remaining five rank-unbounded types (the "Classical Groups") and the way 2), ► regarding (locally) finite and p-soluble groups, and ► regarding **our new perceptions** of the pioneering contributions by Cauchy and by Galois to Sylow theory in finite groups. We much hope to enthuse group theorists with these suggestions and are ready to contribute to, to support and to coördinate all related work.

It follows from our two theorems that simple locally finite groups which satisfy the Strong Sylow theorem for even one Prime p are linear and hence countable if they have a local system of countable simple subgroups each having a Kegel cover "*of alternating type*" or "*of projective special linear type*".

References

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[44] O.H. Kegel: "Four lectures on Sylow theory in locally finite groups". In: Group Theory, Proceedings of the Singapore Group Theory Conference held at the National University of Singapore, June 8–19, 1987, *Walter de Gruyter & Co.*, Berlin & New York (January 1989, reprinted November 2016), 3-27. ISBN 3-11-011366-X. https://www.degruyter.com/view/book/9783110848397/10.1515/9783110848397-004.xml.

[45] O.H. Kegel: "Remarks on uncountable simple groups". In: Proceedings of Ischia Group Theory 2016, *Int. J. Group Theory* **7** (March/June/September 2018). http://www.dipmat2.unisa.it/ischiagrouptheory/IGT2016/home_2016.html.

Slide 3

The Strong Sylow Theorem for the Prime *p* in Simple Locally Finite Groups

DIPL.-MATH. **FELIX F. FLEMISCH**, M.SC., BACC.MATH.

*Dedicated to **Prof. Otto H. Kegel** on the occasion of his 90th birthday*
Ischia Group Theory 2024 from April 8 to April 13

Let *p* be a prime: 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97, 101, 103, 107, 109, 113, 127, 131, 137, 139, 149, 151, 157, 163, 167, 173, 179, 181, 191, 193, 197, 199, 211, 223, 227, 229, 233, 239, 241, 251, 257, 263, 269, 271, 277, 281, 283, 293, 307, 311, 313, 317, 331, 337, 347, 349, 353, 359, 367, 373, 379, 383, 389, 397, 401, 409, 419, 421, 431, 433, 439, 443, 449, 457, 461, 463, 467, 479, 487, 491, 499, 503, 509, 521, 523, 541, 547, 557, 563, 569, 571, 577, 587, 593, 599, 601, 607, 613, 617, 619, 631, 641, 643, 647, 653, 659, 661, 673, 677, 683, 691, 701, 709, 719, 727, 733, 739, 743, 751, 757, 761, 769, 773, 787, 797, 809, 811, 821, 823, 827, 829, 839, 853, 857, 859, 863, 877, 881, 883, 887, 907, 911, 919, 929, 937, 941, 947, 953, 967, 971, 977, 983, 991, 997, 1009, 1013, 1019, … ☺

[44] O.H. KEGEL: “Four lectures on Sylow theory in locally finite groups”, in: Group Theory, *Walter de Gruyter*, Berlin & New York (1989), 3-27 (see MR0981832 [MR 90c:20037] and Zbl 0659.20024).

In his four workshop lectures on **Sylow theory in locally finite groups** at the Singapore Group Theory Conference of June 1987 (see [44]), **Prof. Kegel** stated as a **theorem** and proved “**by inspection**” what is actually a **conjecture**:

Theorem 2.4 “Let the p-subgroup P be a ***p*-uniqueness subgroup** in the finite simple group S which belongs to one of the seven rank-unbounded families. Then the rank of S is bounded in terms of P.”

The family $\mathcal{T}$ of types of known finite simple groups { abelian $_p$,
$\underline{A}^n$, A = PSL $_n$, B = PΩ $_{odd\ n}$, C = PSp $_n$, D = $P\Omega^+_{even\ n}$, ^2A = PSU $_n$, ^2D = $P\Omega^-_{even\ n}$,
E_6 , E_7 , E_8 , F_4 , G_2 , $^2B_2$, $^3D_4$, $^2E_6$, $^2F_4$, $^2G_2$, sporadic$_\star$ } is **beautiful**.
It contains 18 infinite families and one finite family:
the abelian groups, seven rank-unbounded (infinite) families,
ten infinite families with a fixed rank, and 26 sporadic groups.

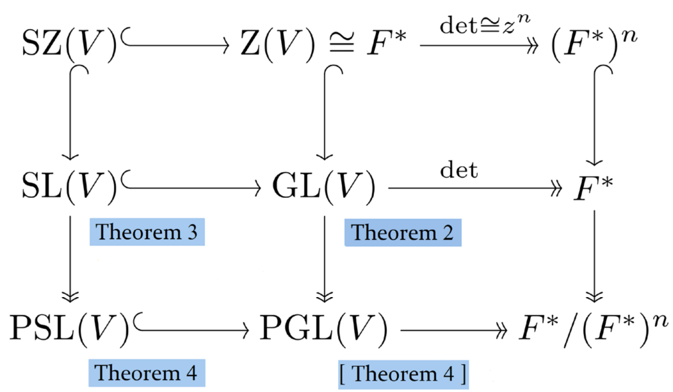
In this paper we prove **Kegel’s conjecture** for $\underline{A}^n$ and for A = PSL $_n$.
It continues [15] F.F. FLEMISCH: “Characterising Locally Finite Groups Satisfying the Strong Sylow Theorem for the Prime *p*”, *Adv. Group Theory Appl.* **13** (June 2022), 13-39 (see MR0981832 and Zbl 0659.20024).
We have included that **beautiful** predecessor paper as **Appendix 1**, although it is open access, since this paper cannot be understood without that predecessor paper – so one needs to have it present when reading this paper – and included as well the MR Review and the Zbl Review and an important comment ☺.

