

The Strong Sylow Theorem for the Prime p in Simple Locally Finite Groups – De Luxe Edition

Felix F. Flemisch

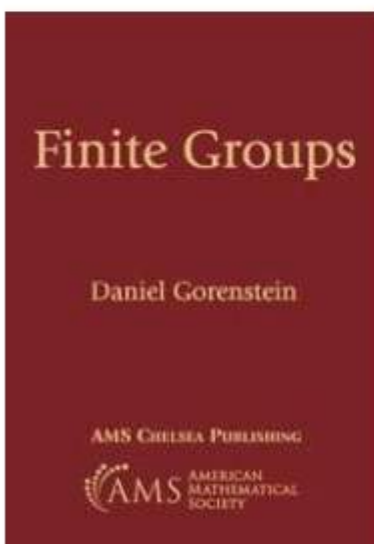
Mitterweg 4e, 82211 Herrsching a. Ammersee, Bavaria, Germany

ABSTRACT

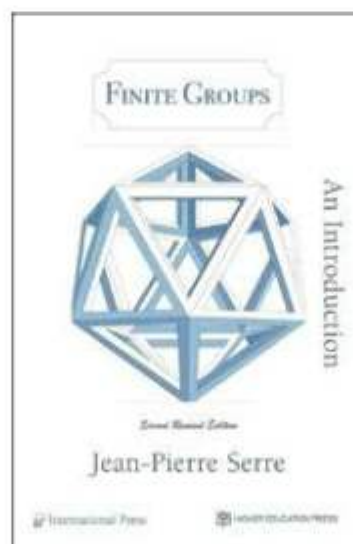
This **Research Article** extends the **JMCA Research Article** “The Strong Sylow Theorem for the Prime p in Simple Locally Finite Groups” (see <https://www.onlinescientificresearch.com/articles/the-strong-sylow-theorem-for-the-prime-p-in-simple-locally-finite-groups.pdf>) from 71 pages to 220 pages by adding **Pages i to viii, -1 to -18, 72 to 266, and ix to x** (see **Page v**).

Pages i to -18 start with this Abstract and then honour Prof. Otto H. Kegel’s beautiful paper [44] thereby explaining its relationship to its main result and to the **71-pages Research Article**, and commenting on its final considerations. They then give an overview of the **220-pages Research Article**, including the Front Cover and a Table of Contents, and explain in great detail the relationships of the **71-pages Research Article** to JMCA and to the Ischia Group Theory (IGT) 2024 Conference and its upcoming Proceedings, centred around the **Talk** given by the author at the Conference on April 11, the 100th birthday of Prof. Philip Hall. They close with a List of Open Issues some of which the author has solved already in yet unpublished work.

Pages 72 to x start with a dedication to **Helga** (see **Page 15**), continue with the PowerPoint Presentation at IGT 2024, honour Ludvig M. Sylow, the discoverer and explorer of Sylow Theory, recall that Ischia was twice an Artist Colony, remember Philip Hall’s and Graham Higman’s fundamental paper “On the p -Length of p -Soluble Groups and Reduction Theorems for Burnside’s Problem” thereby explaining its relationship to the **71-pages Research Article**, recall Some Historicals on Group Theory, in particular Philip Hall’s hand-written Lecture Notes on Group Theory and show these beautiful Lecture Notes, show a presentation of WIKIPEDIA’s Classification of finite simple groups, remember Prof. Kegel and Prof. Hall at the Mathematisches Forschungsinstitut Oberwolfach (MFO), and finally as Back Cover refer to Philip Hall’s Archive at the London Mathematical Society and again to his Lecture Notes, and close by showing Who is Felix F. Flemisch ? & Another Abstract of the **71-pages Research Article**.



<https://bookstore.ams.org/chel-301.h>
 2nd edition 1980, 519 pages (see [22])



<https://intlpress.com/BDetail?from=book&id=1726537209895026690>
 2nd edition 2022, 192 pages
<https://www.irishmathsoc.org/bull81/Murray.pdf>



<https://webpace.maths.qmul.ac.uk/p.j.cameron/notes/gt.pdf>
 2013, 103 pages (see also [6])

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Four lectures on Sylow theory in locally finite groups

O.H. Kegel

The AGTA **Research Paper** [15] “Characterising Locally Finite Groups Satisfying the Strong Sylow Theorem for the Prime p ” and the JMCA **Research Article** “The Strong Sylow Theorem for the Prime p in Simple Locally Finite Groups” (see [doi.org/10.47363/JMCA/2025\(4\)198](https://doi.org/10.47363/JMCA/2025(4)198)) are both based on this **beautiful** paper [44] by Prof. Otto H. Kegel each one proving a conjecture of it: the AGTA paper answers the question on page 10 and the JMCA paper finds a proof for the “inspection” of (2.4) Theorem on page 13 both being centred around the gay concept of a **p -uniqueness subgroup** which is a finite p -subgroup being contained in a unique Sylow p -subgroup. We show the main result of Kegel’s paper and comment his final considerations. See also our comments on **Page 3** and **Page 4**.

Introduction

In their wording some theorems about finite groups also make sense for infinite groups; however, since there are counter examples, most of them cannot be proved. Narrowing down the class of groups considered more theorems of finite origin will become provable. In general, assuming the validity of some theorem about finite groups for some infinite group is a strong finiteness condition, and one might wonder what other finiteness conditions may be deduced from it. A good class in which to study such phenomena is the class of locally finite groups, i.e. the class of those groups in which every finite set of elements generates a finite subgroup.

The basic result on finite groups is the *Sylow Theorem* stating that for a fixed prime p the maximal p -subgroups of a finite group G are conjugate in G . This statement makes sense in arbitrary groups, but it is false in general, as we shall see even in locally finite groups. We shall refer to the discussion of the validity of the Sylow Theorem for the prime p or some variation or generalisation of it in some class of infinite groups as *Sylow Theory*. More than anyone else, Brian Hartley has contributed to Sylow Theory in locally finite groups. If not only in the locally finite group G itself, but also in every subgroup of G , the maximal p -subgroups are conjugate we shall say that G satisfies the *strong Sylow Theorem* for the prime p . For locally finite, locally p -soluble groups satisfying the strong Sylow Theorem for the prime p Hartley exhibited in [10] a very strong finiteness property, if $p \neq 2$. An extension of a weak form of Hartley’s finiteness result is

If for the prime $p \geq 5$ the locally finite group G satisfies the strong Sylow Theorem then there is a finite series of normal subgroups N_i of G with

$$G = N_0 \supseteq \dots \supseteq N_i \supseteq N_{i+1} \supseteq \dots \supseteq N_k = \langle 1 \rangle$$

such that the factors N_i/N_{i+1} of this series are either a direct product of finitely many linear simple groups or locally p -soluble.

Hartley’s result allows one to refine the series so that the locally p -soluble factors N_i/N_{i+1} are either p -groups or p' -groups.

It is this result that I want to explain in these lectures. I hope that on the way the audience will get a few glimpses of the landscape of locally finite groups in general and of the importance that definite results in the theory of finite groups may have for the theory of locally finite groups. Occasionally, questions on finite groups have been motivated by possible applications to locally finite groups.

Prof. Kegel then presents his four lectures on Sylow theory in locally finite groups

Lecture I : Variations on Sylow's Theorem

Lecture II : Singular p -subgroups and simple locally finite groups

Lecture III : The study of crucial configurations

Lecture IV : The strong finiteness results

which culminate in a detailed refinement of the result from Page ii:

(4.4) Theorem. *If the locally finite group G satisfies the strong Sylow Theorem for the prime $p \geq 5$, then there are characteristic subgroups*

$$\langle 1 \rangle \subseteq \mathbf{O}_p(G) \subseteq \mathbf{O}_{p,p'}(G) \subseteq \mathbf{O}_{p,p',p}(G) \subseteq \mathbf{S}_p(G) \subseteq S \subseteq A \subseteq P \subseteq G$$

such that $S/\mathbf{S}_p(G) = \text{soc}(G/\mathbf{S}_p(G))$ is a direct product of finitely many locally finite simple linear groups, A/S is an abelian group of rank bounded by the number of simple direct factors of $S/\mathbf{S}_p(G)$, the factor group P/A is a finite soluble group of order bounded by the number and a function of the types of the simple direct factors of $S/\mathbf{S}_p(G)$, and the factor group G/P permutes these direct factors faithfully. If none of the characteristics of the underlying locally finite fields of the infinite simple direct factors of $S/\mathbf{S}_p(G)$ is p , then the factor group $G/\mathbf{O}_p(G)$ satisfies the minimum condition for p -subgroups. In any case, the factor group $G/\mathbf{O}_{p,p',p}(G)$ is countable.

Here $\mathbf{S}_p(G)$ denotes the largest normal locally p -soluble subgroup of G .

Making a suitable definition for the p -length of the group $S/\mathbf{S}_p(G)$ – possibly simply the number of simple direct p -perfect factors – one gets that with this extended notion of p -length the locally finite group G satisfying the strong Sylow Theorem for the prime $p \geq 5$ will have finite p -length.

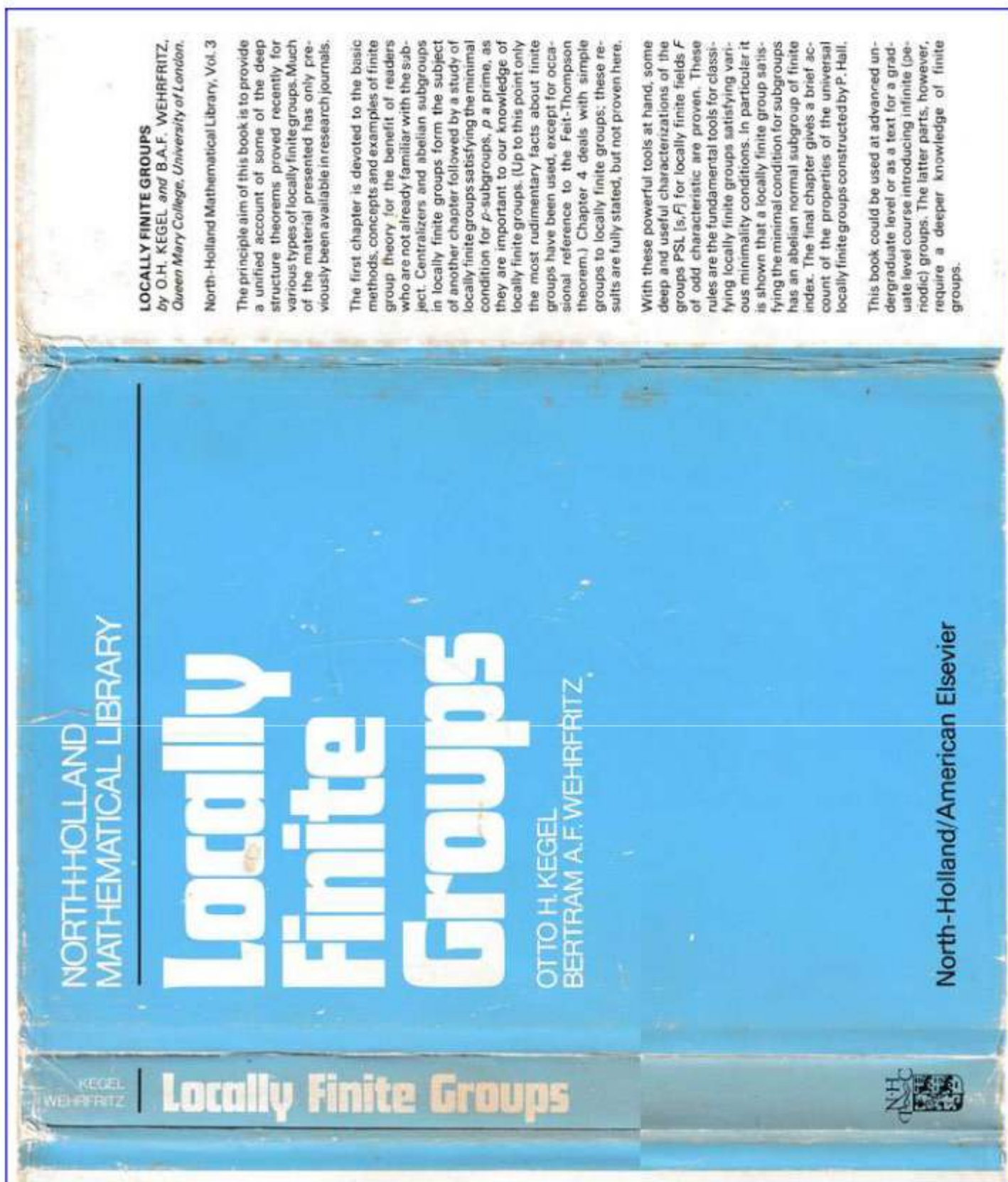
The author has defined in unpublished work for every prime p such a generalised p -length for locally finite groups which is finite if and only if they satisfy the strong Sylow Theorem for p .

It seems desirable to remove the restrictions on the prime p in the above result. This would mean finding an argument in the finite soluble case to prove a qualitative result like (2.1) and finding a rather different argument to prove (3.5) which might use different properties of finite simple groups.

The qualitative result like (2.1) is **Conjecture 3** (see **Pages 9, 12, 13 and 100**). Making a revision of Kegel's (3.5) Theorem thereby relating it to rarely known articles, the author was able to extend Kegel's (4.4) Theorem to the case $p = 3$ without using special properties of finite simple groups. Note also that our results for simple groups are valid for all primes p .

Acknowledgements. For the presentation of these ideas I am indebted to the efforts of my former student Felix Flemisch. – B. Hartley pointed out an embarrassing error in Lecture III.

The author learned **Locally Finite Group Theory** in the 1970ies through the beautiful book [43] and in the 1980ies through personal education by Prof. Otto H. Kegel in the course of developing this paper [44]:



LOCALLY FINITE GROUPS
by O.H. KEGEL and B.A.F. WEHRFRITZ,
Queen Mary College, University of London.

North-Holland Mathematical Library, Vol. 3

The principle aim of this book is to provide a unified account of some of the deep structure theorems proved recently for various types of locally finite groups. Much of the material presented has only previously been available in research journals.

The first chapter is devoted to the basic methods, concepts and examples of finite group theory for the benefit of readers who are not already familiar with the subject. Centralizers and abelian subgroups in locally finite groups form the subject of another chapter followed by a study of locally finite groups satisfying the minimal condition for p -subgroups, p a prime, as they are important to our knowledge of locally finite groups. (Up to this point only the most rudimentary facts about finite groups have been used, except for occasional reference to the Feit-Thompson theorem.) Chapter 4 deals with simple groups to locally finite groups; these results are fully stated, but not proven here.

With these powerful tools at hand, some deep and useful characterizations of the groups $PSL(n, F)$ for locally finite fields F of odd characteristic are proven. These rules are the fundamental tools for classifying locally finite groups satisfying various minimality conditions. In particular it is shown that a locally finite group satisfying the minimal condition for subgroups has an abelian normal subgroup of finite index. The final chapter gives a brief account of the properties of the universal locally finite groups constructed by P. Hall.

This book could be used at advanced undergraduate level or as a text for a graduate level course introducing infinite (periodic) groups. The latter parts, however, require a deeper knowledge of finite groups.

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Locally Finite Groups

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Locally Finite Groups



The Strong Sylow Theorem for the Prime p in Simple Locally Finite Groups – De Luxe Edition

This 220-pages JMCA **Research Article** extends the 71-pages JMCA **Research Article** “[The Strong Sylow Theorem for the Prime \$p\$ in Simple Locally Finite Groups](https://www.onlinescientificresearch.com/journal-of-mathematical-computer-applications-old-articles.php?journal=jmca&v=4&i=1&y=2025&m=February)” (see under <https://www.onlinescientificresearch.com/articles/the-strong-sylow-theorem-for-the-prime-p-in-simple-locally-finite-groups.pdf> & **Page 1 to Page 71**), by its smart **Abstract** (see <https://www.onlinescientificresearch.com/abstract/the-strong-sylow-theorem-for-the-prime-p-in-simple-locally-finite-groups-6110.html> & **Page -12 and Page -13**) and its fine **PowerPoint Presentation** (see https://www.srcscholarsmedia.com/powerpoint-presentations.php?page_no=1 & <https://www.srcscholarsmedia.com/ppt-article/talk-by-dipl-math-felix-f-flemisch-m-sc-bacc-math-at-ischia-group-theory-2024-igt-2024-50.html> & **Page 73 to Page 84**) and a **great lot more** (see **Page -1**) thereby applying not the consecutive page numbering 1 to 220 but a page numbering which keeps the numbering 1 to 71 for the 71-pages Research Article and allows for pages which include two original pages (Pages i to viii, -1 to -18, 1 to 71, 72 to 266, and ix to x):

- Front Cover front – The Strong Sylow Theorem for the Prime p in Simple Locally Finite Groups & Professor Otto H. Kegel on the occasion of his 90th birthday on July 20th, 2024 & Talk on the 120th birthday of Professor Philip Hall & The Periodic Table Of Finite Simple Groups (May 2025 Issue)
- Front Cover back – MÉMOIRE Sur le Nombre des Valeurs qu’une Fonction peut acquérir, lorsqu’on y permute de toutes les manières possibles les quantités qu’elle renferme
- Internet appearance of IGT 2024 (only Home, Agenda and Proceedings)
- Scientific and Organizing Committees of IGT 2024 & Editorial Board of the IGT 2024 Proceedings
- Submissions for the IGT 2024 Conference Proceedings
- JMCA Volume 4, Issue 1 Articles
- Abstract of the following Research Article
- The IGT 2024 Conference Proceedings (May 2025 Issue)
- List of Open Issues
- Research Article
- L’amour immortel, éternel et infini de l’auteur pour [Hélga](#) et les mathématiques
- Talk by Felix F. Flemisch at IGT 2024 – PowerPoint Presentation
- The Discoverer and Explorer of Sylow Theory
- Ischia was twice an Artist Colony
- On the p -Length of p -Soluble Groups and Reduction Theorems for Burnside’s Problem
- Some Historicals on Group Theory
- Philip Hall’s hand-written Lecture Notes on Group Theory
- WIKIPEDIA – Classification of finite simple groups
- Prof. Otto H. Kegel and Prof. Philip Hall at the MFO
- Back Cover front – Philip Hall Archive & Lecture Notes on Group Theory
- Back Cover back – Who is Felix F. Flemisch ? & Research Article Summary



Ischia – Castello Aragonese

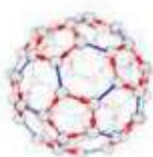


Ischia – Sant’Angelo (see Page 96 and Page 97)



At the wonderfully beautiful Lake Ammersee in Bavaria 😊

Long live Group Theory and in particular Sylow Theory  of Locally Finite Groups!



Gruppentheorie



Studium



München



Freiburg i.Br.



Florenz



Herrsching a.A.



Gauting

Dipl.-Math. **Felix F. Flemisch**, M.Sc., Bacc.Math.

The Strong Sylow Theorem for the Prime p in Simple Locally Finite Groups – De Luxe Edition

May 2025 Issue

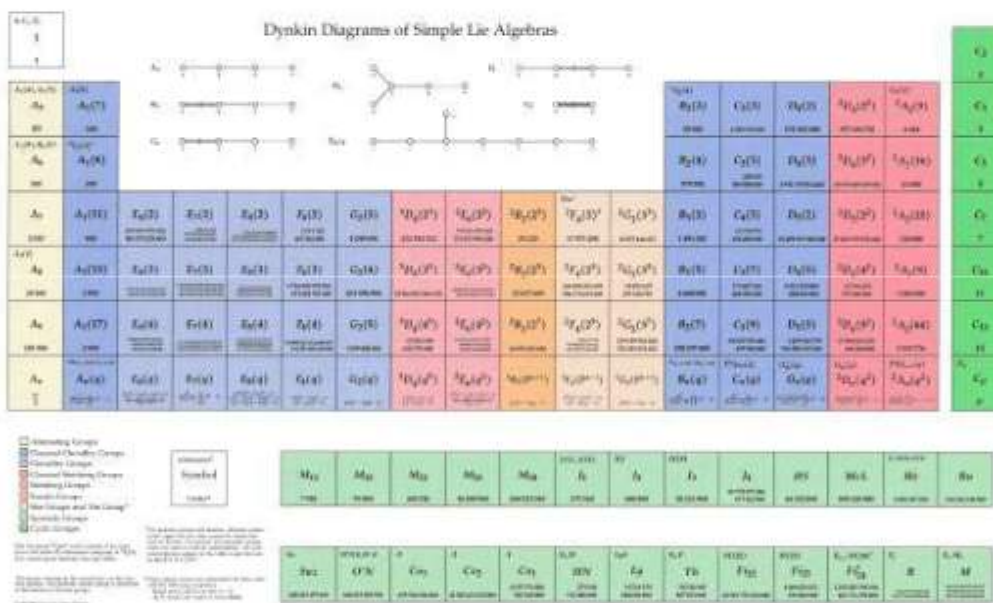


Dedicated to **Professor Otto H. Kegel** on the occasion of his **90th birthday on July 20th** – Ischia Group Theory 2024 from April 8 to April 13
Talk presented at IGT 2024 on Thursday, **April 11th**, that is, on the

120th birthday of Professor Philip Hall



The Periodic Table Of Finite Simple Groups



The family \mathcal{T} of types of known finite simple groups { abelian $_p$, A^n , $A = \text{PSL}_n$, $B = \text{P}\Omega_{\text{odd } n}$, $C = \text{PSp}_n$, $D = \text{P}\Omega_{\text{even } n}^+$, ${}^2A = \text{PSU}_n$, ${}^2D = \text{P}\Omega_{\text{even } n}^-$, $E_6, E_7, E_8, F_4, G_2, {}^2B_2, {}^3D_4, {}^2E_6, {}^2F_4, {}^2G_2$, sporadic $_{\star}$ } is beautiful.

It contains 18 infinite families and one finite family:
the abelian groups, seven rank-unbounded infinite families,
ten infinite families with a fixed rank, and 26 sporadic groups.



JOURNAL
DE L'ÉCOLE POLYTECHNIQUE.

MÉMOIRE

Sur le Nombre des Valeurs qu'une Fonction peut acquérir, lorsqu'on y permute de toutes les manières possibles les quantités qu'elle renferme;

PAR A. L. CAUCHY, INGÉNIEUR DES PONTS ET CHAUSSÉES.

MM. *LAGRANGE* et *VANDERMONDE* sont, je crois, les premiers qui aient considéré les fonctions de plusieurs variables relativement au nombre de valeurs qu'elles peuvent obtenir, lorsqu'on substitue ces variables à la place les unes des autres. Ils ont donné plusieurs théorèmes intéressans relatifs à ce sujet, dans deux mémoires imprimés en 1771, l'un à Berlin, l'autre à Paris. Depuis ce temps, quelques géomètres italiens se sont occupés avec succès de cette matière, et particulièrement M. *Ruffini*, qui a consigné le résultat de ses recherches dans le tome XII des Mémoires de la Société italienne, et dans sa Théorie des équations numériques. Une des conséquences les plus remarquables des travaux de ces divers géomètres, est qu'avec un nombre donné de lettres on ne peut pas toujours former une fonction qui ait un nombre déterminé de valeurs. Les caractères par lesquels cette

XVII. Cahier.



MÉMOIRE. Sur le Nombre des Valeurs qu'une Fonction peut acquérir, lorsqu'on y permute de toutes les manières possibles les quantités qu'elle renferme ;

PAR A. L. CAUCHY, INGÉNIEUR DES PONTS ET CHAUSSÉES.

JOURNAL DE L'ÉCOLE POLYTECHNIQUE., XVII. Cahier, Tome dixième, Janvier 1815, 1-28

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IGT 2024 Home (see <https://sites.google.com/unisa.it/igt/home>)

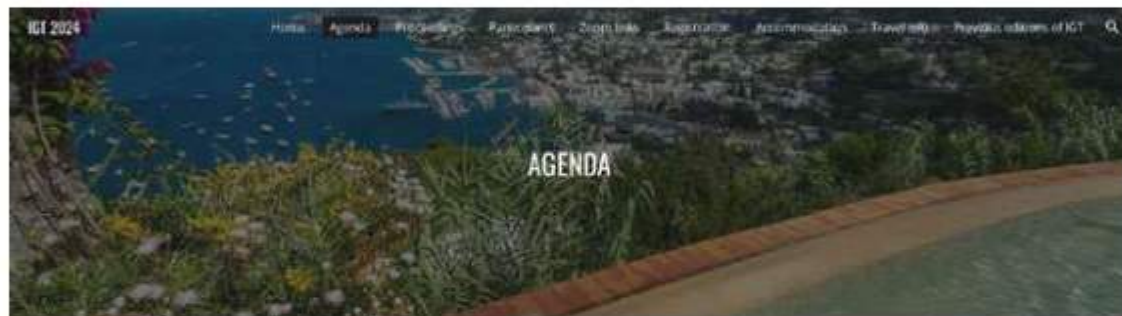


A Conference on Group Theory will be held at "Grand Hotel delle Terme Re Ferdinando", in Ischia (Naples, Italy), from Monday, April 8th, to Sunday, April 13th, 2024. The meeting will start with a Welcome Cocktail and the opening of the permanent poster session on Monday, April 8th, in the late afternoon. Talks will begin on Tuesday, April 9th, in the morning and conclude on Friday, April 12th, in the late afternoon.

The social programme will also include a Festival of Classical Neapolitan songs on Tuesday evening, a Concert of Baroque music for flute and cello on Wednesday evening (18th), the Social Trip to Smeralda Gardens on Thursday morning, and the Conference Dinner on Friday evening.

This edition of Ischia Group Theory is in honour of Otto H. Kegel on the occasion of his 90th birthday, to celebrate his significant role in the international landscape of Group Theory and his fundamental contribution to the Ischia Group Theory conference series.

IGT 2024 Agenda (see <https://sites.google.com/unisa.it/igt/agenda>)



MONDAY, APRIL 8th

15.00 Registration

16.00 Opening permanent Poster Session

18.30 Welcome Cocktail

TUESDAY, APRIL 9th

9.10 Welcome greetings

9.20 Chairman: Dieter Kilsch

9.25 Mahmut Kuzucuoğlu

Uniqueness of κ -existentially closed groups

10.10 Luise-Charlotte Kappe

On the nonabelian tensor product of cyclic groups of odd order

10.50 Viji Z. Thomas

On the size and the exponent of Schur multiplier of p -groups

11.15 Coffee break

Chairman: Ted Hurley

11.45 Peter J. Cameron

Inverse group theory

12.25 Leonid A. Kurdachenko

Groups and left braces: common approaches, different results

13.00 Lunch break

15.00 Chairman: Andrea Caranti

15.05 Bernhard Amberg

Factorized groups and solubility

15.50 Pavel Shumyatsky

Profinite groups whose elements have prime power order

16.25 Coffee break

Chairman: Tullio Ceccherini Silberstein

16.55 Massimiliano Sala

On $NP=coNP$ and its implications to cryptography and group theory

17.35 Stephen Glasby

Classifying finite groups with 3 automorphism orbits

18.15 Elena Bunina

Bi-interpretability and elementary definability of Chevalley groups

18.40 End Session

21.30 Recital of classical Neapolitan songs (Sala Aragonese)

WEDNESDAY, APRIL 10th

9.10 Chairman: Carlo M. Scoppola

9.15 Dan Segal

Axiomatizing groups

10.00 Marston Conder

Soluble quotients of triangle groups

10.40 Evgeny Khukhro

Rank type conditions on commutators in finite groups

11.15 Coffee break

Chairman: Anatolii Tushev

11.45 Benjamin Klopsch

The lower p -series of p -adic analytic pro- p groups and Hausdorff dimension

12.25 Martino Garonzi

On pyramidal groups of prime power degree

12.50 Picture

13.00 Lunch break

15.00 Chairwoman: Rosemary Bailey

15.05 Alexei Stepanov

Nikolai Vavilov and subgroups of Chevalley groups over rings

15.50 Eugene Plotkin

Faces of bounded generation

16.25 Coffee break

Chairman: Tengizi Bokelavadze

16.55 Rostislav I. Grigorchuk

Liftable self-similar groups and scale groups

17.35 Adolfo Ballester-Bolinches

Central nilpotency and solubility of skew left braces

18.15 Natalia Maslova

On Gruenberg–Kegel graphs of finite groups

18.55 Antonio Ioppolo

Groups gradings and graded maps on algebras

19.20 End Session

19.40 Concert of Baroque music for flute and cello (Chiesa Santa Maria di Portosalvo)

THURSDAY, APRIL 11th

9.10 Social trip

13.00 Lunch break

15.00 Chairwoman: Daniela Nikolova

15.05 Wolfgang Willem

In memoriam Prof. Dr. Bertram Huppert

15.50 Gustavo A. Fernández-Alcober

Around conciseness

16.30 Thomas M. Keller

Bounding the number of conjugacy classes of a finite group in terms of a prime

16.55 Coffee break

Chairman: Alessio Russo

17.25 Urban Jezernik

Babai's conjecture for classical groups with random generators

18.05 Felix F. Flemisch
The strong Sylow theorem for the prime p in simple locally finite groups
18.30 End Session

FRIDAY, APRIL 12th

9.10 Chairman: Primož Potočnik

9.15 Gernot Stroth

Zvonimir Janko and the finite simple groups

10.00 Eugenio Giannelli

On Sylow branching coefficients

10.40 Rachel Camina

The Amit and Amit–Ashurst conjectures for finite nilpotent groups

11.15 Coffee break

Chairman: Paolo Papi

11.45 Delaram Kahrobaei

Post-quantum Blockchains using hash functions using higher dimensional special linear groups over finite fields as platforms

12.25 Gunnar Traustason

Some special classes of powerful p -groups

13.00 Lunch break

15.00 Chairman: Péter P. Pálfi

15.05 M. Dolores Pérez-Ramos

Together with Rex and Army: an adventure in the world of Fitting classes

15.50 Cindy (Sin Yi) Tsang

Factorizations of groups and skew braces

16.30 Mark L. Lewis

A lower bound on the size of maximal abelian subgroups

17.10 Gerard Hiss

The eigenvalue one property of finite groups

17.50 Closing

20.30 Social dinner (Sala Aragonesa)

Here you can download all the **Abstracts**:

https://drive.google.com/file/d/1u5veTKR_MzLk5nByaOqqE0QcBEM1qNI/view.

Here you can download **some Slides of the Talks**:

https://drive.google.com/drive/folders/1WqjFnFJzrqVU3A4YqP7hc5_B2FrKXGZ?usp=sharing.



IGT 2024 Proceedings (see <https://sites.google.com/unisa.it/igt/proceedings>)

Conference proceedings will be published in the [Springer Proceedings in Mathematics & Statistics](#) (PROMS). The PROMS series and its content are indexed in MathSciNet/MathReview, zbMATH, and Scopus. The submission **deadline** is **January 31, 2025 (23:59 CET)**.

Here you can find more details about the preparation of your article and about the submission.

Article template: https://drive.google.com/drive/folders/1XQSNITfU6JlhqMse5I4nwaf_UT7wHi_H.

Link to submit your article: <https://equinocs.springernature.com/service/igt2024>.

If you have any doubt, please contact us: [Patrizia Longobardi](mailto:plongobardi@unisa.it) [plongobardi@unisa.it],

[Carmine Monetta](mailto:cmonetta@unisa.it) [cmonetta@unisa.it] or [Marialaura Noce](mailto:mnoce@unisa.it) [mnoce@unisa.it].



Scientific Committee of IGT 2024 & Organizing Committee of IGT 2024

The screenshot shows the website for IGT 2024. The navigation menu includes: Home, Agenda, Proceedings, Participants, Zoom links, Registration, Accommodation, Travel info, and Previous editions of IGT. The page is divided into two main sections:

- Scientific Committee:**
 - Marina Bisi - University of Milan, Italy
 - Andrea Caranti - University of Turin, Italy
 - Costantino DeLizia - University of Salerno, Italy
 - Paolo Longobardi - University of Salerno, Italy
 - Maria Paola Ma - University of Salerno, Italy
 - Carlo Mito Scoppola - University of Bari, Italy
- Organizing Committee:**
 - Federico Di Concilio - University of Salerno, Italy
 - Michela Gualdi - University of Salerno, Italy
 - Mikel Eguzki Garcialeta Perez - University of Basque Country, University of Salerno, Spain
 - Valentina Giacomini - University of Salerno, Italy
 - Carolina Murillo - University of Salerno, Italy
 - Chiara Nicotri - University of Salerno, Italy
 - Maria Laura Nicosia - University of Salerno, Italy
 - Carolina Sica - University of Bari, Italy
 - Arianna Totola - University of Campania "Luigi Vanvitelli", Italy
 - Maria Rita - University of Salerno, Italy
 - Maria Virginia - University of Salerno, Italy

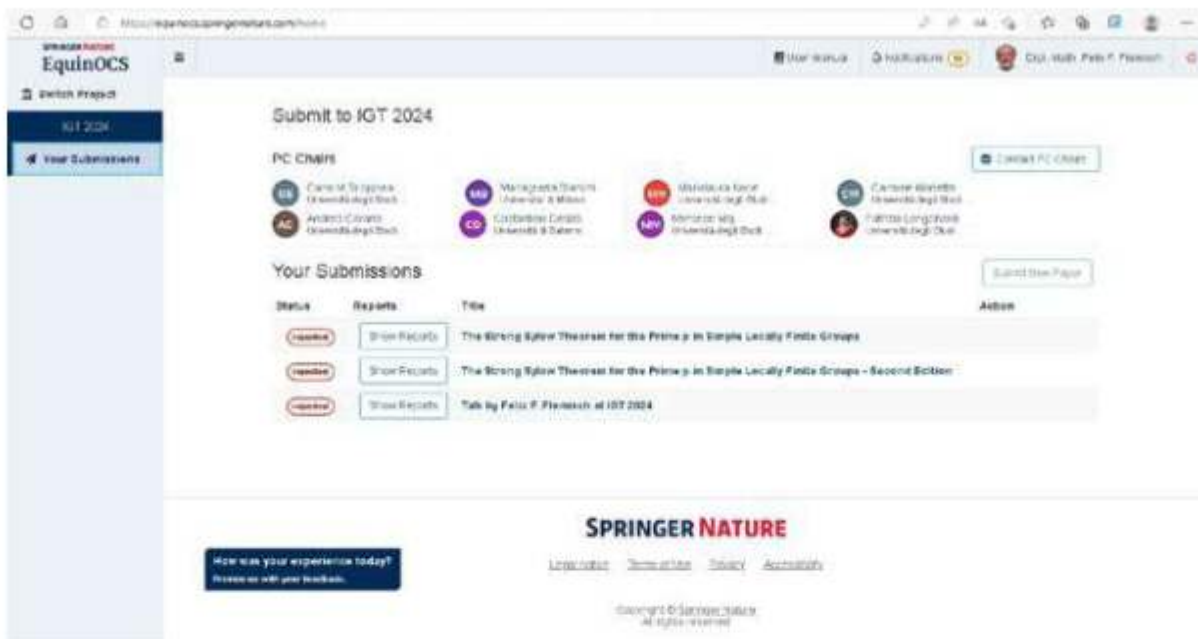
Editorial Board (PC Chairs) of the IGT 2024 Proceedings

It consists of the six members of the Scientific Committee of IGT 2024 and two members of the Organizing Committee of IGT 2024, Carlo M. Scoppola being the Editor-in-Chief.

The screenshot shows the EquinOCS website for IGT 2024. The page title is "IGT 2024 - Ischia Group Theory 2024". The content includes:

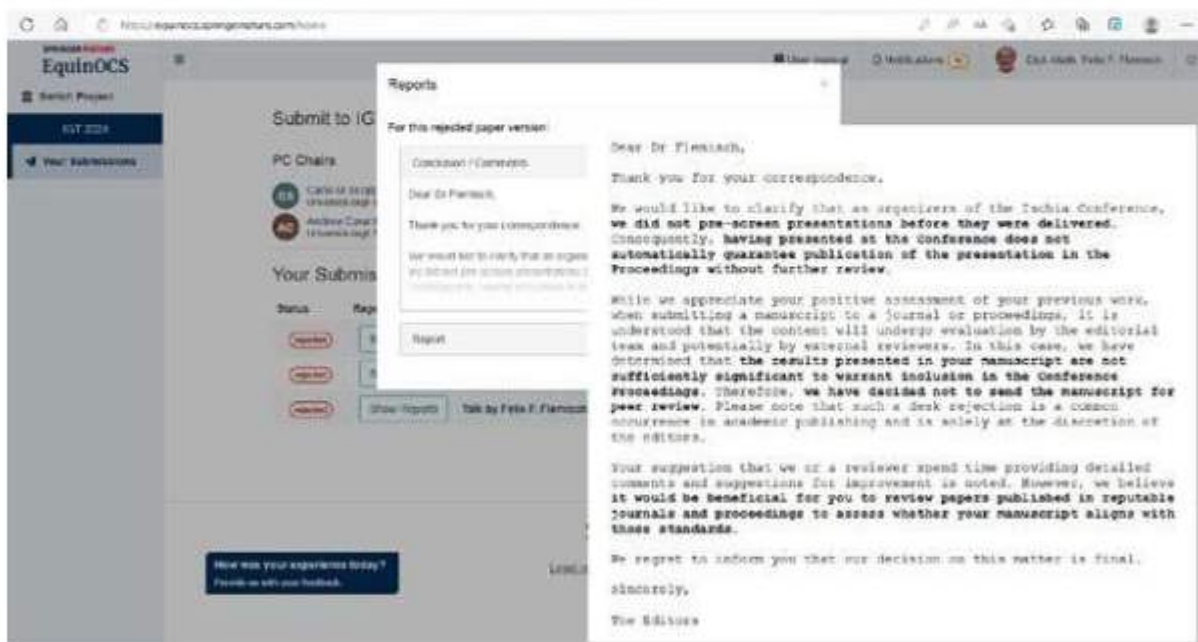
- Submission closed**
- Ischia Group Theory 2024** is a Conference on Group Theory held at "Grand Hotel delle Terme di Ferdinand", in Ischia (Naples, Italy), from Monday, April 6th, to Saturday, April 13th, 2024. The meeting begins with a Welcome Cocktail and the opening of the permanent sector session on Monday, April 6th, in the late afternoon. Talks started on Tuesday, April 6th, in the morning and concluded on Friday, April 12th, at the late afternoon.
- The social programme also included a Festival of Classical Neapolitan Songs on Tuesday evening, a Concert of Baroque music for lute and cello on Wednesday evening, the Sicilian Trio to Mozart's Gassone on Thursday morning, and the Confarone's Dinner on Friday evening.
- This edition of Ischia Group Theory is in honor of Otto H. Kegel on the occasion of his 100th birthday, to celebrate his significant role in the innovative landscape of Group Theory and his fundamental contribution to the Ischia Group Theory conference series.
- Location:** Ischia (Naples, Italy) 6-13 April 2024
URL: <http://www.ischia-conference.org>
- PC Chairs:**
 - Carlo M. Scoppola (University of Bari)
 - Maria Laura Nicosia (University of Salerno)
 - Costantino DeLizia (University of Salerno)
 - Maria Paola Ma (University of Salerno)
 - Carolina Sica (University of Bari)
 - Carolina Murillo (University of Salerno)
 - Paolo Longobardi (University of Salerno)

Submissions for the IGT 2024 Conference Proceedings



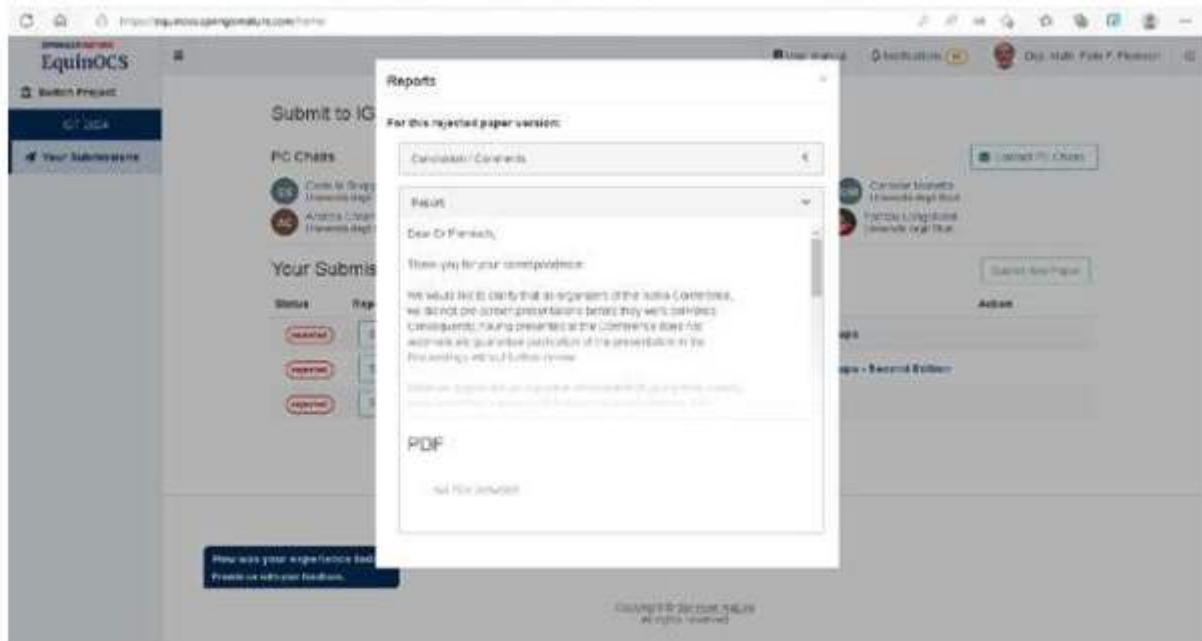
We submitted three submissions for the IGT 2024 Proceedings. All three were rejected by the Editorial Board.