Slide 4

We sketch the proof for $\underline{A}^n$. Let the finite p-group P act on $\underline{A}^n$. Let α be a point and $P_\alpha := \{ x \in P \mid \alpha^x = \alpha \} \subseteq P$ be the stabiliser of α. We denote by $\mathsf{U}(P)$ the set of all subgroups of P and for $U \in \mathsf{U}(P)$ by $\mathsf{R}(P, U) := \{ Ux \mid x \in P \}$ the set of all right cosets of U in P. Then P operates by multiplication from the right for $U \in \mathsf{U}(P)$ transitively on $\mathsf{R}(P, U)$ with $\mathsf{Cor}_P\, U := \bigcap \{ U^x \mid x \in P \}$ as kernel.

The **classification of transitive *P*-sets** reads as follows (see [48], Chapter 6): *Every transitive P-set $\Omega \neq \emptyset$ is P-isomorphic to $\mathsf{R}(P, P_\alpha)$ for all $\alpha \in \Omega$, and for any $U, V \in \mathsf{U}(P)$ the two sets $\mathsf{R}(P, U)$ and $\mathsf{R}(P, V)$ are P-isomorphic if and only if U and V are conjugate in P*. Hence for the action of P we have a bijection between the class $\mathcal{J}(\boldsymbol{P})$ of all P-isomorphism types of transitive P-sets and the set of all conjugacy classes (in P) of subgroups of P, and so $|\mathcal{J}(P)| = \mathbf{g}_p(|\boldsymbol{P}|) :=$ the number of conjugacy classes of subgroups of P. Hence for every P-set Ω the class $\mathcal{J}(\boldsymbol{P}, \boldsymbol{\Omega})$ of P-isomorphism types of P-orbits on Ω has at most $g_p(|P|)$ elements and since every subgroup of P is a subset containing 1, we can summarise $|\mathcal{J}(P, \Omega)| \leq g_p(|P|) \leq |\mathsf{U}(P)| \leq 2^{|P|-1}$. If P is a p-subgroup of $\underline{S}^n$, which is contained in exactly $k \in \mathbb{N}$ Sylow p-subgroups of $\underline{S}^n$, and if $m := k + p + 1$, then $n \leq m \bullet |P| \bullet g_p(|P|) - 1 \leq m \bullet |P| \bullet 2^{|P|-1} - 1$, and in particular $n \leq (p + 2) \bullet |P| \bullet 2^{|P|-1} - 1$ for $k = 1$ (see **Page 5** of the **Research Article**), whence, if not so, P has at least m many P-isomorphic P-orbits on $\Omega := \{ 1, 2, \ldots, n \}$ (see **Page 5**). We deduce from this basic fact the central observation that $\{ S \in \mathsf{Syl}_p \underline{S}^\Omega \mid S \text{ is } P\text{-invariant} \} =: \mathsf{Syl}_p(\underline{S}^\Omega, P) \geq |\mathsf{Syl}_p \underline{S}^m| \geq m - 2 \geq k + 1$ by using **beautiful new ideas** (see **Page 6**). □

We sketch the proof for $\mathsf{A} = \mathsf{PSL}_n$. We apply a three-stage-approach whilst **first** proving the theorem for the **General Linear Groups** over (commutative) locally finite fields (**Theorem 2**), **then** for the **Special Linear Groups** over locally finite fields (**Theorem 3**) and **finally** for the **Projective Special Linear (PSL) Groups** over locally finite fields (**Theorem 4**), thereby using $\mathsf{GL}(n, F) = \mathsf{SL}(n, F) \bullet F^*$ and $\mathsf{PSL}(n, F) = \mathsf{SL}(n, F) / \mathsf{Z}(\mathsf{SL}(n, F))$. This approach can be presented with a **beautiful** diagram:



Slide 5

The major work is required for the **General Linear Groups** with two different and both **beautiful new ideas** for characteristic $\neq p$ and characteristic p. In characteristic $\neq p$ we use that, if for a finite p-group P operating on a finite-dimensional vector space V over a locally finite field and a direct decomposition of V into irreducible P-submodules, there are k many of the P-submodules P-isomorphic, then at least $|\mathrm{Syl}_p\underline{\mathrm{S}}^{\mathrm{k}}|$ Sylow p-subgroups of $\mathrm{GL}(V)$ are P-invariant (see **Proposition 7 a)**). In characteristic p we use that, if k is the dimension of the P-submodule $\mathrm{C}_V(P) := \{ v \in V \mid v^x = v \text{ for all } x \in P \}$ of a non-trivial modular P-module V, then again there are at least $|\mathrm{Syl}_p\underline{\mathrm{S}}^{\mathrm{k}}|$ many P-invariant Sylow p-subgroups of $\mathrm{GL}(V)$ (see **Proposition 7 b)**). We then argue that from **Proposition 7** follows that $\mathrm{n} \leq (p+2) \bullet |P|^2 - 1$ for a p-uniqueness subgroup P of $\mathrm{GL}(\mathrm{n}, F)$ (see **Lemma 2** on **Page 11**).

For the transition from $\mathrm{GL}(\mathrm{n}, F)$ to $\mathrm{SL}(\mathrm{n}, F)$ we are using that a p-uniqueness subgroup of $\mathrm{SL}(\mathrm{n}, F)$ is a p-uniqueness subgroup of $\mathrm{GL}(\mathrm{n}, F)$ as well. From $\mathrm{SL}(\mathrm{n}, F)$ to $\mathrm{PSL}(\mathrm{n}, F)$ we use that $P := Q \bullet \mathrm{D}(\mathrm{SL}(\mathrm{n}, F)) \,/\, \mathrm{D}(\mathrm{SL}(\mathrm{n}, F))$ is a p-uniqueness subgroup of $\mathrm{PSL}(\mathrm{n}, F)$ when Q is a p-uniqueness subgroup of $\mathrm{SL}(\mathrm{n}, F)$, and conversely, together with **Proposition 4** and **Proposition 6**. □