Reports of the first Submission



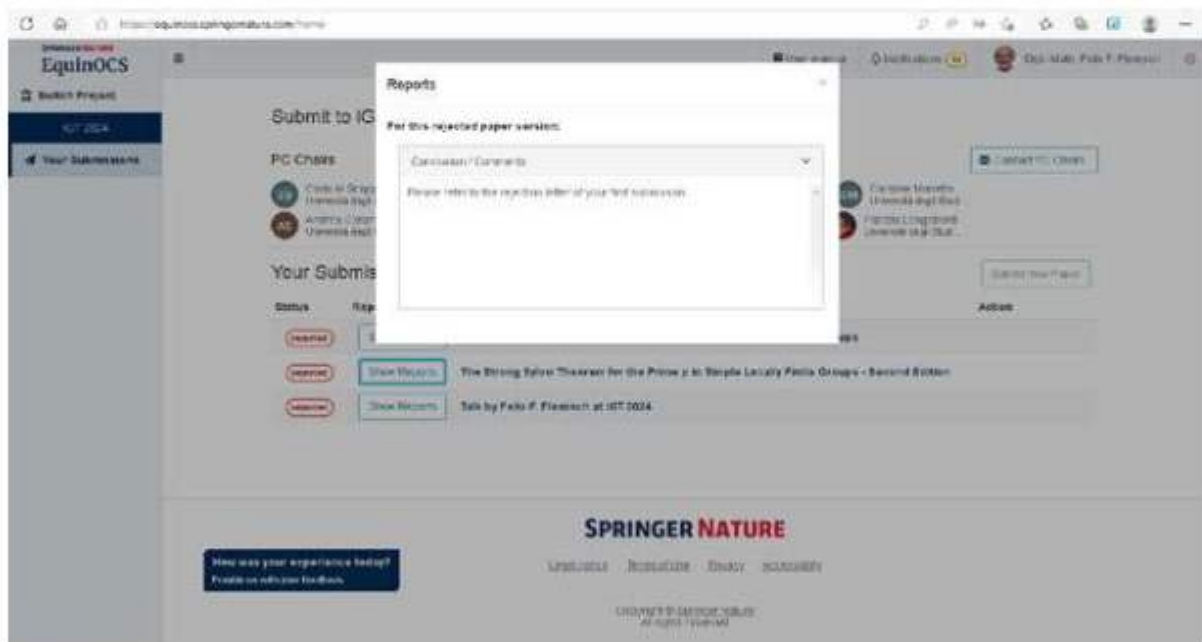
We protested against and even complained about the Rejection not as such but because it • did not give any reason, • claimed wrongly to have not pre-screened our presentation so that we are allegedly not entitled to have it published in the Conference Proceedings, • claimed unjustified (!) that the results presented are not sufficiently significant to warrant inclusion in the Conference Proceedings, and • did in fact by arrogant and overbearing attitude decide not to send our paper to peer review. The Rejection gets crowned by the incredibly impertinent suggestion to review papers published in reputable journals and proceedings, without being able to quote some of those (!), to assess whether our paper aligns with those standards. Our protest and complaint remained without any reaction by the Editorial Board. We then sent a Reprimand for the Rejection supported by some group theorists. No reaction again 😞 !

Report of the first Submission again



We nevertheless submitted a **Second Edition** of our beautiful paper where we corrected all presumed shortcomings thereby adding even more beauty and complied fully with the submission guidelines of Springer's PROMS series (see <https://www.springernature.com/gp/authors>).

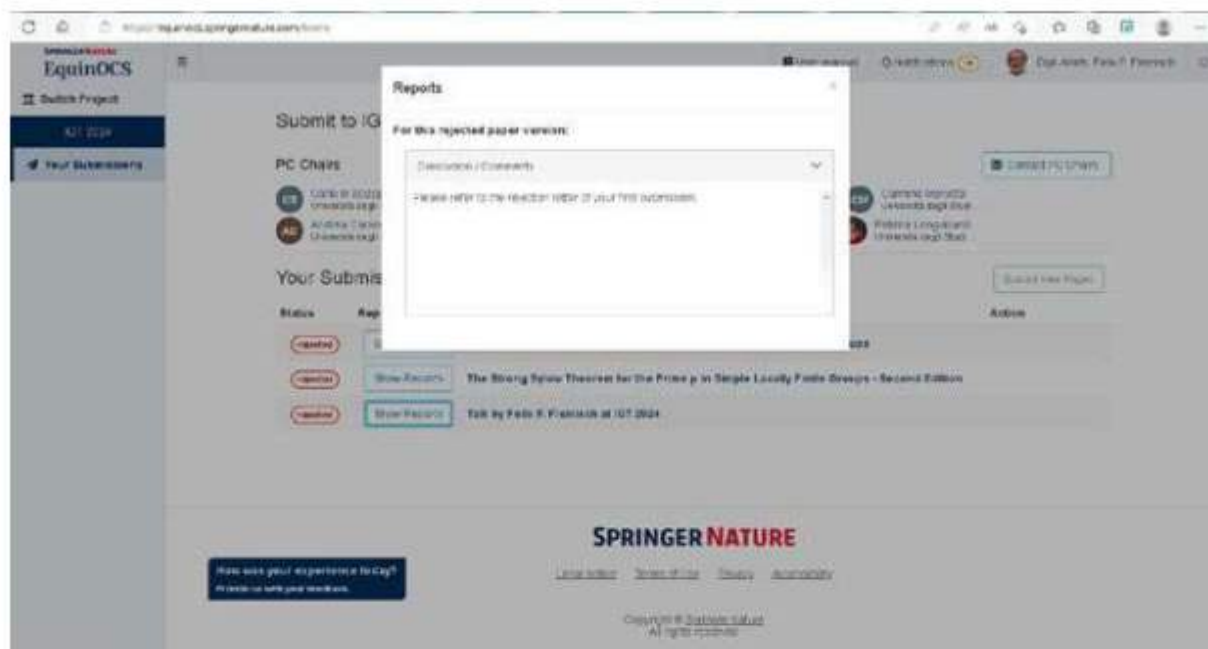
Reports of the second Submission



This **Second Edition** was not even read let alone reviewed but simply rejected with the comment “Please refer to the rejection letter of your first submission.”, a sheer incredible impertinence. We presented the paper in a **Talk** at IGT 2024 (see Page -4) and submitted an **Abstract** (see Page 57) to be included at https://drive.google.com/file/d/1u5veTKR_MzLlk5nByaOqqE0QcBEM1qNI/view (see Page -4). But the Committees did not include it Also, we were not included in the list of **Main Speakers** (see <https://sites.google.com/unisa.it/igt/home>).

Since we did not receive any reaction at all by the Editorial Board of IGT 2024 again, we reworked the **Second Edition** from Springer's PROMS format to the format for a submission for the Persian **International Journal of Group Theory (IJGT)**. They made a slovenly review of the first few pages, claimed wrongly that it has been published before (see Page 7), and rejected it without giving reasons as well instead of requiring an update. We submitted nevertheless a **Third Edition** which remained without any reaction ☹ .

Reports of the third Submission



We then submitted as the third Submission just our **Talk at IGT 2024**, which has 18 slides in contrast to the underlying **research paper** which has about 70 pages, and requested to find a place in the Conference Proceedings. But it was treated as our second Submission irrespective of that the rejection letter of our first Submission cannot apply to the third Submission. Getting not any reaction we remained speechless.

The third Submission – Slide 1 to Slide 9



The third Submission – Slide 10 to Slide 18



While the third Submission to the IGT 2024 Conference Proceedings, the **beautiful Talk by Felix F. Flemisch at IGT 2024**, was rejected by its Editorial Board **without giving any reason**, it was accepted by the Journal of Mathematical & Computer Applications (JMCA) (see Page 73 to Page 84).

We puzzle over the disrespectful treatment of our **Three beautiful Submissions** for the IGT 2024 Conference Proceedings. Possible answers are given by the following quotations:



Realizing that we were not welcome neither at the IGT 2024 Proceedings nor at the IJGT, we decided to terminate our king-size patience with both of them and to try to publish our beautiful paper as a **Research Article** in the great Journal of Mathematical & Computer Applications (JMCA). We and the JMCA know each other since 2023 when we could publish in September our **Review Article** "**Three Beautiful Books about Sylow Theory in Locally Finite Groups**" in **Volume 2 Issue 3** of JMCA (see <https://www.onlinescientificresearch.com/journal-of-mathematical-computer-applications-old-articles.php?journal=jmca&&v=2&&i=3&&y=2023&&m=September>). We reworked and extended and grew more beautiful our paper "**The Strong Sylow Theorem for the Prime p in Simple Locally Finite Groups**" into a **Fourth Edition** which, after a number of fine improvements, was accepted for publication in **Volume 4 Issue 1** of JMCA (see Page -10 and Page -11 & Page 1 to Page 71) ☺. It comes with an **Abstract** (see Page -12 and Page -13 & Page 1) which can well serve as a **detailed overview**.

JMCA Volume 4, Issue 1 Articles

The screenshot displays the homepage of the Journal of Mathematical & Computer Applications (JMCA). At the top, there is a search bar with a 'Google' button. Below it, a navigation menu includes links for Home, Members, PPTs, Video Articles, Journals, e-books, Article Process, Guidelines, Membership, Submit Online, and Contact. The journal's title, 'Journal of Mathematical & Computer Applications', is prominently displayed in red. The ISSN is 2155-4515, and the logo 'SCIENTIFIC Research and Community' is visible. A 'Journal Menu' on the right lists various sections like Journal Home, Aims and Scope, Call for Papers, Editorial Board, Impress, Current Issue, Archive, Journal Guidelines, and Submit Manuscript. The main content area features three article listings, each with a 'View Abstract' and 'View/Download PDF' button. The first article is a research article by Felix F. Flemisch, the second is a review article by Mohrish Neelapu, and the third is a review article by Pascal Stiefenhofer.

<https://www.onlinescientificresearch.com/journal-of-mathematical-computer-applications-oid-articles.php?journal=jmca&lv=4&db=1&by=2025&em=February>

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Journal of Mathematical & Computer Applications

Volume 4, Issue 1 Articles

Research Article
The Strong Sylow Theorem for the Prime p in Simple Locally Finite Groups
Author(s): Felix F. Flemisch
Published Date: 2025-01-27
[View Abstract](#) [View/Download PDF](#)
DOI: doi.org/10.47363/JMCA/2025(4)198

Review Article
Impact of Automation in Software Testing on Defect Discovery Rates
Author(s): Mohrish Neelapu
Published Date: 2025-02-18
[View Abstract](#) [View/Download PDF](#)
DOI: doi.org/10.47363/JMCA/2025(4)203

Review Article
Machine Learning Insights into the Dynamics of Cusp Catastrophe Instability Region
Author(s): Pascal Stiefenhofer
Published Date: 2025-01-07
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DOI: doi.org/10.47363/JMCA/2025(4)190

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Review Article

A New Matrix Decomposition Method for Inverting the Comrade Matrix

Author(s): B. Tolibi*, A. Aïni (Lead) and D. Sarai

Published Date: 2025-02-04

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DOI: doi.org/10.47363/JMCA/2025(4)206

Our Pubmed Indexed Articles

Detecting Peripheral Neuropathy in Patients with Diabetes, Prediabetes and other High-Risk Conditions: An Advanced Practice Nurse's Perspective
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An Analysis of Peripheral Neuropathy Symptom Characteristics in HIV
PMID: 35174365

Overview of Neurotrauma and Sensory Loss
PMID: 35692955

A mobile app providing individually-tailored psychoeducation about sleep for older adults with chronic health conditions and low health literacy
PMID: 38770111

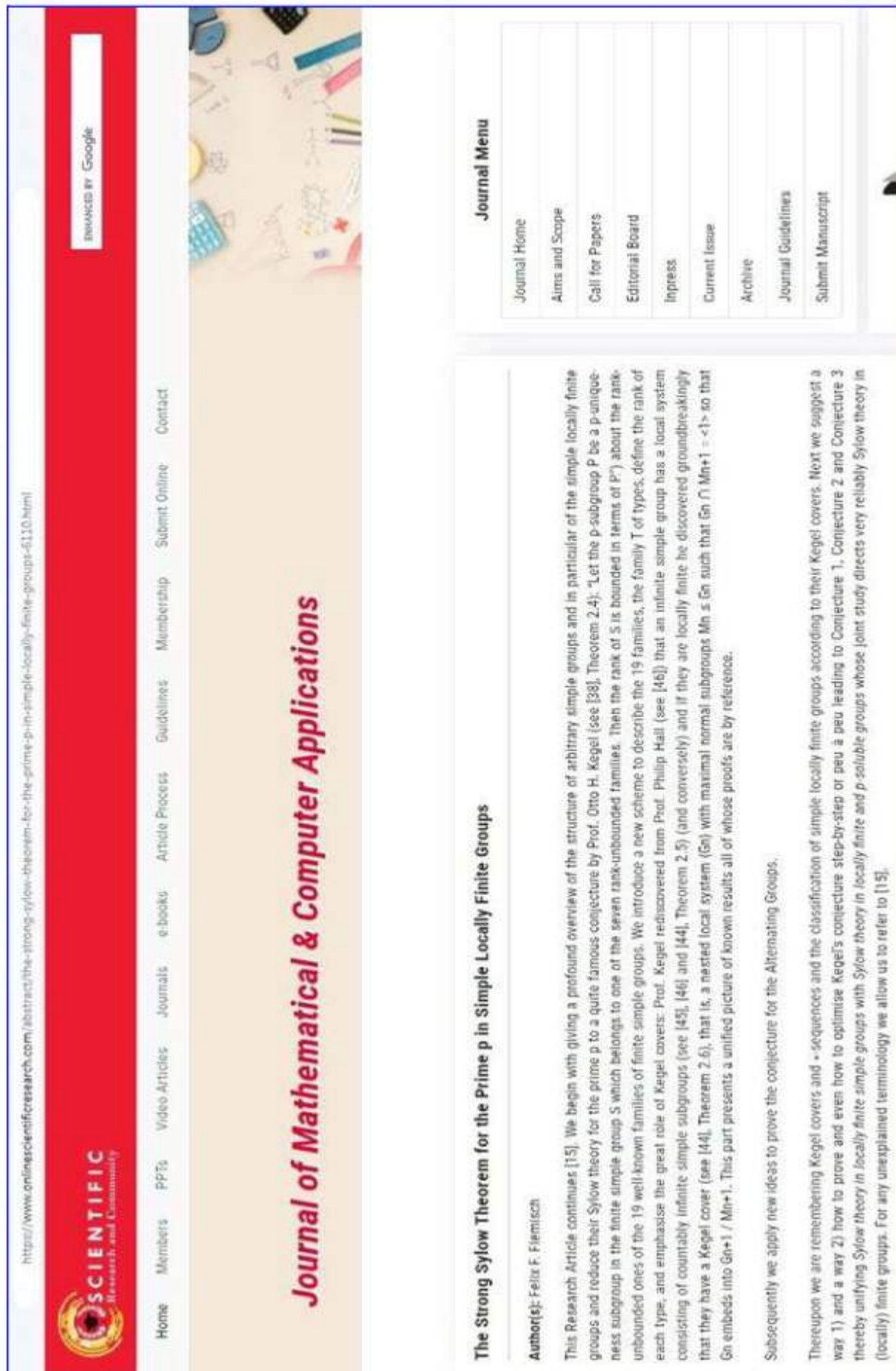
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Abstract of the following Research Article



The screenshot displays the journal's website interface. At the top, there is a search bar with the text "search by Google" and a URL: <https://www.onlinescientificresearch.com/abstract/the-strong-sylow-theorem-for-the-prime-p-in-simple-locally-finite-groups-6110.html>. Below the search bar is a navigation menu with links: Home, Members, PPTs, Video Articles, Journals, e-books, Article Process, Guidelines, Membership, Submit Online, and Contact. The main content area features the journal's logo, "SCIENTIFIC Research and Community", and the title "Journal of Mathematical & Computer Applications" in a large, bold, red font. Below the title, the article title "The Strong Sylow Theorem for the Prime p in Simple Locally Finite Groups" is displayed, followed by the author's name, "Author(s): Felix F. Flemisch". The abstract text begins with "This Research Article continues [15]. We begin with giving a profound overview of the structure of arbitrary simple groups and in particular of the simple locally finite groups and reduce their Sylow theory for the prime p to a quite famous conjecture by Prof. Otto H. Kegel (see [38], Theorem 2.4). Let the p -subgroup P be a p -unique-subgroup in the finite simple group S which belongs to one of the seven rank-unbounded families. Then the rank of S is bounded in terms of P about the rank-unbounded ones of the 19 well-known families of finite simple groups. We introduce a new scheme to describe the 19 families, the family T of types, define the rank of each type, and emphasise the great role of Kegel covers: Prof. Kegel rediscovered from Prof. Philip Hall (see [46]) that an infinite simple group has a local system consisting of countably infinite simple subgroups (see [45], [46] and [44], Theorem 2.5) (and conversely) and if they are locally finite he discovered groundbreakingly that they have a Kegel cover (see [44], Theorem 2.6), that is, a nested local system (G_n) with maximal normal subgroups $M_n \leq G_n$ such that $G_n \cap M_{n+1} = \langle 1 \rangle$ so that G_n embeds into G_{n+1} / M_{n+1} . This part presents a unified picture of known results all of whose proofs are by reference. Subsequently we apply new ideas to prove the conjecture for the Alternating Groups. Thereupon we are remembering Kegel covers and p -sequences and the classification of simple locally finite groups according to their Kegel covers. Next we suggest a way 1) and a way 2) how to prove and even how to optimise Kegel's conjecture step-by-step or p -step leading to Conjecture 1, Conjecture 2 and Conjecture 3 thereby unifying Sylow theory in locally finite simple groups with Sylow theory in locally finite and p -soluble groups whose joint study directs very reliably Sylow theory in (locally) finite groups. For any unexplained terminology we allow us to refer to [15].

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PMID: 35445219

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PMID: 35174365

Overview of Neurotrauma and Sensory Loss
PMID: 35692955

A mobile app providing individually-tailored psychoeducation about stress for older adults with chronic health conditions and low health literacy
PMID: 38770111

We then continue the program begun above to optimise along the way 1) the theorem about the first type $\Xi = \Delta^{n^0}$ of infinite families of finite simple groups step-by-step to further types by proving it for the second type $\Xi = \Delta = \text{PSL}_n$. We apply new ideas to prove Conjecture 2 about the General Linear Groups over locally finite fields, stating that their rank is bounded in terms of their p -uniqueness, and then break down this insight to the Special Linear Groups and to the Projective Special Linear (PSL) Groups over locally finite fields. We close with good suggestions for future research ▶ regarding the remaining rank-unbounded types (the "Classical Groups") and the way 2), ▶ regarding (locally) finite and p -soluble groups, and ▶ regarding Cauchy's and Galois' contributions to Sylow theory in finite groups. We much hope to enthuse group theorists with these suggestions and are ready to support and to coordinate all related work.

It follows from our two theorems that simple locally finite groups which satisfy the Strong Sylow theorem for even one Prime p are linear and hence countable if they have a local system of countable simple subgroups each having a Kegel cover "of alternating type" or "of projective special linear type".

We include the beautiful predecessor Research Article [15] as the First Appendix for good reasons. This Research Article had been presented as a slideshow in a Talk at IGT 2024 on April 11. We include its 16 slides as the Second Appendix. Slide 1 to Slide 12 had been permanently installed during IGT 2024 as a Permanent Poster.

The Research Article consists of the following seventeen beautiful Chapters:

- Sketch of proof for Δ^n . ● Sketch of proof for $\Delta = \text{PSL}_n$. ● Introduction; ● Proof of Theorem 1;
- About Kegel covers; ● Planning future research – Part 1; ● Proof of Theorem 2;
- Proof of Theorem 3; ● Proof of Theorem 4; ● Planning future research – Part 2;
- The First Trilogy and The Second Trilogy and their reviews; ● Acknowledgements;
- Postscript; Luciano De Crescenzo, Felix F. Flemisch, Conflicts of Interest, Pablo Picasso's *Les Femmes d'Alger*;
- About the author in Munich, in Freiburg i.Br., in London, in Weiden i.d.OPf., and in Florence in Tuscany in Italy;
- 75 References; ● Appendix 1 – Reference [15] with MR Review and Zbl Review;
- Appendix 2 – Talk by Felix F. Flemisch at Ischia Group Theory 2024.

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The IGT 2024 Conference Proceedings

Since IGT 2024 is dedicated to Prof. Otto H. Kegel on the occasion of his 90th birthday on 20 July 2024, we asked the Editorial Board twice how he is honoured in the Conference Proceedings:



We did not get any answer Hence we shall be able to answer our question ourselves only after publication of the IGT 2024 Conference Proceedings.