Let G be a countably infinite locally finite simple group. Then there will exist a nested local system $\{ R_{\mathrm{n}} \mid \mathrm{n} \in \mathbb{N} \}$ for G of finite subgroups such that for each $\mathrm{n} \in \mathbb{N}$ the group R_{n} is perfect and there exists a maximal normal subgroup $M_{\mathrm{n+1}}$ of $R_{\mathrm{n+1}}$ satisfying $M_{\mathrm{n+1}} \cap R_{\mathrm{n}} = <1>$, whence $R_{\mathrm{n+1}} / M_{\mathrm{n+1}}$ is simple and $R_{\mathrm{n}} \precsim R_{\mathrm{n+1}} / M_{\mathrm{n+1}}$; such a nested local system is called ***Kegel cover for G***. We call G to be ***of type*** $\Xi \in \mathcal{T}$, if it has a Kegel cover $\Sigma = \{ (R_{\mathrm{k}}, M_{\mathrm{k}}) \mid \mathrm{k} \in \mathbb{N} \}$ such that infinitely many $R_{\mathrm{k+1}} / M_{\mathrm{k+1}}$'s belong to Ξ (wherefore we can replace Σ by these infinitely many $R_{\mathrm{k+1}}$'s), and call G to be ***of alternating type*** if it is of type $\underline{\mathrm{A}}^{\mathrm{n}}$ and to be ***of projective special linear type*** if it is of type $\mathrm{A} = \mathrm{PSL}_{\mathrm{n}}$.

Theorem 1 (see [14]) *Let* $\mathrm{n} \in \mathbb{N}$ *and let* p *be a prime such that* $p \leq \mathrm{n}$. *Let* P *be a finite* p*-group acting on* $\underline{\mathrm{A}}^{\mathrm{n}}$. *Let* $\mathbf{g}_p(|P|)$ *be the number of conjugacy classes of subgroups of* P *and let* k *be the number of* P*-invariant Sylow* p*-subgroups of* $\underline{\mathrm{A}}^{\mathrm{n}}$. *Then* $\mathrm{g}_p(|P|) \leq 2^{|P|-1}$.

a) *If isomorphic subgroups of* P *are conjugate and* $\mathrm{b} := \log_p|P|$ *(so that* $|P| =: p^{\mathrm{b}}$*), then*

$$\mathrm{g}_p(|P|) \leq p^{((\mathrm{b}-2)^4 + 2(\mathrm{b}-2)^3 + (\mathrm{b}-2)^2)/4 - ((\mathrm{b}-2)^2 + \mathrm{b} - 2)/2 - 90 + (|P|-1)/(p-1) + 25}.$$

b) *Let* $\mathrm{m} := \mathrm{k} + p + 1$. *Then* $\mathrm{n} \leq \mathrm{m} \bullet |P| \bullet \mathrm{g}_p(|P|) - 1$. *If* $\mathrm{k} = 1$, *then* $\mathrm{n} \leq \mathrm{f}_p(|P|) := (p+2) \bullet |P| \bullet 2^{|P|-1} - 1$.

A periodic linear group G is locally finite and satisfies the strong Sylow Theorem for ***every*** prime p, and hence $\mathbf{a}_p(G)$ is defined (see **Slide 7** below). We first prove **Conjecture 2** (see **Slide 7**) regarding the **General Linear Groups** over locally finite fields:

Slide 6

Theorem 2 *Let* $\mathrm{n} \in \mathbb{N}$ *and let* p *be a prime. Let* $\mathcal{F}$ *be a locally finite (commutative) field.*
a) *If* $\mathcal{F}$ *has characteristic* p *and* $\mathrm{a}_p = \mathrm{a}_p(\mathsf{GL}(\mathrm{n},\mathcal{F}))$ *then* $\mathrm{n} \leq (p+2) \bullet p^{\mathrm{a}_p} - 1$.
b) *If* $\mathcal{F}$ *has characteristic* $\neq p$ *and* $\mathrm{a}_p = \mathrm{a}_p(\mathsf{GL}(\mathrm{n},\mathcal{F}))$ *then* $\mathrm{n} \leq (p+2) \bullet p^{2\,\mathrm{a}_p} - 1$.

We then break down **Theorem 2** to the **Special Linear Groups** over locally finite fields:

Theorem 3 *Let* $\mathrm{n} \in \mathbb{N}$ *and let* p *be a prime. Let* $\mathcal{F}$ *be a locally finite (commutative) field.*
a) *If* $\mathcal{F}$ *has characteristic* p *and* $\mathrm{a}_p = \mathrm{a}_p(\mathsf{SL}(\mathrm{n},\mathcal{F}))$ *then* $\mathrm{n} \leq (p+2) \bullet p^{\mathrm{a}_p} - 1$.
b) *If* $\mathcal{F}$ *has characteristic* $\neq p$ *and* $\mathrm{a}_p = \mathrm{a}_p(\mathsf{SL}(\mathrm{n},\mathcal{F}))$ *then* $\mathrm{n} \leq (p+2) \bullet p^{2\,\mathrm{a}_p} - 1$.

We continue with breaking down **Theorem 3** to the **Projective Special Linear (PSL) Groups** over locally finite fields:

Theorem 4 *Let* $\mathrm{n} \in \mathbb{N}$ *and let* p *be a prime. Let* $\mathcal{F}$ *be a locally finite field and* P *be a minimal* p*-unique subgroup of* $\mathsf{PSL}(\mathrm{n},\mathcal{F})$ *so that* $|P| = p^{\mathrm{a}_p}$.
a) *If* $\mathcal{F}$ *has characteristic* p *and* $\mathrm{a}_p = \mathrm{a}_p(\mathsf{PSL}(\mathrm{n},\mathcal{F}))$ *then* $\mathrm{n} \leq \mathrm{f}_p(|P|) := (p+2) \bullet p^{\mathrm{a}_p} - 1$.
b) *If* $\mathcal{F}$ *has characteristic* $\neq p$ *and* $\mathrm{a}_p = \mathrm{a}_p(\mathsf{PSL}(\mathrm{n},\mathcal{F}))$ *then* $\mathrm{n} \leq \mathrm{f}_p(|P|) := (p+2) \bullet p^{2\,\mathrm{a}_p} - 1$.

Let G be an infinite simple group. G has a local system consisting of countably infinite simple subgroups (see [45] O.H. Kegel: "Remarks on uncountable simple groups". In: Proceedings of Ischia Group Theory 2016, *Int. J. Group Theory* **7** [March/June/September 2018]). Let each of these be locally finite **of alternating type** or **of projective special linear type**. Then **Theorem 1** and **Theorem 4** imply the following intriguing consequences of the Strong Sylow Theorem for the Prime p:

Theorem 5 *Let* G *be a simple locally finite group* ***of alternating type*** *or* ***of projective special linear type*** *satisfying the Strong Sylow Theorem for the even one Prime* p. *Then* G *is linear and countable.* □

Planning future research

Our **Theorem 1** could be optimised in two ways:

1) Extend it from type $\underline{\mathsf{A}}^{\mathrm{n}}$ **step-by-step** to further types Ξ with an appropriate (similar) function f_p, that is, the rank $\mathrm{r}(G)$ of a group G of type Ξ is bounded by $\mathrm{f}_p(|P|)$ for a p-uniqueness subgroup P of G.

2) Determine for the type $\underline{\mathsf{A}}^{\mathrm{n}}$ and **peu à peu** for further types Ξ the minimal p-unique subgroups, that is, the p-uniqueness subgroups of the non-abelian simple groups of type $\underline{\mathsf{A}}^{\mathrm{n}}$ and of type Ξ which are minimal with respect to order (see [15]).

Slide 7

Let G be a locally finite group satisfying the strong Sylow p-Theorem and let $S \in \mathrm{Syl}_p G$. Then S contains some (w.r.t. S) minimal p-unique subgroup F. We define $\mathbf{a}_p = \mathbf{a}_p(G) \in \mathbb{N}_0$ by $|F| =: p^{a_p}$, that is, we let a_p be the composition length of F. This definition is independent of the choice of the Sylow p-subgroup, so a_p is a (numerical) Sylow p-invariant of G. We call a_p the ***p*-uniqueness of *G***.

Conjecture 1 *Let* $\mathcal{T} := \{ \text{abelian}_p, \underline{A}^n, A = PSL_n, B = P\Omega_{\text{odd } n}, C = PSp_n, D = P\Omega^+_{\text{even } n}, {}^2A = PSU_n, {}^2D = P\Omega^-_{\text{even } n}, E_6, E_7, E_8, F_4, G_2, {}^2B_2, {}^3D_4, {}^2E_6, {}^2F_4, {}^2G_2, \text{sporadic}_\star \}$ *be the family of types of known finite simple groups and let G be a finite simple group of type* $\Xi \in \mathcal{T}$. *Then the rank* $r(G)$ *of G is bounded in terms of the p-uniqueness* $a_p(G)$.

Conjecture 2 *Let* $n \in \mathbb{N}$ *and let p be a prime. Let* $\mathcal{F}$ *be a locally finite (commutative) field.*

a) *If* $\mathcal{F}$ *has characteristic p and* $a_p = a_p(\mathrm{GL}(n,\mathcal{F}))$ *then* $n \le (p+2) \bullet p^{a_p} - 1$.

b) *If* $\mathcal{F}$ *has characteristic* $\neq p$ *and* $a_p = a_p(\mathrm{GL}(n,\mathcal{F}))$ *then* $n \le (p+2) \bullet p^{2a_p} - 1$.

We give a brief attention to (locally) p-soluble groups since it is the reliable joint study of the (locally) simple and the (locally) p-soluble groups which directs the Sylow theory in (locally) finite groups. The central observation is the following best possible claim:

Conjecture 3 *Let p be a prime. Let G be a p-soluble finite group,* $\lambda_p(G)$ *be its p-length, and* $a_p(G)$ *be its p-uniqueness.* *Then* $\lambda_p(G) \le a_p(G) + 1$.

The **classical Hall-Higman theory**, created by *P. Hall*, *G. Higman*, *A.H.M. Hoare*, *T.R. Berger*, *F. Gross* and *E.G. Bryukhanova*, introduces for finite p-soluble groups (best possible) inequalities between their p-length $\boldsymbol{\lambda_p}$ and the order $\boldsymbol{p}^{b_p}$ of a Sylow p-subgroup, its nilpotency class $\mathbf{c}_p$, its solubility length $\mathbf{d}_p$, its exponent $\boldsymbol{p}^{e_p}$, or the rank $\mathbf{r}_p$ of a maximal elementary abelian subgroup. Our true and ambitious aim is **to extend the Hall-Higman theory** to the **beautiful** p-uniqueness $\boldsymbol{p}^{a_p}$ of a Sylow p-subgroup, a truly Herculean  endeavour ☺. The real challenge is to prove **Conjecture 3**. It is much expected that the cases $p \ge 5$, $p = 3$ and $p = 2$ must be treated fairly separately and that $p = 3$ and $p = 2$ will require rather special methods as is already indicated by the available literature.

Slide 8

Our proofs of **Conjecture 1** for the types $\underline{A}^n$ and $A = PSL_n$, that is, to carve out **the optimising way 1)**, are characterised by the fact that *we need not at all know their Sylow p-subgroups*. There is no doubt that we can extend those proofs straightforwardly to the further five classical groups $B = P\Omega_{odd\ n}$, $C = PSp_n$, $D = P\Omega^{+}_{even\ n}$, $^2A = PSU_n$ and $^2D = P\Omega^{-}_{even\ n}$ by considering the respective bilinear form defining these groups of Lie type, resp. the vector spaces they act upon as isometries, and their resulting Sylow *p*-subgroups (*without knowing them*). They can well be considered proved which we shall confirm in the **follow-up paper** " The Strong Sylow Theorem for the Prime *p* in the Locally Finite Classical Groups " considering the locally finite classical groups which are *the linear*, *symplectic*, *unitary* and *orthogonal groups* over locally finite fields. The linear groups are dealt with in this paper and the others are subgroups of the linear groups which are defined through a non-singular bilinear form (or scalar product) being either skew-symmetric (or alternate) or Hermitian or symmetric (defining a quadratic form) as *the group of isometries of the form*. They were nicely introduced to us in the classical books [1] and [58] and are further studied in [6], [24] and [50]. We do not refer to the groups of Lie type resp. the Chevalley groups and the twisted Chevalley groups defined through a Dynkin diagram automorphism followed by a field automorphism, which correspond to the classical groups (see [24], pp. 151-152) and whose fine introductory references are the "Lecture Notes on Chevalley Groups" by **Robert Steinberg** (1967 and 2016) together with the book "Simple Groups of Lie type" by **Roger W. Carter** (1972 and 1989). Therefore we study $P\Omega_{odd\ n}$, PSp_n , $P\Omega^{+}_{even\ n}$, PSU_n and $P\Omega^{-}_{even\ n}$ and not B, C, D, ^2A and ^2D. Consequently the proofs for **the further five types of Classical Groups** can and will be based successfully on our **very beautiful** **Theorem 2** about the **General Linear Groups**. We are preparing to publish our first follow-up paper in 2025.