The publication of the IGT 2024 Conference Proceedings has been announced since mid of December 2024 at the Web page: <https://sites.google.com/uniwa.it/igt/proceedings> (see also Page -4):

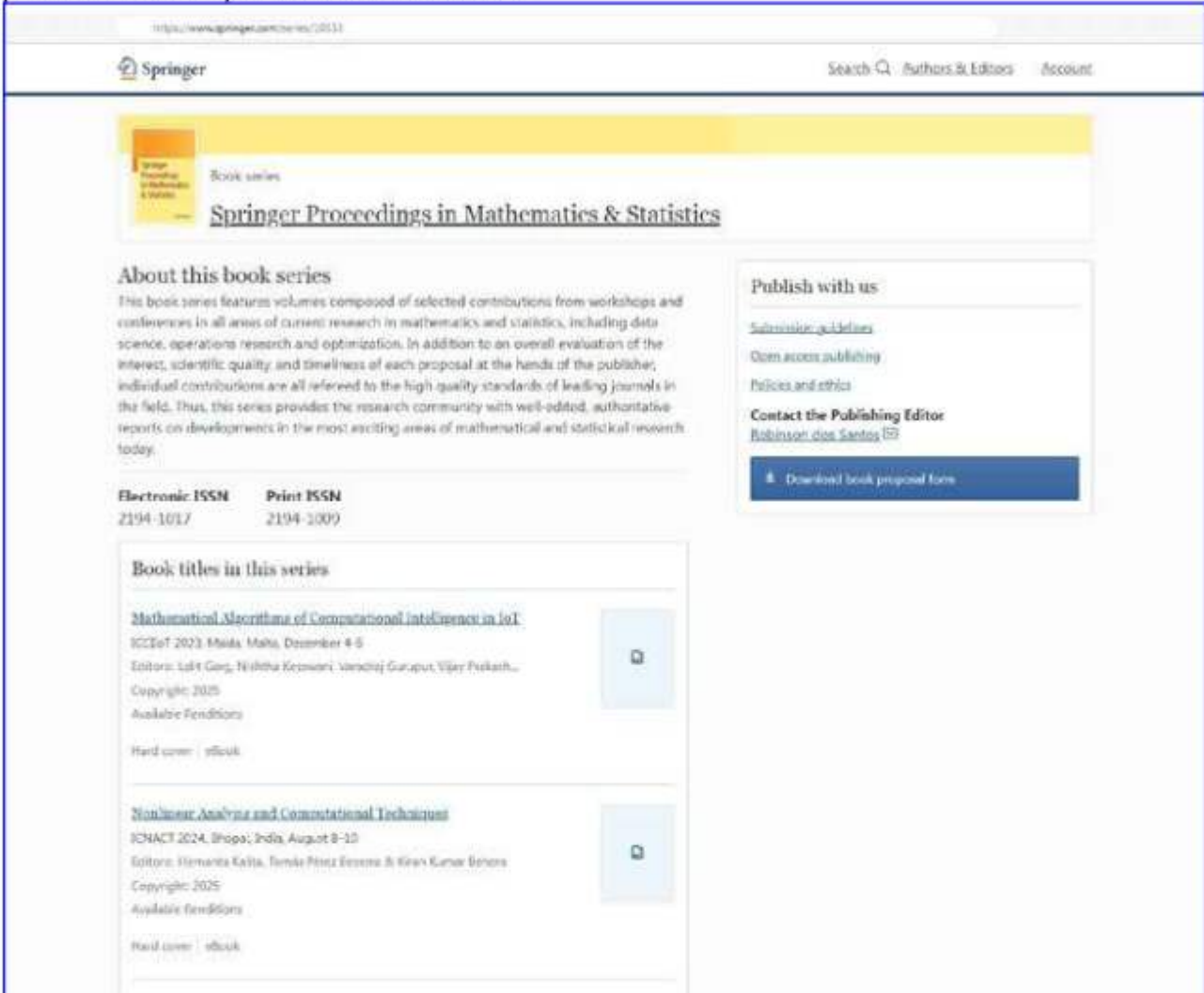


After the deadline this Web page was maintained (instead of reporting the progress of the publication preparations). We asked for the publication date: "According to <https://sites.google.com/uniwa.it/igt/proceedings> the deadline for the IGT 2024 Conference Proceedings was 31 January 2025. When do you now plan to publish the Proceedings at <https://www.springer.com/series/106337>" and got the answer: "The volume is not yet ready, we are still waiting for some reports. I hope it will be published before Summer, but I am not sure." On <day> <month> 2025 the Web page changed to



and at the same time appeared at <https://www.springer.com/series/10533> the publication announcement:

placeholder for the publication announcement



The screenshot shows the Springer website for the 'Springer Proceedings in Mathematics & Statistics' book series. The page features a search bar, navigation links for 'Authors & Editors' and 'Account', and a 'Publish with us' section. The 'Publish with us' section includes links for 'Submission guidelines', 'Open access publishing', and 'Policies and ethics', along with contact information for the Publishing Editor, Robinson dos Santos, and a 'Download book proposal form' button. Below this, there is a section for 'Book titles in this series' with two entries: 'Mathematical Algorithms of Computational Intelligence in IoT' and 'Nonlinear Analysis and Computational Techniques'.

When clicking on the publication announcement, one gets all the details of the announcement at <https://link.springer.com/book/978-3-031-abcde-f>, divided into the six sections Overview, About this book, Keywords, Editors and Affiliations, Bibliographic Information and Publish with us:

placeholders for the details of the publication announcement



The screenshot shows the details of the book 'Addressing Modern Challenges in the Mathematical, Statistical, and Computational Sciences'. The page includes a book cover, title, subtitle, and a list of sections: Overview, About this book, Keywords, Editors and Affiliations, Bibliographic Information, and Publish with us. The 'Overview' section includes a list of bullet points: 'Compiles contributions from experts worldwide', 'Investigates mathematical methods across natural, social, life, and applied sciences', and 'Promotes a truly interdisciplinary approach to addressing real-world problems'. The 'Publish with us' section includes a link to 'Download book proposal form'.

About this book

This proceedings volume features a selection of peer-reviewed papers presented at the 6th AMMCS-International Conference on Applied Mathematics, Modeling, and Computational Science, held in Waterloo, Canada, from August 14–18, 2023. The papers delve into topics where mathematical modeling and applications play a pivotal role, including computational models in physics and chemistry, statistical models in life science, analysis in science and engineering, and finance and social science methods, among others.

Since 2011, the AMMCS conference series has provided a unique platform for technical discussions and the exchange of ideas in all areas related to mathematical, statistical, and computational sciences, modeling, and simulation. Esteemed researchers, industrialists, engineers, and students have presented their latest research and engaged with experts in the field, fostering interdisciplinary collaborations that address the challenges of modern science, technology, and society. This book is a valuable resource for academics and practitioners who are interested in the latest developments in these fields.

Keywords

Computer science Math conference proceedings Coupled systems modeling
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Editors and Affiliations

Wilfrid Laurier University, Waterloo, Canada
D. Marc Kilgour, Roman Makarov, Roderick Melnik, Xu Wang

University of Guelph, Guelph, Canada
Herb Kunze

Bibliographic Information

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Series Title Springer Proceedings in Mathematics & Statistics	Publisher Springer Cham	eBook Packages Mathematics and Statistics, Mathematics and Statistics (RO)
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After publication of the IGT 2024 Conference Proceedings we shall not only answer our question from Page -14 but also provide quite a number of comments (see Page -17 and Page -18).

Had our submissions to the IGT 2024 Proceedings been well accepted (see Pages -6 ff.), we could have presented them with the following Front Cover and Basic Information about the IGT 2024 Conference:

<div style="background-color: #f96; padding: 5px;"> <p>Springer Proceedings in Mathematics & Statistics Volume 132</p> <p>Carlo Maria Scoppola Mariagrazia Bianchi Andrea Caranti Costantino Dell'isa Patrizia Longobardi Mercedes Ma Carmine Monetta Mariagrazia Noce</p> <p style="text-align: center;">Editors</p> </div> <div style="background-color: #fff9c4; padding: 5px; text-align: center;"> <p>Ischia Group Theory 2024</p> <p>Ischia off Naples in Italy at Grand Hotel delle Terme Re Ferdinando, from Monday, April 8, to Saturday, April 13</p> </div> <p>The Strong Sylow Theorem for the Prime p in Simple Locally Finite Groups</p> <p>Felix F. Fleisch</p>  <p><i>Dedicated to Prof. Otto H. Kegel on the occasion of his 90th birthday on July 29th, 2024 – Ischia Group Theory 2024 from April 8th to April 13th. Talk presented at IGT 2024 on Thursday, April 11th, 2024, from 18.00 to 18.30.</i></p> <div style="display: flex; justify-content: space-around; align-items: center;">   </div> <p><i>this is, on the 120th birthday of Prof. Philip Hall (see https://mathshistory.st-andrews.ac.uk/biographies/hall/)</i></p> <p>Felix F. Fleisch (✉) Mühlerweg 4e, 82211 Hevringing a, Ammersee, Bavaria, Germany e-mail: felix.fleisch@hotmail.com</p> <p>© Springer Nature Singapore Pte Ltd. 2025 Carlo Maria Scoppola et al. (eds.), Ischia Group Theory 2024, Springer Proceedings in Mathematics & Statistics 132, https://doi.org/10.1007/978-3-abc-defgh</p>	<p style="text-align: center;">ISCHIA GROUP THEORY 2024</p> <div style="text-align: center;">  <p>Ischia (Naples, Italy) April, 8th – 13th</p> <p>https://sites.google.com/umisa.it/igt</p> </div> <p>It is a great pleasure for us to present the Conference "Ischia Group Theory 2024", an important periodic appointment for the Group Theory international community, held in Ischia (Naples, Italy) at "Grand Hotel delle Terme Re Ferdinando". The meeting will consist of Talks given by invited speakers and a great permanent Poster Session. The social programme will include the Welcome Cocktail on Monday late afternoon, a Buffet of Classical Neapolitan Songs on Tuesday evening, a beautiful Concert of Baroque music for flute and cello on Wednesday evening, a Social Trip to Merella Gardens on Thursday morning, and the Conference Dinner on Friday evening.</p> <p>The 11th edition of Ischia Group Theory will be in honour of Otto H. Kegel on the occasion of his 90th birthday, to celebrate his significant role in the international landscape of Group Theory and his many fundamental contributions to the Ischia Group Theory Conference series.</p> <p>The great Scientific Committee is formed by Mariagrazia Bianchi (Università di Milano), Andrea Caranti (Università di Trento), Costantino Dell'isa (Università di Salerno), Patrizia Longobardi (Università di Salerno), Mercedes Ma (Università di Salerno) and esteemed Carlo Maria Scoppola (Università dell'Aquila), while the great Organizing Committee consists of Federico & Concilio (Università di Salerno), Michele Gatta (Università di Salerno), Mikol Ejzaki, Gerisena Perez (Università di Salerno & University of the Balearic Islands, Italia & Spain), Valentin Gassan (Università di Salerno), Carmine Monetta (Università di Salerno), Oreste Nicosia (Università di Salerno), Mariagrazia Noce (Università di Salerno), Carmela Iuso (Università Federal de Bahia, Brazil), Antonio Toranzo (Università della Campania "Luigi Vanvitelli"), Maria Tota (Università di Salerno) and Marius Vigiata (Università di Salerno).</p> <table border="0" style="width: 100%; background-color: #e0ffe0;"> <tr> <td>The Strong Sylow Theorem for the Prime p in Simple Locally Finite Groups</td> <td style="text-align: right;">1</td> </tr> <tr> <td>Talk on 11 April 2024 (Introduction and Poster)</td> <td style="text-align: right;">73</td> </tr> <tr> <td>Agenda (Monday, April 8th, 2024, to Friday, April 12th, 2024)</td> <td style="text-align: right;">89</td> </tr> <tr> <td>Participants and Speakers and Venue</td> <td style="text-align: right;">93</td> </tr> </table>	The Strong Sylow Theorem for the Prime p in Simple Locally Finite Groups	1	Talk on 11 April 2024 (Introduction and Poster)	73	Agenda (Monday, April 8th, 2024, to Friday, April 12th, 2024)	89	Participants and Speakers and Venue	93
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List of Open Issues

- **OI 1** Extend Theorems 2, 3 and 4 to the remaining Locally Finite Classical Groups (i.e., the Symplectic, Unitary, Orthogonal in char $\neq 2$ and Orthogonal in char 2 Groups).
- **OI 2** Summarise the work by B. Hartley and A. Rae regarding λ_p and p^{2p} and the foregoing work on the classical Hall-Higman theory regarding λ_p and p^{2p} , c_p , d_p , p^{2p} and r_p by P. Hall, G. Higman, A.H.M. Hoare, T.R. Berger, F. Gross, E.G. Bryukhanova and A. Turell.
- **OI 3** Let p be a prime. Let G be a p -soluble finite group, $\lambda_p(G)$ be its p -length, and $a_p(G)$ be its p -uniqueness. Then (best possible) $\lambda_p(G) \leq a_p(G) + 1$.
- **OI 4** Deduce the three rectangles/tableaux of Page 13 and use them to prove Lagrange's theorem and Cauchy's concealed second and third group theorems.
- **OI 5** Deduce Cauchy's theorem of 1812/1815 from $[H:\langle x \rangle] \geq |G|$, if x is a p -blank of G in H , that is, an element of H of order p with $x \notin G$.
- **OI 6** Determine all the (minimal) p -uniqueness subgroups for the known finite simple groups and their natural overgroups, the symmetric and the linear groups, and for the (locally) p -soluble groups, distinguishing $p \geq 5$, $p = 3$ and $p = 2$.
- **OI 7** Introduce for every prime p a generalised p -length for locally finite groups which is finite if and only if they satisfy the strong Sylow Theorem for p .
- **OI 8** Making a revision of Kegel's (3.5) Theorem [44] thereby relating it to rarely known articles, extend Kegel's (4.4) Theorem [44] to the case $p = 3$.
- **OI 9** Summarise Kegel's Sylow paper [44] thereby integrating the AGTA paper [15] and this JMCA paper and extending his main Theorem (4.4) to $p = 3$ and $p = 2$ by using OI 8 and OI 3.

Issues of the De Luxe Edition

This is the **May 2025 Issue** of the **De Luxe Edition** being characterised by still waiting for the announcement of the IGT 2024 Conference Proceedings. Future (monthly) issues, in particular the **<month> 2025 Issue** when the IGT 2024 Proceedings have been announced to be published, will consist of updates only of Page -1 to Page -18 and the eighth-page Front Cover and possibly the Back Cover, that is, of 28+-pages documents, and **for Page 1 to Page 266 the De Luxe Edition will refer to the May 2025 Issue**. The simple reason is that the **Research Article** itself, its **Abstract** and its **PowerPoint Presentation**, as published by JMCA, will not change.

The Strong Sylow Theorem for the Prime p in Simple Locally Finite Groups

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ABSTRACT

This Research Article continues [15]. We begin with giving a profound overview of the structure of arbitrary simple groups and in particular of the simple locally finite groups and reduce their Sylow theory for the prime p to a quite famous conjecture by Prof. Otto H. Kegel (see [44], Theorem 2.4: “Let the p -subgroup P be a p -uniqueness subgroup in the finite simple group S which belongs to one of the seven rank-unbounded families. Then the rank of S is bounded in terms of P .”) about the rank-unbounded ones of the 19 well-known families of finite simple groups. We introduce a new scheme to describe the 19 families, the family \mathcal{T} of types, define the rank of each type, and emphasise the great rôle of Kegel covers: Prof. Kegel rediscovered from Prof. Philip Hall (see [46]) that an infinite simple group has a local system consisting of countably infinite simple subgroups (see [45], [46] and [44], Theorem 2.5) (and conversely) and if they are locally finite he discovered groundbreakingly that they have a Kegel cover (see [44], Theorem 2.6), that is, a nested local system $\{G_n\}$ with maximal normal subgroups $M_n \leq G_n$ such that $G_n \cap M_{n+1} = \langle 1 \rangle$ so that G_n embeds into G_{n+1}/M_{n+1} . This part presents a unified picture of known results all of whose proofs are by reference.

Subsequently we apply new ideas to prove the conjecture for the Alternating Groups.

Thereupon we are remembering Kegel covers and \star -sequences and the classification of simple locally finite groups according to their Kegel covers. Next we suggest a way 1) and a way 2) how to prove and even how to optimise Kegel’s conjecture step-by-step or peu à peu which leads to Conjecture 1, Conjecture 2 and Conjecture 3 thereby unifying Sylow theory in locally finite simple groups with Sylow theory in locally finite and p -soluble groups whose joint study directs very reliably Sylow theory in (locally) finite groups. For any unexplained terminology we allow us to refer to [15].

We then continue the program begun above to optimise along the way 1) the theorem about the first type $\Xi = “A^n”$ of infinite families of finite simple groups step-by-step to further types by proving it for the second type $\Xi = “A = \text{PSL}_n”$. We apply new ideas to prove Conjecture 2 about the General Linear Groups over locally finite fields, stating that their rank is bounded in terms of their p -uniqueness, and then break down this insight to the Special Linear Groups and to the Projective Special Linear (PSL) Groups over locally finite fields. We close with good suggestions for future research ► regarding the remaining rank-unbounded types (the “Classical Groups”) and the way 2), ► regarding (locally) finite and p -soluble groups, and ► regarding Cauchy’s and Galois’ contributions to Sylow theory in finite groups. We much hope to enthuse group theorists with these suggestions and are ready to support and to coordinate all related work.

It follows from our two theorems that simple locally finite groups which satisfy the Strong Sylow Theorem for even one Prime p are linear and hence countable if they have a local system of countable simple subgroups each having a Kegel cover “of alternating type” or “of projective special linear type”.

We include the beautiful predecessor Research Article [15] as the First Appendix for good reasons. This Research Article had been presented as a slideshow in a Talk at IGT 2024 on April 11. We include its 16 slides as the Second Appendix. Slide 1 to Slide 12 had as well been permanently installed during IGT 2024 as a Permanent Poster.

The Research Article consists of the following seventeen beautiful Chapters:

- Sketch of proof for A^n ; • Sketch of proof for $A = \text{PSL}_n$; ① Introduction; ② Proof of Theorem 1;
- ③ About Kegel covers; ④ Planning future research – Part 1; ⑤ Proof of Theorem 2;
- ⑥ Proof of Theorem 3; ⑦ Proof of Theorem 4; ⑧ Planning future research – Part 2;
- ⑨ The First Trilogy and The Second Trilogy and their reviews; • Acknowledgements;
- Postscript, Luciano De Crescenzo, Felix F. Flemisch, Conflicts of Interest, Pablo Picasso’s *La Joie de vivre*;
- About the author in Munich, in Freiburg i.Br., in London, in Weiden i.d.OPf., and in Florence in Tuscany in Italy;
- 75 References; • Appendix 1 – Reference [15] with MR Review and Zbl Review;
- Appendix 2 – Talk by Felix F. Flemisch at Ischia Group Theory 2024.

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Dedicated to **Prof. Otto H. Kegel** on the occasion of his 90th birthday on 20 July 2024 – Ischia Group Theory 2024 from April 8 to April 13 (see <https://www.advgrouptheory.com/IGTArchivum/Pictures/igtphotos/OttoKegel.jpg>)

Talk presented at IGT 2024 on 11 April 2024, that is,



on the 120th birthday of **Prof. Philip Hall** (see <https://mathshistory.st-andrews.ac.uk/Biographies/Hall/>)

Keywords • singular (Sylow) p -subgroup • (very) good Sylow p -subgroup • p -uniqueness subgroup • minimal p -unique subgroup • **very beautiful** (numerical) Sylow p -invariant p -uniqueness a_p • locally finite group satisfying the Strong Sylow Theorem for the Prime p , equivalently, the Strong Sylow p -Theorem • simple group • nested local system • family \mathcal{T} of types of known finite simple groups • simple locally finite group of type $\Xi \in \mathcal{T}$, of alternating type and of projective special linear type • rank of a locally finite simple group • classification of the transitive G -sets • **beautiful** Kegel cover • \star -sequence • Kegel sequence • simple locally finite group which is finitary, of 1-type, of p -type, and of ∞ -type • P -invariant Sylow p -subgroup • conjugacy class • P -isomorphic P -orbit • **beautiful** p -length of a p -soluble finite group • classical Hall-Higman Theory • locally finite field \mathcal{F} • algebraic closure of the **beautiful** prime field in characteristic p • General Linear Group • Special Linear Group • Projective Special Linear (PSL) Group • G -module over some (locally finite) field \mathcal{F} • irreducibility • complete reducibility • (non-)modular G -module • G -isomorphic G -modules • Jordan normal form • Classical Group • Group of Lie type • twisted Chevalley Group

Note – The **rank** of a known locally finite simple group is defined below. For $\text{PSL}(n, \mathcal{F})$, and hence for $\text{GL}(n, \mathcal{F})$ and $\text{SL}(n, \mathcal{F})$, it is simply $n = \dim(\mathcal{F}^n)$. So we have a **rather simple** concept of rank of a linear group which, however, does not contradict any of the elaborate concepts of rank in the excellent book [13].

Let p be a prime: 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97, 101, 103, 107, 109, 113, 127, 131, 137, 139, 149, 151, 157, 163, 167, 173, 179, 181, 191, 193, 197, 199, 211, 223, 227, 229, 233, 239, 241, 251, 257, 263, 269, 271, 277, 281, 283, 293, 307, 311, 313, 317, 331, 337, 347, 349, 353, 359, 367, 373, 379, 383, 389, 397, 401, 409, 419, 421, 431, 433, 439, 443, 449, 457, 461, 463, 467, 479, 487, 491, 499, 503, 509, 521, 523, 541, 547, 557, 563, 569, 571, 577, 587, 593, 599, 601, 607, 613, 617, 619, 631, 641, 643, 647, 653, 659, 661, 673, 677, 683, 691, 701, 709, 719, 727, 733, 739, 743, 751, 757, 761, 769, 773, 787, 797, 809, 811, 821, 823, 827, 829, 839, 853, 857, 859, 863, 877, 881, 883, 887, 907, 911, 919, 929, 937, 941, 947, 953, 967, 971, 977, 983, 991, 997, 1009, 1013, 1019, 1021, 1031, 1033, 1039, 1049, 1051, 1061, 1063, ... ☺

In this paper we prove **Kegel's conjecture** for \mathbb{A}^n and for $\mathbb{A} = \text{PSL}_n$. It continues [15] F.F. FLEMISCH: “**Characterising Locally Finite Groups Satisfying the Strong Sylow Theorem for the Prime p** ”, *Adv. Group Theory Appl.* **13** (June 2022),

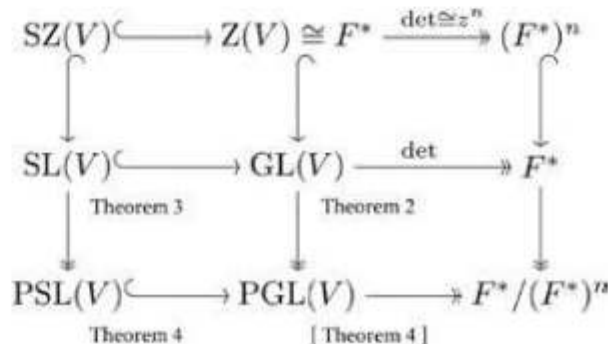
13-39 (see MR4441631 and Zbl 1496.20065). We included that **beautiful** predecessor paper completely as an **Appendix**, although it is open access, since the current paper cannot be understood without that predecessor paper – so one needs to have it present when reading the current paper – and included as well the MR Review and the Zbl Review and an important comment ☺.