Our **second follow-up paper** "The Strong Sylow Theorem for the Prime *p* in Locally Finite and *p*-Soluble Groups" considers (locally) finite and *p*-soluble groups. It summarises the work by **B. Hartley** and **A. Rae** regarding $\boldsymbol{\lambda}_{\boldsymbol{p}}$ and $\boldsymbol{p}^{a_p}$ (see **Page 37** of [15] and the **References** of [44]) and the foregoing work on Hall-Higman theory regarding $\boldsymbol{\lambda}_{\boldsymbol{p}}$ and $\boldsymbol{p}^{b_p}$, $\mathbf{c}_{\boldsymbol{p}}\mathbf{d}_{\boldsymbol{p}}\boldsymbol{p}^{e_p}$ and $\mathbf{r}_{\boldsymbol{p}}$ by **P. Hall**, **G. Higman**, **A.H.M. Hoare**, **T.R. Berger**, **F. Gross**, **E.G. Bryukhanova** and last but not least by **A. Turull** as indicated on **Page 8** and **Page 9**. It then proves **Conjecture 3** (see the **Slide 7** above) not only in **English** but partly in **Portuguese** for historical reasons.

Our **beautiful** **third follow-up paper** "Augustin-Louis Cauchy's and Évariste Galois' Contributions to Sylow Theory in Finite Groups" pays sincere tribute to **Augustin-Louis Cauchy**'s and **Évariste Galois**' pioneering contributions to Sylow theory in finite groups by working out their new perceptions. It proves in a unified way **Lagrange's theorem** and **Cauchy's concealed second and third group theorems** by exploring three **beautiful** rectangles/tableaux. We show the second rectangle and the third tableau to raise inquisitiveness:

Slide 9

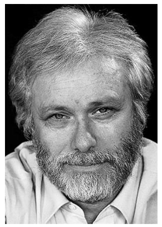
set of ***certain*** orbits of H under G acting by left translation	the **first row** consists of ***all*** right cosets Gx_1^k of G in H $(0 \le k \le p\text{-}1)$ with the powers of some p-blank $\boldsymbol{x_1}$ of G in H; the **following rows** consist of right cosets of G in H with the powers of left conjugates of $\boldsymbol{x_1}$				correspondence	$\boldsymbol{X} := <\boldsymbol{x_1}>$; set of ***all*** orbits of H under $G \circlearrowleft \circlearrowright \boldsymbol{X}$, the **simultaneous** actions of G by left translation and of $\boldsymbol{X}$ by right translation
$Gx_1^0\, t_1 = G$	Gx_1	Gx_1^2	…	$Gx_1^{p\text{-}1}$	↔	cosets $G<\boldsymbol{x_1}> = G\boldsymbol{X}$ = double coset $G\,1\,\boldsymbol{X}$
$Gx_2^0\, t_2 = Gt_2$	$Gx_2\, t_2$	$Gx_2^2\, t_2$	…	$Gx_2^{p\text{-}1}\, t_2$	↔	cosets $G<\boldsymbol{x_2}>t_2$ = double coset $Gt_2\boldsymbol{X}$
$Gx_3^0\, t_3 = Gt_3$	$Gx_3\, t_3$	$Gx_3^2\, t_3$	…	$Gx_3^{p\text{-}1}\, t_3$	↔	cosets $G<\boldsymbol{x_3}>t_3$ = double coset $Gt_3\boldsymbol{X}$
…	…	…	…	…	…	…
$Gx_S^0\, t_S = Gt_S$	$Gx_S\, t_S$	$Gx_S^2\, t_S$	…	$Gx_S^{p\text{-}1}\, t_S$	↔	cosets $G<\boldsymbol{x_S}>t_S$ = double coset $Gt_S\boldsymbol{X}$

set of ***certain*** orbits of H under G acting by left translation	the **first row** consists of ***all*** right cosets $G\,\boldsymbol{x_{1c}}$ of G in H $(0 \le c \le \lvert H\rvert_p\text{-}1)$ with the elements of some Sylow p-subgroup $\boldsymbol{X}$ of H, all of whose **elements of order** p are p-blanks of G in H; the **following rows** consist of right cosets of G in H with the elements of left conjugates of $\boldsymbol{X}$				correspondence	$\lvert \boldsymbol{X}\rvert = \lvert H\rvert_p = p^b$; set of ***all*** orbits of H under $G \circlearrowleft \circlearrowright \boldsymbol{X}$, the **simultaneous** actions of G by left translation and of $\boldsymbol{X}$ by right translation
$G\,x_{10}\, t_1 = G$	$G\,\boldsymbol{x_{11}}$	$G\,\boldsymbol{x_{12}}$	…	$G\,\boldsymbol{x_{1\,p^b\text{-}1}}$	↔	cosets $G\,\{\boldsymbol{x_{1c}} \mid 0 \le c \le p^b\text{-}1\}$ = $G\boldsymbol{X}$ = double coset $G\,1\,\boldsymbol{X}$
$G\,x_{20}\, t_2 = G\,t_2$	$G\,\boldsymbol{x_{21}}\, t_2$	$G\,\boldsymbol{x_{22}}\, t_2$	…	$G\,\boldsymbol{x_{2\,p^b\text{-}1}}\, t_2$	↔	cosets $G\,\{\boldsymbol{x_{2c}} \mid 0 \le c \le p^b\text{-}1\}\, t_2$ = double coset $G\,t_2\,\boldsymbol{X}$
$G\,x_{30}\, t_3 = G\,t_3$	$G\,\boldsymbol{x_{31}}\, t_3$	$G\,\boldsymbol{x_{32}}\, t_3$	…	$G\,\boldsymbol{x_{3\,p^b\text{-}1}}\, t_3$	↔	cosets $G\,\{\boldsymbol{x_{3c}} \mid 0 \le c \le p^b\text{-}1\}\, t_3$ = double coset $G\,t_3\,\boldsymbol{X}$
…	…	…	…	…	…	…
$G\,x_{T0}\, t_T = G\,t_T$	$G\,\boldsymbol{x_{T1}}\, t_T$	$G\,\boldsymbol{x_{T2}}\, t_T$	…	$G\,\boldsymbol{x_{T\,p^b\text{-}1}}\, t_T$	↔	cosets $G\,\{\boldsymbol{x_{Tc}} \mid 0 \le c \le p^b\text{-}1\}\, t_T$ = double coset $G\,t_T\,\boldsymbol{X}$

For an outline of this **very beautiful** paper see **Page 13** and **Page 14** of the **Research Article** and **Slide 11**.

Siamo angeli con un'ala soltanto e possiamo volare solo restando abbracciati. [Italian]
We are angels who have but a single wing and we can only fly if we cling to one another. [English]
Wir sind Engel mit nur einem Flügel, um fliegen zu können müssen wir uns umarmen. [German]
Nous sommes des anges à une seule aile, nous ne pouvons voler qu'en restant enlacés. [French]
Somos ángeles con una única ala y sólo podemos volar abrazados. [Spanish]
Nós somos anjos com apenas uma asa e só podemos voar quando nos abraçamos. [Portuguese]

Luciano De Crescenzo
(★ 18 August 1928 in **Naples** until ✞ 18 July 2019 in **Rome**)
Così parlò Bellavista. Napoli, amore e libertà.
XXIII Piedigrotta. 1977 e settembre 2019


Slide 10

A MATHEMATICIAN, like a painter or a poet, is a maker of patterns. If his patterns are more permanent than theirs, it is because they are made with ***ideas***. ... The mathematician's patterns, like the painter's or the poet's, must be ***beautiful***; the ***ideas***, like the colours or the words, must fit together in a harmonious way. ***Beauty*** is the first test: there is no permanent place in the world for ugly mathematics.
Godfrey Harold Hardy. **A Mathematician's Apology.** § 10. July 18, 1940.

L'autore è appassionatamente curioso del futuro.
The author is passionately curious about the future.
Der Autor ist sehr leidenschaftlich neugierig auf die Zukunft.
L'auteur est passionnément curieux de l'avenir.
O autor é muito apaixonadamente curioso sobre o futuro.
Felix Fortunatus Flemisch. **Firenze.** April 11, 1992.

The **Research Article** has the following seventeen **beautiful** Chapters: **Sketch** of proof for $\underline{A}^{\mathbf{n}}$; **Sketch** of proof for A = PSL_n; **1 Introduction**; **2 Proof** of Theorem 1; **3 About Kegel covers**; **4 Planning future research** – Part 1; **5 Proof** of Theorem 2; **6 Proof** of Theorem 3; **7 Proof** of Theorem 4; **8 Planning future research** – Part 2; **9 The First Trilogy** and **The Second Trilogy** and their reviews; **Acknowledgements**; **Postscript, Luciano De Creszenzo, Felix F. Flemisch, Conflicts of Interest, Pablo Picasso's** *La Joie de vivre* ; **About the author** in **Munich**, in **Freiburg i.Br.**, in **London**, in **Weiden i.d.OPf.**, and in **Florence** in Tuscany in Italy; 75 **References**; **Appendix 1** – Reference [15] with MR Review and Zbl Review; **Appendix 2 – Talk by Felix F. Flemisch at Ischia Group Theory 2024**.

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Slide 11

We are planning to revise thoroughly Sylow theory starting with **a really new proof** for **Cauchy**'s known as fundamental theorem in group theory (look at https://en.wikipedia.org/wiki/Cauchy%27s_theorem_(group_theory)) based on **beautiful** ideas by **Galois**. In the forthcoming (third) follow-up **Research Article** "Augustin-Louis Cauchy's and Évariste Galois' Contributions to Sylow Theory in Finite Groups" beyond our **First Trilogy** (look at the **Postscript** on **Page 15**) we first describe and then provide new but classical and rather unified proofs for the fundamental theorems by **Lagrange** and by **Cauchy** on finite groups of – in our modest opinion – considerable historical relevance.

We can describe in detail consequences of **the absence of group elements of prime order *p***, in spite of their availability in overgroups, thereby providing a much unified and also heretofore undiscovered approach to the theorems of Lagrange and of Cauchy and their implications for p-groups. Since this approach uses only ideas from a very well-known paper by **Augustin-Louis Cauchy** presented first in 1812 and then published in 1815, this bears considerable historic relevance. While it is widely acknowledged that Cauchy had ***published*** his fundamental group theorem not until 1845/1846 and had there based it on double cosets of the finite permutation group and some Sylow p-subgroup of its symmetric overgroup, one could henceforth well argue that he had ***presented*** his theorem ***in a truly concealed way*** already a good thirty years earlier. **Évariste Galois** knew both Cauchy's paper of 1815 and – based on his own rather perceptive considerations – Cauchy's group theorem and even already Sylow's existence theorem. Cauchy's and Galois' ideas are particularly lucid in the embryonic case of permutation groups of prime degree p (≥ 5) when Sylow p-subgroups of the symmetric overgroup obviously exist. If $G \subseteq H$ with H being finite, then **the unified method of proof** consists in arranging the elements of H in **a rectangle** with $|G|$ columns and $[H:G]$ rows resp. the (right) cosets of G in H in a rectangle with p resp. with $|H|_p$ columns and $[H:G]/p$ resp. $[H:G]/|H|_p$ rows to obtain information about $[H:G]$ (see the three rectangles above).