Sketch of proof for \mathbb{A}^n

Let the finite p -group P act on \mathbb{A}^n . Let a be a point and let $P_a := \{x \in P \mid a^x = a\} \subseteq P$ be the stabiliser of a . We denote by $U(P)$ the set of all subgroups of P and for every $U \in U(P)$ by $R(P, U) := \{Ux \mid x \in P\}$ the set of all right cosets of U in P . Then P operates by multiplication from the right for every $U \in U(P)$ transitively on $R(P, U)$ with $\text{Cor}_P U := \{U^x \mid x \in P\}$ as the kernel. The classification of transitive P -sets reads as follows: *Every transitive P -set $\Omega \neq \emptyset$ is P -isomorphic to $R(P, P_a)$ for all $a \in \Omega$, and for any $U, V \in U(P)$ the two sets $R(P, U)$ and $R(P, V)$ are P -isomorphic if and only if U and V are conjugate in P .* Hence for the action of P we have a bijection between the class $\mathcal{J}(P)$ of all P -isomorphism types of transitive P -sets and the set of all conjugacy classes (in P) of subgroups of P , and therefore $|\mathcal{J}(P)| = g_P(P)$:= the number of conjugacy classes of subgroups of P . Therefore for every P -set Ω the class $\mathcal{J}(P, \Omega)$ of P -isomorphism types of P -orbits on Ω has at most $g_P(P)$ elements and since every subgroup of P is a subset containing 1, we can now summarising deduce $|\mathcal{J}(P, \Omega)| \leq g_P(P) \leq |U(P)| \leq 2^{|P|-1}$. If P is a p -subgroup of \mathbb{S}^n which is contained in exactly $k \in \mathbb{N}$ Sylow p -subgroups of \mathbb{S}^n and if $m := k + p + 1$, then $n \leq m \cdot |P| \cdot g_P(P) - 1$ and $n \leq (p + 2) \cdot 2^{|P|-1} - 1$ for $k = 1$ (see **Page 5**), whence, if not so, P has at least m many P -isomorphic P -orbits on $\Omega := \{1, 2, \dots, n\}$ (see **Page 5**). We are then able to deduce from this fact the central observation that $\{S \in \text{Syl}_p(\mathbb{S}^\Omega) \mid S \text{ is } P\text{-invariant}\} =: \text{Syl}_p(\mathbb{S}^\Omega, P) \geq |\text{Syl}_p(\mathbb{S}^m)| \geq m - 2 \geq k + 1$ by using **beautiful new ideas** (see **Page 6**). □

Sketch of proof for $\mathbb{A} = \text{PSL}_n$

We are applying a three-stage-approach whilst **first** proving the theorem for the **General Linear Groups** over (commutative) locally finite fields (**Theorem 2**), then for the **Special Linear Groups** over locally finite fields (**Theorem 3**) and finally for the **Projective Special Linear (PSL) Groups** over locally finite fields (**Theorem 4**), thereby using that $\text{GL}(n, \mathcal{F}) = \text{SL}(n, \mathcal{F}) \cdot \mathcal{F}^*$ and $\text{PSL}(n, \mathcal{F}) = \text{SL}(n, \mathcal{F}) / Z(\text{SL}(n, \mathcal{F}))$ (see **Page 11** and **Page 12**). This can be shown with a **very beautiful** diagram:



The major work is required for the **General Linear Groups** with two different and both **very beautiful** approaches for characteristic $\neq p$ and characteristic p . In characteristic $\neq p$ we use that, if for a finite p -group P which is operating on a finite-dimensional vector space V over a locally finite field and a direct decomposition of V into irreducible P -submodules, there are k many of the P -submodules P -isomorphic, then at least $|\text{Syl}_p S^k|$ Sylow p -subgroups of $\text{GL}(V)$ are P -invariant (see **Proposition 7 a**). In characteristic p we use that, if k is the dimension of the P -submodule $C_V(P) := \{v \in V \mid v^x = v \text{ for all } x \in P\}$ of a non-trivial modular P -module V , then again there are at least $|\text{Syl}_p S^k|$ many P -invariant Sylow p -subgroups of $\text{GL}(V)$ (see **Proposition 7 b**). We then are able to argue that from **Proposition 7** follows that $n \leq (p+2) \cdot |P|^2 - 1$ for a p -uniqueness subgroup P of $\text{GL}(n, \mathbb{F})$ (see **Lemma 2** on **Page 11**). For the transition from $\text{GL}(n, \mathbb{F})$ to $\text{SL}(n, \mathbb{F})$ we are using that a p -uniqueness subgroup of $\text{SL}(n, \mathbb{F})$ is a p -uniqueness subgroup of $\text{GL}(n, \mathbb{F})$ as well. For the transition from $\text{SL}(n, \mathbb{F})$ to $\text{PSL}(n, \mathbb{F})$ we use that $P := Q \cdot D(\text{SL}(n, \mathbb{F})) / D(\text{SL}(n, \mathbb{F}))$ is a p -uniqueness subgroup of $\text{PSL}(n, \mathbb{F})$ when Q is a p -uniqueness subgroup of $\text{SL}(n, \mathbb{F})$, and conversely, together with the **Proposition 4** and the **Proposition 6** to get the lower bound $p+2$ whence P lies in at least $|\text{Syl}_p S^{p+2}|$ Sylow p -subgroups of $\text{PSL}(n, \mathbb{F})$. \square

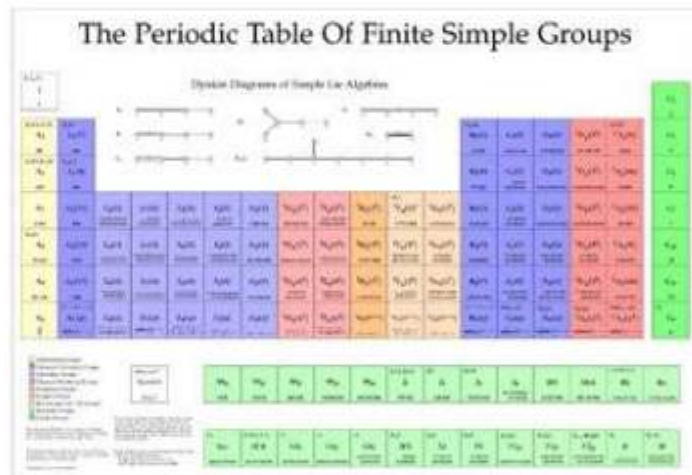
1. Introduction

For any unexplained notation we refer to [15].

Bring to mind that a group is called *simple* if itself and $\langle 1 \rangle$ are its sole normal subgroups and that a *local system* for a group G is a family Σ of subgroups such that every element of G lies in a Σ -group and for every two Σ -groups there exists another Σ -group which contains both. The local system Σ for the group G is said to be *nested* if there exists a sequence $\{U_n \mid n \in \mathbb{N}\}$ of subgroups of G such that $U_n \subseteq U_{n+1}$ for all $n \in \mathbb{N}$ and $\Sigma = \{U_n \mid n \in \mathbb{N}\}$. If G is a countable group and $\{x_n \mid n \in \mathbb{N}\}$ an enumeration of G , let $U_n := \langle x_1, x_2, \dots, x_n \rangle$ ($n \in \mathbb{N}$); then $\{U_n \mid n \in \mathbb{N}\}$ is a nested local system for G . If the locally finite group G has such a nested local system, then G is countable. If an infinite group $G = \langle U \mid U \in \Sigma \rangle$ possesses a local system Σ consisting of simple subgroups, it is simple: suppose $N \neq \langle 1 \rangle$ is a normal subgroup of G ; if $N \cap U = \langle 1 \rangle$ for all $U \in \Sigma$ then $N = \langle N \cap U \mid U \in \Sigma \rangle = \langle 1 \rangle$; hence $N \cap U = U$ for some $U \in \Sigma$ and so $N \cap V = V$ for all $V \in \Sigma$ since $U, V \subseteq W$ for each $V \in \Sigma$ with some $W \in \Sigma$; thus $N = G$. An infinite simple group has, according to Philip Hall (see [46], p. 137, which introduces the **beautiful** term “*bountiful*”), some local system consisting of countably infinite simple subgroups (see [42], p. 18, [43], Theorem 4.4, [44], Theorem 2.5, and [45] O.H. KEGEL: “Remarks on uncountable simple groups”, in: Proceedings of Ischia Group Theory 2016, *Int. J. Group Theory* 7 (2018)). Thus simplicity is definitely a countably recognisable group theoretic property (see [2]). Periodic linear groups are locally finite (see [43], Theorem 1.L.1) and satisfy the Strong Sylow Theorem for every Prime p (see [54] and [44], 1.7). Simple periodic linear groups are countable (see [43], Theorem 1.L.2).

If G is a countably infinite locally finite simple group, then there will exist a nested local system $\{R_n \mid n \in \mathbb{N}\}$ for G of finite subgroups such that for each $n \in \mathbb{N}$ the group R_n is perfect and there exists some maximal normal subgroup M_{n+1} of R_{n+1} satisfying $M_{n+1} \cap R_n = \langle 1 \rangle$, so that R_{n+1} / M_{n+1} is simple and $R_n \cong R_{n+1} / M_{n+1}$ (see **Chapter 3**); such a nested local system is called **Kegel cover** (or **\star -sequence**) for G . We define the **family \mathcal{T} of types of known finite simple groups** by using some assumed well-known symbols: $\mathcal{T} := \{\text{abelian}_p, \mathbb{A}^n, \mathbb{A} = \text{PSL}_n, \mathbb{B} = \text{P}\Omega_{\text{odd } n}, \mathbb{C} = \text{PSp}_n, \mathbb{D} = \text{P}\Omega_{\text{odd } n}^+, {}^2\mathbb{A} = \text{PSU}_n, {}^2\mathbb{D} = \text{P}\Omega_{\text{even } n}^-, \mathbb{E}_6, \mathbb{E}_7, \mathbb{E}_8, \mathbb{F}_4, \mathbb{G}_2, {}^2\mathbb{B}_2, {}^3\mathbb{D}_4, {}^2\mathbb{E}_6, {}^2\mathbb{F}_4, {}^2\mathbb{G}, \text{sporadic } \star\}$. If G is a known finite simple group of type $\Xi \in \mathcal{T}$, we call p resp. n resp. 2 resp. 4 resp. 6 resp. 7 resp. 8 resp. \star ($\star :=$ the order of G) the **rank $r(G)$ of G** . A countably infinite locally finite simple group is called to be **of type $\Xi \in \mathcal{T}$** , if it just has a Kegel cover $\Sigma = \{(R_k, M_k) \mid k \in \mathbb{N}\}$ in such a way that infinitely many of the R_{k+1} / M_{k+1} 's belong to Ξ (wherefore we can replace Σ by these infinitely many R_{k+1} 's), and is called to be **of alternating type** if it is of type \mathbb{A}^n . Note that such a group could a priori be of several types but we may placidly assume by the well-known pigeonhole principle (see https://en.wikipedia.org/wiki/Pigeonhole_principle) that in fact **all R_{k+1} / M_{k+1} 's belong to the same of the 19 known families**.

The following figure (© 2012 by Iván Andrus [see <https://irandrus.files.wordpress.com/2012/06/periodic-table-of-groups.pdf> and <https://irandrus.wordpress.com/2012/06/17/the-periodic-table-of-finite-simple-groups/>]) depicts the **19 families** of known finite simple groups in a **beautiful** arrangement called “*Periodic Table*”:



If the locally finite group G satisfies the Strong Sylow Theorem for the Prime p it contains a p -uniqueness subgroup (see [15], Theorem 3.9, and [44], Theorem 1.5, in conjunction with [15], Proposition 2.3). Thus, if for a countably infinite locally finite simple group G with Kegel cover $\{(R_k, M_k) \mid k \in \mathbb{N}\}$ and p -uniqueness subgroup P we could prove that **the ranks of the R_{k+1} / M_{k+1} 's are bounded in terms of P** , then we could very straightforwardly deduce Prof. Otto H. Kegel's **Theorem 2.7** (see [44]: “For the locally finite simple group G the following are equivalent: (i) Every countable simple subgroup of G contains a p -uniqueness subgroup; (ii) G satisfies the Strong Sylow Theorem for the Prime p ; (iii) G is linear.”) and his central **Theorem 3.4** (see [44]: “If $\{F_i\}_{i \in \mathbb{N}}$ is a smooth simple straight

split sequence of finite p -perfect subgroups of the locally finite group G , then the countably infinite group $U = \langle F_i; i \in \mathbb{N} \rangle$ has 2^{\aleph_0} maximal p -subgroups.”).

Note – To study crucial configurations, Kegel developed in [44] the quite excogitated concept of “(smooth simple straight) split sequences of finite p -perfect subgroups with their associated ascending sequences of subgroups” which is related to his equally very fine concept of the “Sylow-separated (ascending) sequences of p -subgroups with associated sequences of Sylow p -subgroups” he had developed already nearly ten years earlier in “O.H. KEGEL: ‘Chain conditions and Sylow’s theorem in locally finite groups’, in: Symposia Mathematica, Volume XVII, Convegno sui Gruppi Infiniti, Istituto Nazionale di Alta Matematica (INdAM) ‘Francesco Severi’, Roma, 11-14 Dicembre 1973, Academic Press, London-New York (1976), 251-259. ISBN 978-0-12612-217-6.”

So, in his four workshop lectures on Sylow theory in locally finite groups at the famed and such eminent Singapore Group Theory Conference of June 1987, Kegel stated as a theorem and proved “by inspection” what is actually a **conjecture** (see [44], **Theorem 2.4**): “Let P be a p -uniqueness subgroup of the finite simple group S which belongs to one of the seven rank-unbounded families. Then the rank of S is bounded in terms of P .” In this paper we prove the conjecture for the case that the finite simple group S is some A^n ($n \in \mathbb{N}$) thereby getting Kegel’s Theorem 2.7 and Theorem 3.4 for the case that the countably infinite locally finite simple group is **of alternating type**.

If Σ is a local system of countably infinite simple subgroups of the simple locally finite group G with (G countable $\Rightarrow \Sigma = \{G\}$) and P^U for each Σ -group U is a p -uniqueness subgroup of U , which exists if G satisfies the Strong Sylow Theorem for the Prime p (see [15]), and $\{(R_k, M_k) \mid k \in \mathbb{N}\}$ is for each $U \in \Sigma$ a Kegel cover for U of alternating type, then for each $U \in \Sigma$ will exist a $k = k(U) \in \mathbb{N}$ with $P^U \subseteq R_k^U$, whence $P^U \cdot M_m^U / M_m^U \approx P^U / P^U \cap M_m^U$ is a p -uniqueness subgroup of R_m^U / M_m^U for all $m \geq k(U)$, and we could deduce easily from the following **Theorem 1** that the ranks $\{r(R_m^U / M_m^U) \mid m \geq k(U)\}$ are bounded by $f_p(P^U)$ for all $U \in \Sigma$, so that all Σ -groups would be linear (see [47]) and so G would be linear, too, and so also countable.

Theorem 1 (see [14]). Let $n \in \mathbb{N}$ and let p be a prime such that $p \leq n$. Let P be a finite p -group acting on A^n . Let $g_p(P)$ be the number of conjugacy classes of subgroups of P and let k be the number of P -invariant Sylow p -subgroups of A^n . Then $g_p(P) \leq 2^{|P|-1}$.

- a) If isomorphic subgroups of P are conjugate and $b := \log_p |P|$ (so that $|P| = p^b$), then $g_p(P) \leq p^{((b-2)^4 + 2(b-2)^3 + (b-2)^2) / 4 - ((b-2)^2 + b - 2) / 2 - 90} + (|P| - 1)(p - 1) + 25$.
- b) Let $m := k + p + 1$. Then $n \leq m \cdot |P| \cdot g_p(P) - 1$.
If $k = 1$, then $n \leq f_p(P) := (p + 2) \cdot |P| \cdot 2^{|P|-1} - 1$.

Having proved **Theorem 1** we state **a way 1)** and **a way 2)** how to optimise **Theorem 1**, make a couple of remarks and suggestions on **Planning future research** and state three conjectures.

A periodic linear group is locally finite (see [43], Theorem 1.L.1) and satisfies the Strong Sylow Theorem for every Prime p (see [54] and [44], 1.7). As the next undertaking we are proving **Conjecture 2** of **Page 8** regarding the **General Linear Groups** over locally finite fields (see [14]):

- Theorem 2.** Let $n \in \mathbb{N}$ and let p be a prime. Let F be a locally finite (commutative) field.
- a) If F has characteristic p and $a_p = a_p(\text{GL}(n, F))$ then $n \leq (p + 2) \cdot p^{2p} - 1$.
 - b) If F has characteristic $\neq p$ and $a_p = a_p(\text{GL}(n, F))$ then $n \leq (p + 2) \cdot p^{2p} - 1$.

Afterwards we are breaking down **Theorem 2** to the **Special Linear Groups** over locally finite fields:

- Theorem 3.** Let $n \in \mathbb{N}$ and let p be a prime. Let F be a locally finite (commutative) field.
- a) If F has characteristic p and $a_p = a_p(\text{SL}(n, F))$ then $n \leq (p + 2) \cdot p^{2p} - 1$.
 - b) If F has characteristic $\neq p$ and $a_p = a_p(\text{SL}(n, F))$ then $n \leq (p + 2) \cdot p^{2p} - 1$.

We continue with breaking down **Theorem 3** to the **Projective Special Linear (PSL) Groups** over locally finite fields:

- Theorem 4.** Let $n \in \mathbb{N}$ and let p be a prime. Let F be a locally finite (commutative) field and let P be a minimal p -unique subgroup of $\text{PSL}(n, F)$.
- a) If F has characteristic p and $a_p = a_p(\text{PSL}(n, F))$ then $n \leq f_p(P) := (p + 2) \cdot p^{2p} - 1$.
 - b) If F has characteristic $\neq p$ and $a_p = a_p(\text{PSL}(n, F))$ then $n \leq f_p(P) := (p + 2) \cdot p^{2p} - 1$.

An infinite simple locally finite group G always has a local system Σ consisting of countably infinite simple locally finite subgroups and each Σ -group U has a Kegel cover $\{(R_k^U, M_k^U) \mid k \in \mathbb{N}\}$ (see **Page 3**). If all the factors R_k^U / M_k^U of the Kegel covers for all Σ -groups U are of type $\Xi = “A = \text{PSL}_n”$, then G is called to be **of projective special linear type**. If G satisfies the Strong Sylow Theorem for the Prime p , then each Σ -group U has a p -uniqueness subgroup P^U (see [15]).

For each $U \in \Sigma$ exists some $k = k(U) \in \mathbb{N}$ with $P^U \subseteq R_k^U$, whence $P^U \cdot M_m^U / M_m^U \approx P^U / P^U \cap M_m^U$ is a great p -uniqueness subgroup of R_m^U / M_m^U for all $m \geq k(U)$. If G is of projective special linear type, it follows from **Theorem 4** that the ranks $\{r(R_m^U / M_m^U) \mid m \geq k(U)\}$ will be bounded by $f_p(P^U)$ for all the Σ -groups U , which hence are linear and so G will be linear and therefore also countable (see [47]). Summarising we can see the consequences of the Strong Sylow Theorem for the Prime p according to **Theorem 1** and **Theorem 4**:

Theorem 5. *Let G be a simple locally finite group of alternating type or of projective special linear type satisfying the Strong Sylow Theorem for the even one Prime p . Then G is linear and countable.* \square

Having proved Theorems 1, 2, 3 and 4 we make a couple of further remarks and suggestions on **Planning future research** and announce very beautifully **The Second Trilogy**.