Cauchy's theorem of 1812/1815 is a direct consequence of $[H:\langle x\rangle] \geq |G|$ if x is an element of H of prime order p with $x \notin G$ which we call a ***p*-blank of *G* in *H***. We find that Lagrange's theorem and Cauchy's theorem are just like two sides of a coin where "Lagrange" represents the case $p^0 = 1$ and "Cauchy" the case $p^1 = p$ thereby offering a unified approach to both theorems. Hence, "Cauchy" is not only a partial converse of "Lagrange" but it is in fact a smart "swapping" of p for 1 as well: $\boldsymbol{p^0 = 1} \circlearrowleft \boldsymbol{p = p^1}$.

Cauchy depicts 1815 a p-cycle for some prime p as a regular p-gon  and studies p-cycles in considerable detail.

We present Cauchy's ***classical proof*** of **Lagrange's theorem** and supplement it with a ***modern proof***. We then present Cauchy's ***classical proofs*** of his **published first theorem**, of his **concealed second theorem** and of his **concealed third theorem**. Subsequently we introduce double cosets and show how they lead to a ***modern proof*** of Cauchy's second and third theorems what Cauchy did **beautifully** as well but not until 1845/1846 after reconsidering, impressed by a paper of **Joseph Bertrand**, his work of 1812/1815, that is, after – believe it or not – 30 years.

We continue with **first** correcting a great misunderstanding of Cauchy's work of 1845/1846 in the literature and **then** presenting Cauchy's work of 1812/1815 in the sincere succession of the earlier work of **Joseph-Louis de Lagrange** (Giuseppe Luigi Lagrangia), **Alexandre-Théophile Vandermonde** and the pioneer **Paolo Ruffini**, as indicated by Cauchy himself, and identify the crucial parts of Cauchy's first publication on group theory. **Finally** we present what **Évariste Galois** knew already about **Cauchy's group theorems** and about **Sylow's famous theorems** by referring to his published papers and to his posthumously published papers as well. However, this requires rather considerable further (historical) research. We would be inestimably delighted if several group theory researchers would help us with this tedious but very suspenseful work and are ready to cöordinate all the work. We are closing with fairly comprehensive **Acknowledgements** and a greatly sizeable list of **References**.

Augustin-Louis Cauchy
(21 August 1789 until 23 May 1857)

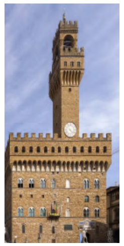
Évariste Galois
(25 October 1811 until 31 May 1832)

Slide 12

About the author

Felix F. Flemisch was born on 17 May 1951 in wonderful **Munich** in Bavaria in Germany. In **June 1971** he received his **Abitur** ☺ whose subject Mathematics was taught in a pioneering spirit by **Dr. Helmut Bergold**. Afterwards he received his first-ever degree **Baccalaureus der Mathematik** (**Bacc.Math.**) in **July 1974** with the alas unpublished **very beautiful bachelor's thesis** "Über einfache Punkte affiner Varietäten" from the most venerable Albert-Ludwigs-Universität at **such beautiful Freiburg im Breisgau** in **green** Baden-Württemberg in Germany under the rather thorough supervision of **Akadem. Rat Dr. Herbert Götz**, and then his degree **Master of Science** (**M.Sc.**) from the Faculty of Science of the highly esteemed University of **London**, Bedford College, United Kingdom, in **August 1975** under the supervision of greatly adored **Prof. Paul Moritz Cohn**. From October 1975 until – very regrettably – only July 1976 he was employed as ***a fairly diligent Teaching Assistant with two graduations*** by the hoar Mathematische Fakultät of **Freiburg i.Br.**'s Albert-Ludwigs-Universität. Subsequently he quite enthusiastically continued his postgraduate mathematical studies in such marvellous and fabulous **Freiburg i.Br.** – with decent interruptions as ***a teacher*** and as ***a tutor*** – and received his degree **Diplom-Mathematiker** (**Dipl.-Math.**) in **April 1985** under the impressive supervision of **Prof. Otto Helmut Kegel**. The Research Article [15] publishes the most essential and partly well corrected portions of his German **Diplomarbeit** [14] of **October 1984** and a "sprinkling" of new considerations and results as well which try to propose coming directions of research for Sylow theory in (locally) finite groups. The publication at hand continues [15] with theorems about simple locally finite groups "***of alternating type***" and "***of projective special linear type***" and makes quite a number of thorough suggestions for future research ☺. From February 1981 until April 1985 the author was enormous happily affiliated to the **Institut für Medizinische Biometrie und Statistik (IMBI)** at lovely **Freiburg im Breisgau** as ***a considered Wissenschaftlicher Mitarbeiter***. Since **May 1985** he was based in **Munich** and devotedly working with greatest joy for the telecom industry first as ***a System Software Developer***, then as ***a Systems Engineer***, and in closing as ***a Director for the International Standardisation of telecom software and concepts***. On the very **11 April 1992** he so blissful happily married the most fabulous and wonderful-ever woman **Helga** in **beautiful Florence** in Tuscany in Italy, which was such a memorable marriage celebrated along with about twenty friends and uniting the most venerable





city **Weiden** in ***Upper Palatinate*** (i.d.OPf.) (Helga) with the huge cosmopolitan city **Munich** in ***Upper Bavaria*** (Felix). That unforgetful event was built really for eternity: **Helga and Felix were meant to last forever** ♡. Since **October 2016** the author is retired and is still resp. is again much loving to work for Mathematics, in particular for the **very beautiful** Group Theory ☺.

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The Ancient University City Freiburg im Breisgau

These are some **beautiful** views of the ancient university city **Freiburg i.Br.** (since 1457) which allow you to assess why students like and even love it so much up to the present day, of course apart from the challenging research work having and being done there in mathematics and statistics.