2. Proof of Theorem 1

Proof. We begin with some general remarks. For any group G we denote by $U(G)$ the set of all its subgroups and for every $U \in U(G)$ by $R(G, U) := \{Ux \mid x \in G\}$ the set of all right cosets of U in G . Then G operates by multiplication from the right for every $U \in U(G)$ transitively on $R(G, U)$ with $\text{Cor}_G U := \{U^x \mid x \in P\}$ as the kernel. If G acts (from the right) on a set Ω , so that Ω is a G -set, and $\alpha \in \Omega$ is any point, then $G_\alpha := \{x \in G \mid \alpha^x = \alpha\} \subseteq G$ is the stabiliser of α . Another G -set Ψ is said to be G -isomorphic to Ω in case there exists a bijection $\xi: \Omega \rightarrow \Psi$ such that $\xi(\alpha^x) = \xi(\alpha)^x$ for all the $\alpha \in \Omega$ and $x \in G$. The **classification of transitive G -sets** reads as follows (see [50], Chapter 6): *Every transitive G -set $\Omega \neq \emptyset$ is G -isomorphic to $R(G, G_\alpha)$ for all $\alpha \in \Omega$, and for any two $U, V \in U(G)$ the two sets $R(G, U)$ and $R(G, V)$ are G -isomorphic if and only if U and V are conjugate in G .* Hence for the action of P we will have a bijection between the class $\mathcal{J}(P)$ of P -isomorphism types of transitive P -sets and the set of all conjugacy classes (in P) of subgroups of P , and so $|\mathcal{J}(P)| = g_P(|P|)$. Thus for every P -set Ω the class $\mathcal{J}(P, \Omega)$ of P -isomorphism types of P -orbits on Ω has at most $g_P(|P|)$ elements and since every subgroup of P is a subset containing 1, we can summarising deduce that $|\mathcal{J}(P, \Omega)| \leq g_P(|P|) \leq |U(P)| \leq 2^{|P|-1}$.

Consider now some $n \in \mathbb{N}$ and a p -subgroup P of \mathbb{S}^n for some prime p which is contained in exactly $k \in \mathbb{N}$ Sylow p -subgroups of \mathbb{S}^n . Then $n \leq (k + p + 1) \cdot |P| \cdot g_P(|P|) - 1$ \star .

RATIONALE – Suppose $n \geq (k + p + 1) \cdot |P| \cdot g_P(|P|)$. Then $G := P$ is a finite group which operates on the set $\Omega := \{1, 2, \dots, n\}$ with $|\Omega| \geq (k + p + 1) \cdot |G| \cdot g_P(|G|)$. We show that the number of G -isomorphic G -orbits on Ω must be at least $k + p + 1$. The group G partitions Ω into r orbits $\Psi_1, \Psi_2, \dots, \Psi_r$. Since the orbit lengths $|\Psi_1|, |\Psi_2|, \dots, |\Psi_r|$ divide the group order $|G|$, it follows that $|\Omega| = \sum_{1 \leq i \leq r} |\Psi_i| \leq r \cdot |G|$; hence if $r \geq (k + p + 1) \cdot |\mathcal{J}(G, \Omega)|$, then by the pigeonhole principle there will be **at least $k + p + 1$ many G -isomorphic G -orbits on Ω** . \blacksquare Therefore P has at least $k + p + 1$ many P -isomorphic P -orbits on Ω . This implies, as we show below, that there are at least $|\text{Syl}_p \mathbb{S}^{k+p+1}|$ many P -invariant Sylow p -subgroups of \mathbb{S}^Ω . Since also $|\text{Syl}_p \mathbb{S}^n| \geq n - 2$ for $n \in \mathbb{N}$ (see Lemma 1 below), $|\text{Syl}_p \mathbb{S}^{k+p+1}| \geq (k + p + 1) - 2 = (k + 1) + (p - 2) \geq k + 1$ follows. \blacksquare

a) For all $0 \leq k \leq b$ let j_k denote the number of conjugacy classes of subgroups of index p^{b-k} in P . Then clearly $j_0 = 1, j_1 = 1$ and $j_b = 1$, but also $j_{b-1} \leq (|P| - 1) / (p - 1)$: the Frattini subgroup $\Phi(P)$ of P has an elementary abelian factor group of rank $\leq b$,

since a maximal subgroup of a finite p -group is normal of index p , whence j_{b-1} represents the number of the one-dimensional subspaces of the $\text{GF}(p)$ -vectorspace $P/\Phi(P)$. **Now suppose that the isomorphic subgroups of P are conjugate.** Then $j_2 = 2$, since there are two isomorphism types of groups of order p^2 , the cyclic group and the elementary abelian group, $j_3 = 5$, since there are five isomorphism types of groups of order p^3 , and $j_4 \leq 15$, since there are 14 isomorphism types of groups of order 2^4 and 15 isomorphism types of groups of order p^4 for $p \neq 2$ (see [23]). It follows that $j_0 + j_1 + j_2 + j_3 + j_4 + j_{b-1} + j_b \leq (|P| - 1) / (p - 1) + 25$. Considering a chief series for a group of order p^k ($k \in \mathbb{N}$) one can determine the number of maximal possible multiplication tables of groups of order p^k and thus obtain rather simply the estimate $i_{pk} \leq p^{(k^3-k)/6}$ for the number i_{pk} of isomorphism types of groups of order p^k (see [28], Theorem 3.1). Since we can calculate $\sum_{5 \leq k \leq b-2} (k^3 - k) / 6 =$ (see under <https://www.numberempire.com/seriescalculator.php>) $((b-2)^4 + 2(b-2)^3 + (b-2)^2) / 4 - ((b-2)^2 + b - 2) / 2 - 90$, it now follows the rather cool inequality $\sum_{5 \leq k \leq b-2} p^{(k^3-k)/6} \leq p^{((b-2)^4 + 2(b-2)^3 + (b-2)^2) / 4 - ((b-2)^2 + b - 2) / 2 - 90}$. Summarising we get $g_P(|P|) \leq$

$$p^{((b-2)^4 + 2(b-2)^3 + (b-2)^2) / 4 - ((b-2)^2 + b - 2) / 2 - 90} + (|P| - 1) / (p - 1) + 25. \quad \square$$

b) We may assume that the group P operates faithfully on \mathbb{A}^n which is a normal subgroup of index 2 in \mathbb{S}^n . If $n \leq 5$ or $n \geq 7$ the automorphism group $\text{Aut}(\mathbb{A}^n)$ of \mathbb{A}^n is known to be isomorphic to the group of inner automorphisms of \mathbb{S}^n which is isomorphic to \mathbb{S}^n (see [51], Satz 1.9). $\text{Aut}(\mathbb{A}^6)$ is the semidirect product of a group C_2 of order 2 with \mathbb{S}^6 (see [32]). Thus P is (isomorphic to) a p -subgroup of \mathbb{S}^n or of $C_2 \cdot \mathbb{S}^6$ which normalises k Sylow p -subgroups of \mathbb{A}^n . Every Sylow 2-subgroup of \mathbb{A}^n lies in only one Sylow 2-subgroup of \mathbb{S}^n , since \mathbb{A}^n contains for $n \geq 5$ just as many Sylow 2-subgroups as has \mathbb{S}^n , and a Sylow 2-subgroup of \mathbb{A}^n is its own normaliser in \mathbb{S}^n (see [59]). **Thus the p -subgroup P of \mathbb{S}^n (or of $C_2 \cdot \mathbb{S}^6$, if $p = 2$) lies in exactly k many Sylow p -subgroups of \mathbb{S}^n .** (If $k \geq 2$ then even $k \geq p + 1$ because the number of all Sylow p -subgroups of the semidirect product $P \cdot \mathbb{S}^n$ is congruent to 1 modulo p .) We digress now and permit a short **memory parenthesis**: **When G is a finite group, P a p -subgroup of G and $S \in \text{Syl}_p G$, then the operation of P by conjugation on $C(G, S) := \{S^x \mid x \in G\}$ has at least one fixed point, that is $(\exists x \in G)(P^x \subseteq S)$, and for $P \in \text{Syl}_p G$ exactly one, that is, $|\text{Syl}_p G| = |G : N_G S| = |C(G, S)| \equiv 1 \pmod{p}$; hence G satisfies the Strong Sylow Theorem for the Prime p , that is, every $U \in U(G)$ conjugates transitively on $\text{Syl}_p U$, and thus we have the **Frattini argument** for G (and p), that is, if N is a normal**

\star If P is a p -uniqueness subgroup of \mathbb{S}^n , then $n \leq (p + 2) \cdot |P| \cdot 2^{|P|-1}$. If the countable group $\mathbb{S}^{(\mathbb{N})}$ would satisfy the Sylow Theorem for the prime p , then by Theorem 3.4 of [15] it would even satisfy the Strong Sylow p -Theorem, and thus it would by Theorem 3.9 of [15] contain a p -uniqueness subgroup P . Now $\mathbb{S}^{(\mathbb{N})}$ has a nested local system $\{U_n \mid n \in \mathbb{N}\}$ with $U_n \approx \mathbb{S}^n$ for all $n \in \mathbb{N}$. Since P is finite, there exists an $m \in \mathbb{N}$ with $P \subseteq U_m$. Then P would be singular in U_n for all $n \in \mathbb{N}$ with $n \geq m$ and we get the rubbish $n \leq (p + 2) \cdot |P| \cdot 2^{|P|-1}$ for all $n \geq m$. Similarly, every finite p -subgroup of $\mathbb{S}^{(\mathbb{N})}$ is contained in at least n_0 Sylow p -subgroups of $\mathbb{S}^{(\mathbb{N})}$ since $\mathbb{S}^{(\mathbb{N})}$ does not satisfy the Sylow p -Theorem.

subgroup of G and $P \in \text{Syl}_p N$, then $N_G P$ covers G/N , that is, $G = N \cdot N_G P$. ■ We now put $m := k + p + 1$ and are supposing $n \geq m \cdot |P| \cdot |g_p(P)|$. Then according to the remarks made at the outset, when arguing for the RATIONALE, there will be at least m many P -isomorphic P -orbits on Ω .

In order to proceed we need a lower bound for $|\text{Syl}_p S^\Omega|$:

Lemma 1. Let p be a prime and let $n \in \mathbb{N}$.

- α) If $p > n$, then $|\text{Syl}_p S^n| = 1$.
- β) If $((p, n) = (p, 1), (2, 2), (2, 3), (3, 3), (2, 4), (3, 4))$, then $|\text{Syl}_p S^n| = (n = 1, n - 1 = 1, n = 3, n - 2 = 1, n - 1 = 3, n = 4)$.
- γ) If $p \leq n$ and $n \geq 5$, then $|\text{Syl}_p S^n| \geq n$.
- δ) If $p \leq n$, then $|\text{Syl}_p S^n| \geq n - 2$.

RATIONALE – α) S^n is a p' -group for $p > n$ since $n! = |S^n|$.

β) $|\text{Syl}_p S^1| = 1$ for all p because of $S^1 = \langle 1 \rangle$ and $\text{Syl}_2 S^2 = \{S^2\}$ because of $|S^2| = 2$. Since $|S^3| = 2 \cdot 3$ and $|S^4| = 2^3 \cdot 3$ we have $|\text{Syl}_2 S^3|, |\text{Syl}_2 S^4| \in \{1, 3\}$ and $|\text{Syl}_3 S^3|, |\text{Syl}_3 S^4| \in \{1, 4\}$ because of $|\text{Syl}_p G| \equiv 1 \pmod{p}$. From $|S^3 \cdot A^3| = 2$ follows that A^3 is a normal subgroup of S^3 whence $|\text{Syl}_2 S^3| = 3$ because S^3 is non-abelian. We know that S^4 has exactly two non-trivial proper normal subgroups, namely the Klein four-group and the A^4 , and therefore has neither a normal Sylow 2-subgroup nor a normal Sylow 3-subgroup, whence $|\text{Syl}_2 S^4| = 3$ and $|\text{Syl}_3 S^4| = 4$.

γ) We show first: (i) If $n \geq 5$ then S^n contains just one non-trivial normal subgroup, namely the A^n . RATIONALE – Let (if possible) $\langle 1 \rangle \neq N \subseteq S^n$ be normal in S^n with $N \neq A^n$, then $N \cap A^n = \langle 1 \rangle$ since A^n is simple, hence $|N \cdot A^n| = |N \cdot A^n|$ divides $|S^n|$, and so $|N| = 2$; as a 2-transitive group S^n is primitive whence N operates trivially or transitively which is clearly impossible for $|N| = 2$. ■ Since $|\text{Syl}_p S^n| \equiv 1 \pmod{p}$ it follows from (i), $|S^n \cdot A^n| = 2$, and $|S^n| = n!$ that $|\text{Syl}_p S^n| \geq 3$. Since $|\text{Syl}_p S^n| = |S^n \cdot N_{S^n} S|$ for $S \in \text{Syl}_p S^n$ it now suffices to show the following: (ii) Let $n \geq 5$ and $3 \leq k \leq n - 1$. Then S^n has not any subgroup of index k at all.

RATIONALE – Suppose there exists a subgroup U of S^n with $|S^n : U| = k$. The transitive operation of S^n on $R(S^n, U)$ via right multiplication gives rise to some homomorphism $\varphi: S^n \rightarrow S^k$. Because of $k \leq n - 1$ we have $\langle 1 \rangle \neq \text{kernel } \varphi \subseteq U$ and since $k \geq 3$ we have $|\text{kernel } \varphi| < |A^n|$. By (i) this is impossible. ■

δ) follows from point β) and point γ). ■

We return to the group P operating on Ω with at least m many P -isomorphic P -orbits. Application of Lemma 1 gives $|\text{Syl}_p S^m| \geq m - 2 = (k + p + 1) - 2 = (k + 1) + (p - 2) \geq k + 1$. Therefore it remains to prove that if $\text{Syl}_p(S^\Omega, P) := \{S \in \text{Syl}_p S^\Omega \mid S \text{ is } P\text{-invariant}\}$ and there are at least m many P -isomorphic P -orbits on Ω , then $|\text{Syl}_p(S^\Omega, P)| \geq |\text{Syl}_p S^m|$. For each $1 \leq i \leq r$ let V_i be the point stabiliser of $\Omega \setminus \Psi_i$ in S^Ω ; hence $V_i \approx S^{\Psi_i}$. Then we truly have $P \subseteq D := \langle V_i \mid 1 \leq i \leq r \rangle$. Let B be the set of permutations on Ω which interconvert in entire blocks the P -isomorphic Ψ_i 's and let the remaining Ψ_i 's pointwise fixed. Then $B \subseteq S^\Omega$ with $B \approx S^m$ and with $B \cap D = \langle 1 \rangle$. Because B interchanges only P -isomorphic P -orbits, it is normalised by D . Hence $K := \langle B, D \rangle$ is the semidirect product $B \cdot D$, and so D is normal in K with $K/D \approx B$. Now let $Q \in \text{Syl}_p K$ with $P \subseteq Q$.

Since D is normal in K and the finite group K satisfies the Sylow Theorem for the Prime p , we have $P \subseteq D \cap Q \in \text{Syl}_p D$ and by the Frattini argument (see above) $N_K(D \cap Q)/N_D(D \cap Q) \approx K/D$. It follows that $|\text{Syl}_p(S, P)| \geq |\{R \in \text{Syl}_p K \mid P \subseteq R\}| \geq |\{R \in \text{Syl}_p K \mid D \cap R = D \cap Q\}| = |\text{Syl}_p(N_K(D \cap Q)/N_D(D \cap Q))| = |\text{Syl}_p(K/D)| = |\text{Syl}_p S|$. □ **Q.E.D. (Quod Erat Demonstrandum)**

Corollary. Let G be a simple locally finite group of alternating type with Kegel covers $\{(R_k^U, M_k^U) \mid U \in \Sigma, k \in \mathbb{N}\}$ as described on Page 4 satisfying the Strong Sylow Theorem for the Prime p and let P^U for each Σ -group U be a p -uniqueness subgroup of U (see [15]). Then we have the inequality $\tau(R_m^U/M_m^U) \leq f_p(P^U) := (p + 2) \cdot |P^U| \cdot 2^{|P^U| - 1} - 1$ for all $m \geq k(U)$ and for all $U \in \Sigma$, and G is a linear group and a countable group.

Proof. Our Theorem 1, [47], and [43], Theorem 1.L.2. □

We keep the overall context of the Corollary and let G be a locally finite group satisfying the Strong Sylow Theorem for the Prime p and let P be a p -uniqueness subgroup of G . In view of Theorem 1 it is of rather considerable interest whether resp. when isomorphic (finite) subgroups of P are conjugate. Therefore let Q and Q^* be isomorphic subgroups of P and also let r be their common index in P . The left regular representation $\lambda_g: h \rightarrow gh$ for all $h \in P$ ($g \in P$) and the right regular representation $\rho_g: h \mapsto hg^{-1}$ for all the $h \in P$ ($g \in P$) both embed P into the symmetric group S^P on P . Now a famous result by Philip Hall (see [26], Lemma 1) establishes that either regular representation maps isomorphic subgroups onto conjugate subgroups: let $x \mapsto x^*$ ($x \in Q$) be an isomorphism of Q onto Q^* ; let $\{y_1, y_2, \dots, y_r\}$ be a complete set of left coset representatives of Q in P and $\{y_1^*, y_2^*, \dots, y_r^*\}$ be such a set of left coset representatives of Q^* in P ; the mapping $\xi: y_i x \mapsto y_i^* x^*$ ($x \in Q \mid i = 1, 2, \dots, r$) is a permutation of P , so that $\xi \in S^P$; if $t \in Q$ and if ρ is any regular representation of P , we then have $y_i^* x^* \xi^{-1} \rho(t) \xi = y_i x \rho(t) \xi = y_i(xt) \xi = y_i^*(xt)^* = y_i^* x^* t^*$, since $*$ is a homomorphism, so that $\xi^{-1} \rho(t) \xi = \rho(t^*)$; hence ξ transforms $\rho(Q)$ into $\rho(Q^*)$. ■ However, we should in fact need conjugacy not only in S^P but in P itself. This is an open problem. Note that if this would be solved without restrictions then in a (locally finite) p -group, the simplest locally finite group satisfying the Strong Sylow Theorem for the Prime $p \dots$, isomorphic finite subgroups would be conjugate, a rather striking property. Hence the solution will probably need restrictions.



Sisyphus by Tiziano - Oil on Canvas, 1548 - 1549 © Museo Nacional del Prado, Madrid

Les dieux avaient condamné Sisyphus à rouler sans cesse un rocher jusqu'au sommet d'une montagne d'où la pierre retombait par son propre poids. ... Il faut imaginer Sisyphus heureux.
Die Götter hatten Sisyphus dazu verurteilt, einen Felsblock unablässig den Berg hinaufzuwälzen, von dessen Gipfel der Stein kraft seines eigenen Gewichts wieder hinunterrollte. ... Wir müssen uns Sisyphus als einen glücklichen Menschen vorstellen.
The gods had condemned Sisyphus to ceaselessly rolling a rock to the top of a mountain, whence the stone would fall back of its own weight. ... One must imagine Sisyphus happy.
Gli dei avevano condannato Sisseo a far rotolare senza posa un macigno sino alla cima di una montagna, dalla quale la pietra ricadeva per azione del suo stesso peso. ... Bisogna immaginare Sisseo felice.



3. About Kegel covers

Let G be a locally finite group. A set of pairs $\{(H_i, M_i) \mid i \in \mathfrak{J}\}$ is called a *Kegel cover* for G if, for all i in \mathfrak{J} , H_i is a finite subgroup of G and M_i is a maximal normal subgroup of H_i , and if for each finite subgroup H of G there exists an $i \in \mathfrak{J}$ with $H \subseteq H_i$ and $H \cap M_i = \langle 1 \rangle$; the groups H_i/M_i ($i \in \mathfrak{J}$) are called *the factors of the Kegel cover* (see [49]). In [14] we introduced the concept of the \star -sequence for G . Let G be a countably infinite simple locally finite group. We then are defining a \star -sequence for G as a set of pairs $\{(R_n, M_n) \mid n \in \mathbb{N}\}$ where $\{R_n \mid n \in \mathbb{N}\}$ is a nested local system for G and for all $n \in \mathbb{N}$ the group R_n is perfect, $R_n \neq R_{n+1}$ and M_{n+1} is some maximal normal subgroup of R_{n+1} with $M_{n+1} \cap R_n = \langle 1 \rangle$, that is, *the factor* R_n/M_n , which is a non-abelian finite simple group, is (isomorphic to) a proper section of the non-abelian simple group R_{n+1}/M_{n+1} , and therefore $\{R_n \mid n \in \mathbb{N}\}$ is totally ordered by involvement. Such a group G has a nice \star -sequence (see [14], and [42], p. 20, and [43], Lemma 4.5, which tough Kegel calls an “approximation principle”, and [44], Theorem 2.6, and the origin as the rather smart concept of a so-called “ α -Folge” introduced in [41], Definition 2.1 and Hilfssatz 2.2 [but see the Remark on p. 116 of [43] regarding Hilfssatz 2.2]; see also [49], Lemma 3.4). Brian Hartley refers to a \star -sequence, where the R_n 's need not to be perfect, as a *Kegel sequence* (see [27], Definition 2.2). He moreover states rather enlightening that the nomenclature of covers and sequences is more recent and even dedicates the entire Chapter 2 of [21] to Kegel sequences and to Kegel covers.

Proposition 1. *Let G be a countably infinite simple locally finite group. If $\{(R_n, M_n) \mid n \in \mathbb{N}\}$ is some \star -sequence for G , then it is a Kegel cover for G .*

Proof. If H is a finite subgroup of G , there exists an R_k of the nested local system $\{R_n \mid n \in \mathbb{N}\}$ for G with $H \subseteq R_k \subseteq R_{k+1}$ ($k \in \mathbb{N}$) and then $H \cap M_{k+1} = \langle 1 \rangle$. \square

U. MEIERFRANKENFELD (see [49]) classified (with the help of S. DELCROIX) simple locally finite groups G according to their Kegel covers (see [10]): **finitary** (there exists a field \mathcal{F} and a faithful $\mathcal{F}G$ -module V such that $V(g-1) = [V, g]$ is finite dimensional for all $g \in G$) (see [25]), **of 1-type** (where each Kegel cover has an alternating factor), **of p -type** for a unique prime p (where each Kegel cover has some classical group in characteristic p as some factor), and **of ∞ -type** (which have a Kegel cover all of whose factors allow embedding of every finite group). He proved earlier pretty much surprisingly that a non-finitary such group is either of alternating type (hence of **1-type** or of **∞ -type**) or (of **p -type** and of **projective special linear type**) (see [48] and the marvellous preprint at <https://users.math.msu.edu/users/meierfra/Preprints/Nflfsg/nflfsg.html>).