Professor Otto H. Kegel that's him all over

This **Research Article** is dedicated to **Prof. Otto H. Kegel** on the occasion of his 90th birthday on 20 July 2024. We therefore are closing the **Research Article** with two **beautiful** photographs of him:

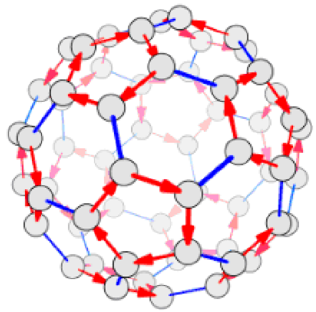
Prof. Otto H. Kegel
am Mathematischen Forschungsinstitut Oberwolfach (MFO)

Prof. Otto H. Kegel
at the Oberwolfach Research Institute for Mathematics
(see https://mfo.de/ and https://opc.mfo.de/related?id=23960 and https://opc.mfo.de/detail?photo_id=12422)

Prof. Kegel was very frequently at the famous MFO near such **beautiful** Freiburg im Breisgau, where he occupied his chair from 1975 to 1999 🎓 , both as a guest and a speaker and as an organiser of fascinating conferences.

Prof. Kegel,
wie er leibt und
glücklich lebt,
auf einem Spaziergang in
Freiburg im Breisgau

Prof. Kegel,
that’s him all over,
his happy spitting image,
on a cosy stroll in
Freiburg im Breisgau



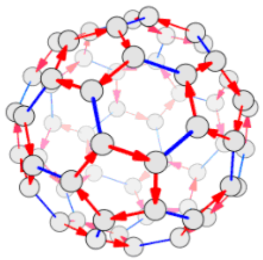
Long live Group Theory and in particular Sylow Theory of Locally Finite Groups!

Am wunderschönen Ammersee in Bayern
At the wonderfully beautiful Lake Ammersee in Bavaria




Thank you very much for your patient attention!

Are there any questions?
I would be very happy to (try to) answer them.
But I know that there are already so many by myself … .



Who is Felix F. Flemisch and Abstract

Felix F. Flemisch fairly proudly received his first degree Bacc.Math. 🎓 in 1974 from the Albert-Ludwigs-Universität at *lovely* Freiburg im Breisgau, his postgraduate degree M.Sc. 🎓 in 1975 from the honourable University of London, UK, and finally his degree Dipl.-Math. 🎓 at *marvellous* and *fabulous* Freiburg i.Br. in 1985. From February 1981 until April 1985 he was quite happily affiliated to the Albert-Ludwigs-Universität Freiburg i.Br., Universitätsklinikum Freiburg, Institut für Medizinische Biometrie und Statistik (IMBI). Since May 1985 he was enthusiastically and with great joy working for the telecom industry. On April 11, 1992, he married beloved **Helga** in beautiful Florence in Tuscany in Italy ♥. Since October 2016 he is retired and is still resp. is again loving to work on mathematics, in particular on the very beautiful Group Theory ☺.

This **Research Article** continues [15]. We begin with giving a quite profound overview of the structure of arbitrary simple groups and then in particular of the simple locally finite groups and reduce their Sylow theory for the prime *p* to a rather famous conjecture by **Prof. Otto H. Kegel** (see [44], Theorem 2.4: "Let the *p*-subgroup *P* be a *p*-uniqueness subgroup in the finite simple group *S* which belongs to one of the seven rank-unbounded families. Then the rank of *S* is bounded in terms of *P*.") about **the rank-unbounded ones** of the 19 known families of finite simple groups. We introduce **a new scheme** to describe the known 19 families, **the family $\mathcal{T}$ of types**, define **the rank** of each type, and emphasise the rôle of **Kegel covers**. This part presents a unified picture of known results all of whose proofs are by reference.

Subsequently we apply **new ideas** to prove the conjecture for the **Alternating Groups**.

Thereupon we are remembering Kegel covers and ★-sequences. Next we suggest some **future research** by stating **a way 1)** and **a way 2)** how to prove and even how to optimise Kegel's conjecture step-by-step or peu à peu which is leading to **Conjecture 1**, to **Conjecture 2** and to **Conjecture 3** thereby unifying *Sylow theory in locally finite simple groups* with *Sylow theory in locally finite and p-soluble groups* whose joint study directs very reliably the Sylow theory in (locally) finite groups. For any unexplained terminology we refer to [15].

We then continue the program begun above to optimise along **the way 1)** the theorem about **the first type** Ξ = "$\underline{A}^n$" of infinite families of finite simple groups step-by-step to further types by proving it for **the second type** Ξ = "A = PSL_n". We start with applying **new ideas** to prove **Conjecture 2** about the **General Linear Groups** over (commutative) locally finite fields, stating that their rank is bounded in terms of their *p*-uniqueness, and then break down this insight to the **Special Linear Groups** and to the **Projective Special Linear (PSL) Groups** over locally finite fields. We close with a number of suggestions for **future research** ► regarding the remaining five rank-unbounded types (the "Classical Groups") and **the way 2)**, ► regarding (locally) finite and *p*-soluble groups, and ► regarding Cauchy's and Galois' contributions to Sylow theory in finite groups. We hope to enthuse group theorists with them and are ready to coördinate related work.

We include the beautiful predecessor research paper [15] as **Appendix 1** for good reasons.

The **Research Article** has the following **seventeen** **beautiful** **Chapters**:

Sketch of proof for $\underline{A}^n$; **Sketch** of proof for A = PSL_n; **1 Introduction**; **2 Proof** of Theorem 1; **3 About Kegel covers**; **4 Planning future research** – Part 1; **5 Proof** of Theorem 2; **6 Proof** of Theorem 3; **7 Proof** of Theorem 4; **8 Planning future research** – Part 2; **9 The First Trilogy** and **The Second Trilogy** and their reviews; **Acknowledgements**; **Postscript**, **Luciano De Creszenzo**, **Felix F. Flemisch**, **Conflicts of Interest**, **Pablo Picasso**'s *La Joie de vivre*; **About the author** in **Munich**, in **Freiburg i.Br.**, in **London**, in **Weiden i.d.OPf.**, and in **Florence** in Tuscany in Italy; **75 References**; **Appendix 1** – Reference [15] with MR Review and Zbl Review; **Appendix 2** – **Talk at IGT 2024** on Thursday, April 11, the 120th birthday of **Prof. Philip Hall**.