It had been inadvertently suggested that the results of this paper were a consequence of [25] and alternatively of “J.I. HALL – B. HARTLEY: ‘A group theoretical characterization of simple, locally finite, finitary linear groups’, Arch. Math. (Basel) 60, Issue 2 (February 1993), 108-114.” since the groups considered were thought to be finitary. However, this thinking is not true.

The joint paper by Hall and Hartley does not refer to Kegel covers and especially do both papers not refer to the p -uniqueness subgroups (Flemisch) resp. to the singular p -subgroups (Kegel). It had then been wrongly argued that the Kegel kernels M_i were not considered which in the given situation were claimed to be $\langle 1 \rangle$ for all $i \in \mathbb{N}$. But the Kegel factors R_i/M_i were considered and not only the kernels M_i nor were the kernels all $\langle 1 \rangle$. By rather vivid imagination it had then been quite wrongly concluded that the groups considered would become finitary linear locally finite simple groups which were classified by [25] (which is true). Even if all this would be true, [25] does not prove the results of this paper nor all the more so the paper by Hall and Hartley.

4. Planning future research – Part 1

We have seen that a simple locally finite group G can be covered by countable simple locally finite groups U each of which possesses a \star -sequence $\{(R^U, M^U) \mid n \in \mathbb{N}\}$ and so is in some sense a limit of the (approximating) sequences R^U/M^U ($n \in \mathbb{N}$) of finite non-abelian simple groups. If all the factors of the Kegel covers for all U , that is, all the R^U/M^U 's, belong to the same family Ξ of the infinite families $\{A^n, A = \text{PSL}_n, B = \text{P}\Omega_{\text{odd } n}, C = \text{PSp}_n, D = \text{P}\Omega_{\text{odd } n}^+, {}^2A = \text{PSU}_n, {}^2D = \text{P}\Omega_{\text{even } n}, E_6, E_7, E_8, F_4, G_2, {}^2B_2, {}^3D_4, {}^2E_6, {}^2F_4, {}^2G_2\}$, we call G to be **of type** Ξ . We propose to prove Kegel’s conjecture for all these types seratim, that is, one type after another in the given succession, and started already with the first type $\Xi = “A^n”$.

Our **Theorem 1** could be optimised in two ways:

- 1) Extend it from type A^n **step-by-step** to further types Ξ with an appropriate (similar) function f_p , that is, the rank $r(G)$ of a finite group G of type Ξ is bounded by $f_p(|P|)$ whenever P is a given p -uniqueness subgroup of G .
- 2) Determine for the type A^n and **peu à peu** for further types Ξ all the minimal p -unique subgroups, that is, the p -uniqueness subgroups of the non-abelian simple groups of type A^n and of type Ξ , which are minimal with respect to order (see [15]).

Note that whilst **way 2)** is of great interest for all types and also for sporadic \star (whereas it is trivial for abelian p), **way 1)** is not of interest for the families $\{E_6, E_7, E_8, F_4, G_2, {}^2B_2, {}^3D_4, {}^2E_6, {}^2F_4, {}^2G_2\}$ because these families have a fixed rank (label) and so are infinite only through the underlying field.

We recall from [15] the **Theorem 4.1** and its consequences:

Theorem 4.1 (see [14]). *Let G be a locally finite group satisfying the Strong Sylow Theorem for the Prime p .*

- a) *Each Sylow p -subgroup of G contains at least one (w.r.t. order) minimal p -unique subgroup of G .*
- b) *Every two (w.r.t. order) minimal p -unique subgroups of G have the same order.* \square

Let G be a **beautiful** locally finite group satisfying the Strong Sylow p -Theorem and let $S \in \text{Syl}_p G$. According to our **Theorem 4.1 a)**, S contains some (w.r.t. S) minimal p -unique subgroup F . We define $a_p = a_p(G) \in \mathbb{N}_0$ by $|F| =: p^{a_p}$, that is, we let a_p be the composition length of F . Then according to our **Theorem 4.1 b)**

this definition is independent of the special choice of the Sylow p -subgroup S of G , whereby in consequence a_p is a (numerical) Sylow p -invariant of G . We call a_p the p -uniqueness of G .

Then the optimising way 1) can be stated as follows:

Conjecture 1. Let $\mathcal{T} := \{\text{abelian}_p, A = \text{PSL}_n, B = \text{P}\Omega_{\text{odd } n}, C = \text{PSp}_n, D = \text{P}\Omega_{\text{odd } n}^+, {}^2A = \text{PSU}_n, {}^2D = \text{P}\Omega_{\text{even } n}^-, E_6, E_7, E_8, F_4, G_2, {}^2B_2, {}^3D_4, {}^2E_6, {}^2F_4, {}^2G_2, \text{sporadic } \star\}$ be the family of types of known finite simple groups and let G be a finite simple group of type $\Xi \in \mathcal{T}$. Then the rank $\tau(G)$ of G is bounded in terms of the p -uniqueness $a_p(G)$.

Brian Hartley (15 May 1939 until 8 October 1994) in his Mathematical Review of [44] (see MR981832 [MR 90c #20037 (March 1990)]) stated the following: “If the simple locally finite group G satisfies the Strong Sylow Theorem for the (even one) Prime p , then G is linear. This depends on the classification of finite simple groups and an assertion about singular p -subgroups of classical groups. Another proof of this result has since been given by the reviewer (not yet published).” The assertion mentioned is Kegel’s conjecture (see [44], Theorem 2.4). However, due to the so very tragic death of Prof. Hartley in 1994, aged 55 (see [14]), this certainly highly insight gaining proof was never prepared for publication. Hartley wrote 1994 a very eye-opening paper on simple locally finite groups (see [27]) which, however, did not refer to Kegel’s work [44] and not even included it in its list of 56 references. The paper could appear only posthumously which most likely is the reason for the full ignorantness of Kegel’s paper. Hartley’s paper was meticulously completed and carefully prepared for publication by Richard E. Phillips (3 December 1936 until 9 November 1999). We consider it much rewarding, even after 30 years, to inspect Hartley’s estate *In Search of not Lost Notes* (see Marcel Proust [10 July 1871 until 18 November 1922]: “À la recherche du temps perdu” / “In Search of Lost Time” / “Auf der Suche nach der verlorenen Zeit” / “Alla ricerca del tempo perduto” / “En busca del tiempo perdido” / “Em busca do tempo perdido”).

Now as a very first step towards solving **Conjecture 1** for the second type $\Xi = “A = \text{PSL}_n”$, we state another conjecture w.r.t. the general linear group over locally finite fields (see [14]):

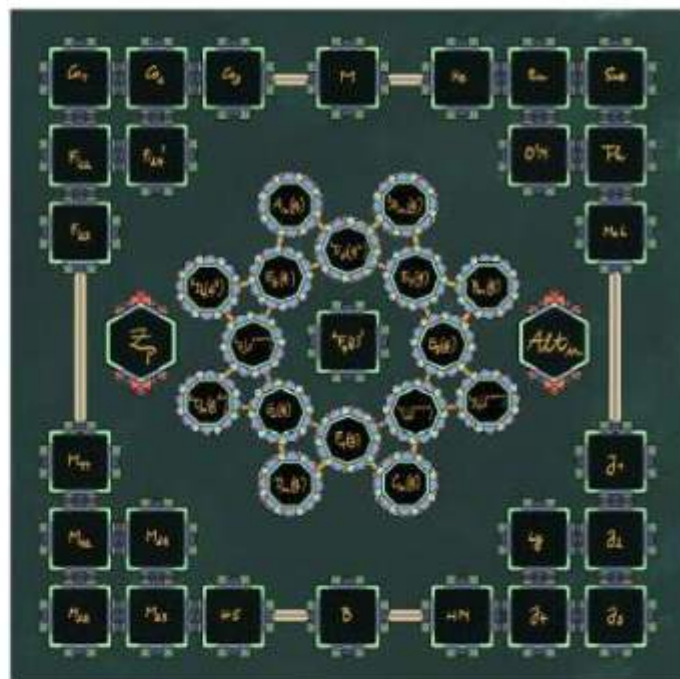
Conjecture 2. Let $n \in \mathbb{N}$ and let p be a prime.

Let F be a locally finite (commutative) field.

- a) If F has characteristic p and $a_p = a_p(\text{GL}(n, F))$ then $n \leq (p + 2) \cdot p^{2p} - 1$.
- b) If F has characteristic $\neq p$ and $a_p = a_p(\text{GL}(n, F))$ then $n \leq (p + 2) \cdot p^{2p} - 1$.

In the entire paper we do not refer to the classification of finite simple groups (see [23], [61] and **Page 3**) but prefer to talk about the 19 families of “known” finite simple groups. Our efforts are directed towards knowing much better their Sylow subgroups. We hope to find useful insights about the Sylow subgroups of classical groups in the ATLAS of Finite Groups [8] and in the comprehensive literature about them.

The classification of the finite simple groups
(13 sporadic groups above 18 infinite families around another “sporadic” group and further 13 sporadic groups below)



(© 2022 by Mathsies – Own work, CC BY-SA 4.0, https://upload.wikimedia.org/wikipedia/commons/archive/a/a9/202111205053%21Classification_of_the_finite_simple_groups.jpg, 28 December 2021, at 15:08 (UTC); [61])

Kegel’s lectures [44] present the very basics of Sylow theory in locally finite groups, give an overview of the prodigious work of Brian Hartley and Andrew Rae on the Sylow theory in locally finite and p -soluble groups, and reveal in great detail the normal structure for groups satisfying the Strong Sylow Theorem for the Prime p in the general case (for $p \geq 5$). Chapters 2 and 4 of [12] give a rather good overview as well but alas without appreciating Kegel’s very insight gaining work properly and avoiding all its beautiful details. We cite from the Preface of [12]: “The condition that all the maximal p -subgroups of a locally finite group are conjugate is a very strong condition indeed; the structure of those groups has been obtained in the locally p -soluble case by Hartley and in the general case by Kegel. The Hartley-Kegel theorem is quite involved so I decided to simply state the results obtained.” Also, simple groups are not in the scope of [12] and therefore [12] must be supplemented by [27].

Although this paper is about simple groups we cannot help to close with a brief attention to p -soluble groups since it is the joint study of the (locally) simple and the (locally) p -soluble groups which directs reliably the Sylow theory in (locally) finite groups.

In Chapter “2 Some length type inequalities” of his rather remarkable contribution [47], Alexandre Turell (see <https://people.clas.ufl.edu/turull/>) states a conjecture of Thomas R. Berger (which dates back to John G. Thompson in the 1970’s):

 (The Foundation for The Great Future) states a conjecture of Thomas R. Berger (which dates back to John G. Thompson in the 1970’s):

Conjecture 2.3 (see [3]). Let p be a prime. There exists a linear function f_p such that if G is a finite p -soluble group with p -length $\lambda_p(G)$ and P is a subgroup of G of order p^k ($k \in \mathbb{N}$) contained in precisely one Sylow p -subgroup then $\lambda_p(G) \leq f_p(k)$.

Having studied the very most of the hereof related literature published by Brian Hartley, by Andrew Rae, and by Thomas R. Berger, we profess to have happily discovered such a linear function, namely our nice a_p . Therefore we can state Thomas R. Berger's conjecture more precisely (and best possible) as our

Conjecture 3. Let p be a prime. Let G be a p -soluble finite group, $\lambda_p(G)$ be its p -length, and $a_p(G)$ be its p -uniqueness. Then $\lambda_p(G) \leq a_p(G) + 1$.

It is much expected that the cases $p \geq 5$, $p = 3$ and $p = 2$ must be treated fairly separately and also that $p = 3$ and $p = 2$ will require fairly special methods as already indicated by the available literature. A. Turell gives in Section 2 of [53] a quite concise overview of the classical Hall-Higman theory created by P. Hall, G. Higman, A.H.M. Hoare, T.R. Berger, F. Gross and E.G. Bryukhanova, which introduces for finite p -soluble groups (best possible) inequalities between their p -length λ_p and the order p^{b_p} of a Sylow p -subgroup, its nilpotency class c_p , its solubility length d_p , its exponent p^{e_p} , or the rank r_p of a maximal elementary abelian subgroup. Our aim is to extend the Hall-Higman theory to the very beautiful p -uniqueness p^{a_p}



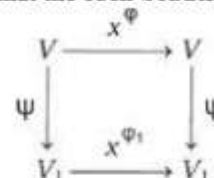
of a Sylow p -subgroup, an Herculean endeavour. It is in this context that A. Turell cites T.R. Berger's conjecture and presents some results up to 1994 with regard to partly solving it but they are very gey far from being complete, in particular concerning the basic results of B. Hartley and A. Rae. But on the other hand T.R. Berger presents in [3] a, as he says, reasonably complete list of references up to 1979, including 15 of his own contributions, where eleven are related to p -length problems, and discusses his method of proof for p -length and other length type problems in a considerably detailed fashion.

5. Proof of Theorem 2

Proof. We begin with some general remarks (see [9], Chapter II, and [11], Chapters 1 and 2). Let \mathcal{F} be a field, $V \neq \{0\}$ be a vector space over \mathcal{F} with its automorphism group $GL(V)$, and let G be a group. V is called a G -module over \mathcal{F} and G operates on V , if a homomorphism of groups $\varphi: G \rightarrow GL(V)$ is declared. φ is then called a linear representation of G on V over \mathcal{F} . Every permutation representation of G on a set $\Omega \neq \emptyset$ now induces a G -module $V(\Omega)$, called the permutation module of (G, Ω) over \mathcal{F} .

Therefore to every subgroup U of G belongs the G -module $V(\mathcal{R}(G, U))$ (see Page 5 and Page 6) with respect to (w.r.t.) multiplication from the right. A subspace W of V is called G -invariant or a G -submodule, if for all $x \in G$ we have $x^\varphi(W) \subseteq W$, that is, φ induces an operation of G on W . We say that G operates on V irreducible, if V contains exactly two G -sub-

modules (namely $\{0\}$ and V), and completely reducible, if to every G -submodule W of V there exists a G -submodule X of V with $V = W \oplus X$, equivalently, if V is decomposable into a direct sum of minimal G -submodules. G operates on V non-modular, if $\text{char } \mathcal{F} = 0$ or $\text{char } \mathcal{F} \neq 0$ and G contains no $\text{char } \mathcal{F}$ -elements $\neq 1$; otherwise G operates modular on V . Now let V_1 be another G -module over \mathcal{F} on which G operates via φ_1 . Then V is called G -isomorphic to V_1 , if there exists an isomorphism of vector spaces $\psi: V \rightarrow V_1$ such that the such beautiful diagram shown



commutes for all $x \in G$:

Every irreducible G -module is G -isomorphic to a factor module of $V(\mathcal{R}(G, \langle 1 \rangle))$: the class $\mathcal{J}(G, \mathcal{F})$ of all G -isomorphism types of irreducible G -modules is a duly set of (finite-dimensional) vector spaces over \mathcal{F} all of which have their dimension $\leq |G|$.

We now start the proof of Theorem 2 by quoting two fairly well-known facts about non-modular linear representations (see [22], Chapter 3, Theorem 3.1, and [11], Theorem 10.8, for point a), as well as [9], Theorem 27.22 with Remark 27.25, for point b)). We denote for point b) by $h(G)$ the class number of G , that is, the number $|\{x^G \mid x \in G\}|$ of conjugacy classes of G .

Proposition 2. Let G be a finite group.

- a) (Heinrich Maschke, 1898) Every non-trivial non-modular finite-dimensional G -module is completely reducible.
- b) Let \mathcal{F} be a field with $(\text{char } \mathcal{F}, |G|) = 1$. Then there are at most $h(G)$ many G -isomorphism types of irreducible G -modules over \mathcal{F} . ■

We use Proposition 2 b) straight away to prove the following:

Proposition 3.

- a) There exists a function $\gamma: \mathbb{N} \rightarrow \mathbb{N}$ with the following property: If G is a finite group, \mathcal{F} a field with $(\text{char } \mathcal{F}, |G|) = 1$ and $\mathcal{J}(G, \mathcal{F})$ the class of all G -isomorphism types of irreducible G -modules over \mathcal{F} , then $\mathcal{J}(G, \mathcal{F})$ is a genuine set with $|\mathcal{J}(G, \mathcal{F})| \leq \gamma(|G|)$.
- b) Let G be a finite group, \mathcal{F} a field with $(\text{char } \mathcal{F}, |G|) = 1$ and V a finite-dimensional G -module over \mathcal{F} . Let $\mathcal{J}(G, V)$ be the set of G -isomorphism types of irreducible G -submodules of V . Then $|\mathcal{J}(G, V)| \leq \gamma(|G|)$, where γ is the function from point a). ■

RATIONALE – a) We define $\gamma: \mathbb{N} \rightarrow \mathbb{N}$ simply by $\gamma(n) := n$. Then $h(G) \leq \gamma(|G|)$. Since by Proposition 2 b) there is an injective mapping of $\mathcal{J}(G, \mathcal{F})$ into $\{x^G \mid x \in G\}$ the assertion follows.

b) follows from point a). ■

Up next we use Proposition 2 a) and Proposition 3 b) to prove

Proposition 4. Let G be a finite group and $k \in \mathbb{N}$. Let V be a finite-dimensional non-modular G -module with $\dim(V) \geq |G| \cdot \gamma(|G|) \cdot k$, where γ is the function from Proposition 3 a). Then there exist at least k many G -isomorphic irreducible G -submodules of V .

RATIONALE – By **Proposition 2 a)** and a trivial induction on $\dim(V)$ there are m irreducible G -submodules U_i of V with $V = \bigoplus \{U_i \mid 1 \leq i \leq m\}$. If $0 \neq v_i \in U_i$ then $\langle v_i^x \mid x \in G \rangle$ is a G -submodule of V and thus $\dim(U_i) \leq |G|$ ($1 \leq i \leq m$). Now let $\mathcal{J}(G, V)$ be the set of G -isomorphism types of irreducible G -submodules of V . If $m \geq k \cdot |\mathcal{J}(G, V)|$ then by the pigeonhole principle there will be just k many G -isomorphic irreducible G -submodules of V . Because of $\dim(V) = \sum \{\dim(U_i) \mid 1 \leq i \leq m\}$ and $m \leq \{\dim(U_i) \mid 1 \leq i \leq m\} \leq m \cdot |G|$, we have $\dim(V)/|G| \leq m \leq \dim(V)$. Thus there are at least k many G -isomorphic irreducible G -submodules of V if only $\dim(V) \geq k \cdot |G| \cdot |\mathcal{J}(G, V)|$. So the assertion follows from **Proposition 3 b)**. ■

Recall that a finite group G operates *modular* on a G -module V if $G = \langle 1 \rangle$ or G operates not non-modular on V . Therefore a finite p -group for the prime p operates modular on every vector space over the field \mathcal{F} if and only if $\text{char } \mathcal{F} = p$. We prove next two elementary facts (see [22], Chapter 2, Lemmata 6.2 and 6.3):

Proposition 5.

- a) Let G be a group, N be a normal subgroup of G and V be a G -module. Then $C_V(N) := \{v \in V \mid v^x = v \text{ for all } x \in N\}$ is a G -submodule of V .
- b) Let P be a finite p -group for the prime p and let V be a non-trivial modular P -module. Then $C_V(P) \neq \{0\}$.

RATIONALE – a) Put $U := C_V(N)$. Then U is a subspace of V . Let $u \in U$ and $x \in G$. For $y \in N$ we have also $y^{x^{-1}} \in N$, since N is normal in G , and so $u^{y^{x^{-1}}} = u$. Therefore we have $u^{x^y} = u^x$ for all $y \in N$, that is, $u^x \in U$.

b) We carry out an induction on $|P|$. For $P = \langle 1 \rangle$ we have $C_V(P) = V$ and nothing to prove. Let $|P| \geq p$ and M be a maximal subgroup of P . Then M is normal in P with $|P:M| = p$. Put $U := C_V(M)$. Then U is by point a) a P -submodule of V and by the induction hypothesis we have $U \neq \{0\}$. Let $y \in P \setminus M$ and $y' \in \text{GL}(U)$ be the restriction of y to U . Then $y^p \in M$ and $\langle M, y \rangle = M \cdot \langle y \rangle = P$ and so $C_V(P) = U \cap C_V(y) = C_U(y')$. It remains for us to prove that $C_U(y') \neq \{0\}$. Let \mathcal{F} be the field over which V is being a vector space and let $\mu(X)$ be the minimal polynomial of y' over \mathcal{F} . Then $\mu(X)$ divides the polynomial $X^p - 1$ of $\mathcal{F}[X]$, since y' has order 1 or p in $\text{GL}(U)$, and $p = \text{char } \mathcal{F} = \text{char } \mathcal{F}[X]$ as well. Therefore $X^p - 1 = (X - 1)^p$. Hence $\mu(\kappa) = 0$ for $\kappa \in \mathcal{F}$ if and only if $\kappa = 1$, that is, 1 is the only eigenvalue of y' with $C_U(y') \neq \{0\}$ as its eigenspace. ■

We are in the very happy position to prove an intriguing toughening of **Proposition 5 b)** which is quite definitely not an elementary insight (see as well [43], p. 41, where, however, this core assertion is not proved properly and even only for an elementary abelian P , and [32], Chapter VIII, Lemma 10.17, where, however, only the very special example is considered that V is an abelian p -group and P has order p):

Proposition 6. Let P be a finite p -group for the prime p , \mathcal{F} be a field of characteristic p and V be a finite-dimensional P -module over \mathcal{F} . Then $\dim(C_V(P)) \geq \dim(V)/|P|$.

RATIONALE – We refine the proof of **Proposition 5 b)** and carry out an induction on $|P|$. For $P = \langle 1 \rangle$ we have nothing to prove. Let $|P| \geq p$ and M be a maximal subgroup of P . Then M is normal in P with $|P:M| = p$. Put $U := C_V(M)$. Let $y \in P \setminus M$ and $y' \in \text{GL}(U)$ be the restriction of y to U . Then $y^p \in M$ and $\langle M, y \rangle = M \cdot \langle y \rangle = P$ and so $C_V(P) = U \cap C_V(y) = C_U(y')$. From **Proposition 5 a)** and the induction hypothesis follows that U is a P -submodule of V with $\dim(U)/p \geq \dim(V)/(M \cdot p) = \dim(V)/|P|$. It thus remains for us to prove the following:

$$(\diamond) \quad p \cdot \dim(C_U(y')) \geq \dim(U).$$

Put $n := \dim(U)$ and $d := \dim(C_U(y'))$. Let $\mu(X)$ be the minimal polynomial of y' over \mathcal{F} . Then $\mu(X)$ will divide the polynomial $X^p - 1$ of $\mathcal{F}[X]$, since y' has order 1 or p in $\text{GL}(U)$. Because of $p = \text{char } \mathcal{F} = \text{char } \mathcal{F}[X]$ we have $X^p - 1 = (X - 1)^p$. Hence 1 is the unique eigenvalue of y' with $C_U(y')$ as related eigenspace. In particular $d \geq 1$. Let $\chi(X) := \det(y' - X \text{id}_U)$ be the characteristic polynomial of y' over \mathcal{F} . Then $\chi(X)$ has degree n and is divided by $\mu(X)$. In particular $U = \text{kernel}(y' - \text{id}_U)^n$ whence y' is unipotent. **RECALL** – Let G be a subgroup of $\text{GL}(n, \mathcal{F})$. We call $x \in G$ *unipotent* if $(x - 1)^n = 0$, that is, if all eigenvalues of x are 1, and call G *unipotent* if each element of G is unipotent. Every unipotent subgroup of $\text{GL}(n, \mathcal{F})$ is some conjugate of a subgroup of $\text{UT}(n, \mathcal{F})$, the group of upper triangular matrices. If $\text{char } \mathcal{F}$ is a prime p , then the unipotent elements of $\text{GL}(n, \mathcal{F})$ are precisely the p -elements and $\text{UT}(n, \mathcal{F})$ is a Sylow p -subgroup of $\text{GL}(n, \mathcal{F})$. ■ Thus there is an \mathcal{F} -basis of U such that the matrix of U w.r.t. this \mathcal{F} -basis will lie in $\text{UT}(n, \mathcal{F})$. This matrix can be decomposed in Jordan normal form as follows. Let $\tau := y' - \text{id}_U$ and for each $m \in \mathbb{N}_0$ let $C_m := \text{kernel}(\tau^m)$. The C_m 's are \mathcal{F} -subspaces of U with $\{0\} = C_0 \subseteq C_1 \subseteq C_m \subseteq C_{m+1} \subseteq \dots$. We have $C_1 = C_U(y')$ and $C_n = U$. Let $k \in \mathbb{N}$ be minimal w.r.t. $C_k = U$ and put $r := \dim(U/C_{k-1})$.

Then $u \mapsto u^{\tau^{k-1}}$ ($u \in U$) induces an isomorphism of U/C_{k-1} onto an \mathcal{F} -subspace of C_1 . It follows that $r \leq d$. We have $\tau^p = (y' - \text{id}_U)^p = y'^p - \text{id}_U = 0$ since y' has order 1 or p in $\text{GL}(U)$ and $p = \text{char } \mathcal{F} = \text{char } \mathcal{F}[X]$ whence $\text{image}(\tau^m) = \{0\}$ for all $m \in \mathbb{N}$ with $m \geq p$. It follows that $k \leq p$. Now for each $u \in U/C_{k-1}$ we define $W_u := \langle u, u^\tau, \dots, u^{\tau^{k-1}} \rangle$ which will be a y' -invariant \mathcal{F} -subspace of U with $\dim(W_u) = k$. The $k \times k$ -matrix $A(y')$ of y' restricted to W_u w.r.t. $\{u^{\tau^{k-1}}, u^{\tau^{k-2}}, \dots, u^\tau, u\}$ has the shape shown:

$$A(y') = \begin{pmatrix} 1 & 1 & \dots & 0 \\ & 1 & 1 & \dots \\ & & \dots & \dots \\ & & & 1 \\ 0 & & & & 1 \end{pmatrix}$$

There exist $u_1, u_2, \dots, u_r \in U \setminus C_{k-1}$ with $U = \bigoplus \{W_{u_i} \mid 1 \leq i \leq r\}$. Then the $n \times n$ -matrix $A(y')$ of y' w.r.t. the basis $\{u_1^{\tau^{k-1}}, \dots, u_1, u_2^{\tau^{k-1}}, \dots, u_2, \dots, u_r^{\tau^{k-1}}, \dots, u_r\}$ of U has the above shape as well. It now follows that $n = k \cdot r$ and hence $n \leq p \cdot d$ by the previous inequalities. This is (\diamond) to be proved. ■

The inequality of **Proposition 6** is best-possible since for every prime p there exists a faithful finite-dimensional C_p -module V over $\text{GF}(p)$ with $\dim(C_V(C_p)) = \dim(V)/p$: let q be a prime such

that p divides $q - 1$; the S^4 is a semidirect product of the $S^3 = C_2 \cdot C_3$ with the four group $C_2 \times C_2$; this operation can be generalised to an operation of $C_p \cdot C_q$ on $V := C_p^{(q-1)}$; if $p = 2$ one gets for every impair prime q the “generalised S^4 ” of order $2^q q$; the (classical) Hall-Higman theory can now be used to show $\dim(C_V(C_p)) = \dim(V)/p$ (see Page 8 and Page 9).

We next apply some of the beautiful new ideas of the proof of Theorem 1 b) and of Proposition 5 to prove for $GL(V)$ a similar statement as for S^Ω where $\Omega := \{1, 2, \dots, n\}$:

Proposition 7. Let V be a finite-dimensional vector space over the locally finite (commutative) field \mathcal{F} . The finite p -group P for the prime p shall operate on V .

- a) Let $\text{char } \mathcal{F} = p$ and let $V = \bigoplus \{U_i \mid 1 \leq i \leq m\}$ be a direct decomposition of V into irreducible P -submodules according to Proposition 2 a). Let k be the number of P -isomorphic U_i 's. Then there exist at least $|\text{Syl}_p S^k|$ many P -invariant Sylow p -subgroups of $GL(V)$.
- b) Let $\text{char } \mathcal{F} = p$ and $k := \dim(C_V(P))$. Then there are at least $|\text{Syl}_p S^k|$ many P -invariant Sylow p -subgroups of $GL(V)$.

RATIONALE – We may suppose without loss of generality (w.l.o.g.) that P is a subgroup of $GL(V)$ and operates by conjugation on $GL(V)$. If $S \in \text{Syl}_p GL(V)$ then $\text{Syl}_p N_{GL(V)}(S) = \{S\}$ and hence P normalises S if and only if $P \subseteq S$. Therefore we have to prove the following:

$$(\star) |\{S \in \text{Syl}_p GL(V) \mid P \subseteq S\}| \geq |\text{Syl}_p S^k|.$$

a) We certainly may suppose w.l.o.g. that the first k of the U_i 's are P -isomorphic. Let $H_i \subseteq GL(V)$ be the point stabiliser of $\bigoplus \{U_j \mid 1 \leq j \leq m, j \neq i\}$; then $H_i \approx GL(U_i)$ ($1 \leq i \leq m$). Put $D := \langle H_i \mid 1 \leq i \leq m \rangle = \prod \{H_i \mid 1 \leq i \leq m\} \subseteq GL(V)$. Then $P \subseteq D$. Let B be the set of automorphisms of V which interconvert in entire blocks the P -isomorphic U_i 's and let the remaining U_i 's pointwise fixed. Then $B \subseteq GL(V)$ with $B \approx S^k$ and $B \cap D = \langle 1 \rangle$. Since B interchanges only P -isomorphic U_i 's, it is normalised by D . Hence $K := \langle B, D \rangle$ is the semidirect product $B \cdot D$, and hence D is normal in K with $K/D \approx B$. Now let $Q \in \text{Syl}_p K$ with $P \subseteq Q$. Since D is normal in K , we have $P \subseteq D \cap Q \in \text{Syl}_p D$ and by the Frattini argument, which follows from the (Strong) Sylow p -Theorem for the finite K , $N_K(D \cap Q)/N_D(D \cap Q) \approx K/D$. It follows that $|\{S \in \text{Syl}_p GL(V) \mid P \subseteq S\}| \geq |\{S \in \text{Syl}_p GL(V) \mid D \cap S = D \cap Q\}| \geq |\text{Syl}_p(N_K(D \cap Q)/N_D(D \cap Q))| \geq |\text{Syl}_p(K/D)| \geq |\text{Syl}_p B|$. This is the inequality of (\star) to be proved.

b) $C := C_V(P)$ is by Proposition 5 a non-trivial P -submodule of V . Let $D := C_{GL(V)}(C)$. Then $P \subseteq D$. Now let C_1 be a (not necessarily P -invariant) complement to C in V , that is, $V = C \oplus C_1$. Let B be the point stabiliser of C_1 . Then $GL(V) \approx B$ and $B \cap D = \langle 1 \rangle$. For all $b \in B$, $d \in D$ and $c \in C$ we have $c^{(d^b)} = (c^{b^{-1}})^{db} = (c^{b^{-1}})^b = c$. Hence B normalises D and so $K := \langle B, D \rangle = B \cdot D$ whence D is normal in K with $K/D \approx B$. Since $k = \dim(C)$, the group B contains a subgroup which is isomorphic to S^k , namely the group of all permutation matrices of rank k over \mathcal{F} (see [11], § 1.3). Therefore $|\text{Syl}_p B| \geq |\text{Syl}_p S^k|$. Now (\star) follows verbatim as in point a). ■

Next we are notably very happy to be able to use the foregoing Propositions 4 & 6 & 7 together with Lemma 1 of Page 9 to prove a core Lemma from which Theorem 2 follows immediately. □

Lemma 2. Let $n \in \mathbb{N}$ and let p be a prime. Let \mathcal{F} be a locally finite (commutative) field and let P be a finite p -subgroup of $GL(n, \mathcal{F})$ which is contained in exactly $k \in \mathbb{N}$ Sylow p -subgroups of $GL(n, \mathcal{F})$.

- a) If \mathcal{F} has characteristic $\neq p$ then $n \leq (k + p + 1) \cdot |P|^2 - 1$.
- b) If \mathcal{F} has characteristic p then $n \leq (k + p + 1) \cdot |P| - 1$.
- c) If P is a p -uniqueness subgroup of $GL(n, \mathcal{F})$ then $n \leq f_p(|P|) := (p + 2) \cdot |P|^2 - 1$.

RATIONALE – a) Let $n \geq (k + p + 1) \cdot |P|^2$. By Proposition 4 and since $\gamma(|P|) = |P|$ by the proof of Proposition 3 a), the space \mathcal{F}^n then has at least $k + p + 1$ many irreducible P -isomorphic P -submodules. Thus P lies by Proposition 7 a) in at least $|\text{Syl}_p S^{k+p+1}|$ many Sylow p -subgroups of $GL(n, \mathcal{F})$. From Lemma 1 b) of Page 6 we can now conclude $|\text{Syl}_p S^{k+p+1}| \geq k + p + 1 - 2 \geq k + 1$.

b) Let $n \geq (k + p + 1) \cdot |P|$. We then have $\dim(C_{\mathcal{F}^n}(P)) \geq k + p + 1$ by Proposition 6. Therefore P lies by Proposition 7 b) in at least $|\text{Syl}_p S|$ many Sylow p -subgroups of $GL(n, \mathcal{F})$. Now follows from Lemma 1 b) of Page 6 that $|\text{Syl}_p S| \geq k + p + 1 - 2 \geq k + 1$.

c) follows from point a) and point b). ■

6. Proof of Theorem 3

A subgroup of $GL(n, \mathcal{F})$ is locally finite if and only if \mathcal{F} is locally finite, that is, if every finitely generated subfield of \mathcal{F} is finite. \mathcal{F} is locally finite if and only if it is isomorphic to a subfield of $\overline{\mathcal{F}_p}$, the algebraic closure of the nice prime field $GF(p) = \mathcal{F}_p$, for some prime p , and hence is countable. Since $\mathcal{F}_p \subseteq \mathcal{F}_p^m$ if and only if m divides n ($m, n \in \mathbb{N}$), we consider the chain $\mathcal{F}_p \subseteq \mathcal{F}_p^m \subseteq \mathcal{F}_p^{(m+1)} \subseteq \mathcal{F}_p^{(m+2)} \subseteq \dots$ of algebraic extensions, where $\mathcal{F}_p^{(m+1)}$ is obtained by just adjoining some root α of an irreducible polynomial of degree $n + 1$ over \mathcal{F}_p^m , that is, $\mathcal{F}_p^{(m+1)} = \mathcal{F}_p^m(\alpha)$ ($n \in \mathbb{N}$). Then $\overline{\mathcal{F}_p} = \bigcup \{\mathcal{F}_p^m \mid m \in \mathbb{N}\} = [\text{since } \mathcal{F}_p^m \subseteq \mathcal{F}_p^n] \bigcup \{\mathcal{F}_p^n \mid n \in \mathbb{N}\}$ (see [4], Section 2.2). All the subfields of \mathcal{F}_p (see [4], Section 2.3) correspond to all the locally finite fields in characteristic p .

Let $\mathcal{F}^* := \mathcal{F} \setminus \{0\}$ be the multiplicative group of \mathcal{F} and let $SL(n, \mathcal{F}) := \{A \in GL(n, \mathcal{F}) \mid \det(A) = 1\}$.

Proof. $GL(n, \mathcal{F}) = SL(n, \mathcal{F}) \cdot \mathcal{F}^*$ is the semidirect product of $SL(n, \mathcal{F})$ with \mathcal{F}^* and the unique Sylow p -subgroup S_p of \mathcal{F}^* is \mathcal{F}^* if $\text{char } \mathcal{F} = p$ and $\langle 1 \rangle$ if $\text{char } \mathcal{F} \neq p$. Thus $\{S \mid S \in \text{Syl}_p GL(n, \mathcal{F})\} = \{T \cdot S_p \mid T \in \text{Syl}_p SL(n, \mathcal{F})\}$ whence every Sylow p -subgroup of $SL(n, \mathcal{F})$ lies in only one Sylow p -subgroup of $GL(n, \mathcal{F})$. Hence if P is a p -uniqueness subgroup of $SL(n, \mathcal{F})$ it is also a p -uniqueness subgroup of $GL(n, \mathcal{F})$. Therefore $n \leq (p + 2) \cdot |P|^2 - 1$ if $\text{char } \mathcal{F} \neq p$ by Lemma 2 a) which is Theorem 3 b) and $n \leq (p + 2) \cdot |P| - 1$ if $\text{char } \mathcal{F} = p$ by Lemma 2 b) which is Theorem 3 a). □

7. Proof of Theorem 4

Let $D(\mathrm{SL}(n, \mathcal{F})) := \{A \in \mathrm{SL}(n, \mathcal{F}) \mid A \text{ is some scalar matrix}\}$ be the subgroup of $\mathrm{SL}(n, \mathcal{F})$ of matrices in which all off-diagonal entries are zero and the diagonal entries are any scalars, that is, elements of \mathcal{F} , but not all zero. It is very well-known that $D(\mathrm{SL}(n, \mathcal{F}))$ is the centre of $\mathrm{SL}(n, \mathcal{F})$ and that $\mathrm{PSL}(n, \mathcal{F}) := \mathrm{SL}(n, \mathcal{F})/D(\mathrm{SL}(n, \mathcal{F}))$.

Proof. If S is a **beautiful** Sylow p -subgroup of $\mathrm{SL}(n, \mathcal{F})$, then $S/D(\mathrm{SL}(n, \mathcal{F})) \cong S/Z(S)$ is a Sylow p -subgroup of $\mathrm{PSL}(n, \mathcal{F})$ where $Z(S)$ denotes the centre (“Zentrum”) of S . If Q is a p -uniqueness subgroup of $\mathrm{SL}(n, \mathcal{F})$ then $P := QD(\mathrm{SL}(n, \mathcal{F}))/D(\mathrm{SL}(n, \mathcal{F})) \cong Q/Z(Q)$ will be a p -uniqueness subgroup of $\mathrm{PSL}(n, \mathcal{F})$ (see [44], 1.6, and [15], Proposition 2.3), and conversely, and $n \leq f_p(|Q|)$ by **Theorem 3**. However, even $n \leq f_p(|P|)$ since otherwise $n \geq (p+2) \cdot |P|$ resp. $n \geq (p+2) \cdot |P|^2$ if $\mathrm{char} \mathcal{F} = p$ resp. if $\mathrm{char} \mathcal{F} \neq p$. Since P operates on the underlying vector space \mathcal{F}^n , we have $\dim(C_{\mathcal{F}^n}(P)) \geq p+2$ by **Proposition 6** resp. the space \mathcal{F}^n has at least $p+2$ many irreducible P -isomorphic P -submodules according to **Proposition 4**. Therefore P lies in at least $|\mathrm{Syl}_p \mathcal{S}^{p+2}|$ Sylow p -subgroups of $\mathrm{PSL}(n, \mathcal{F})$ by **Proposition 7** which is at least 2 by **Lemma 1 δ** of **Page 6**. \square

8. Planning future research – Part 2

Our proofs of the **Conjecture 1** of **Page 8** for the types $\Xi = “\mathbb{A}^n”$ and $\Xi = “A = \mathrm{PSL}_n”$, that is, to carve out the optimising way 1), are characterised by the fact that we need not at all know their Sylow p -subgroups. There is no doubt that we can (easily) extend those proofs rather straightforwardly to the types $\Xi \in “B = \mathrm{P}\Omega_{\mathrm{odd} \, n}, C = \mathrm{P}\mathrm{Sp}_n, D = \mathrm{P}\Omega_{\mathrm{odd} \, n}^+, {}^2A = \mathrm{PSU}_n, {}^2D = \mathrm{P}\Omega_{\mathrm{even} \, n}^-”$ by considering thoroughly the respective bilinear form defining these groups of Lie type, resp. the underlying vector spaces they act upon as isometries, and their resulting Sylow p -subgroups (without knowing them). They can well be considered proved which we shall confirm in a follow-up paper (see below: the **Part 1** of our **Second Trilogy**).

Optimising **Theorem 1**, **Theorem 2**, **Theorem 3** and **Theorem 4** along the way 2) of **Page 7** is much more challenging since it requires to determine the (minimal) p -uniqueness subgroups of \mathbb{A}^n and of all the classical groups. Fortunately, a vast literature about these groups and their Sylow p -subgroups is available, even about the intersections of their Sylow p -subgroups. The starting point for future research into these hugely **beautiful** objects should be the papers by LÉO A. KALOUJNINE (see [32]-[40]) and by ALAN J. WEIR (see [55]-[58]) and **Theorem 1.4 B** of [11] together with [7]. The starting point for Sylow p -intersections could be [5] which has a sizeable list of references and all sorts of historical details.

A MATHEMATICIAN, like a painter or a poet, is a maker of patterns. If his patterns are more permanent than theirs, it is because they are made with ideas. ... The mathematician's patterns, like the painter's or the poet's, must be beautiful; the ideas, like the colours or the words, must fit together in a harmonious way. Beauty is the first test: there is no permanent place in the world for ugly mathematics.

Godfrey Harold Hardy (7 February 1877 until 1 December 1947).

A Mathematician's Apology. § 10. July 18, 1940. ISBN 978-1-68422-185-1.

With a foreword by Charles Percy Snow. ISBN 978-1-107-60463-6.

The author is passionately curious about the future.
Der Autor ist sehr leidenschaftlich neugierig auf die Zukunft.
L'auteur est passionnément curieux de l'avenir.
L'autore è appassionatamente curioso del futuro.
O autor é muito apaixonadamente curioso sobre o futuro.

Felix Fortunatus Flemisch (17 May 1951 until today).

Firenze. April 11, 1992.

We now indicate how to continue the way 1) of **Page 7** for the remaining types $\Xi \in “B = \mathrm{P}\Omega_{\mathrm{odd} \, n}, C = \mathrm{P}\mathrm{Sp}_n, D = \mathrm{P}\Omega_{\mathrm{odd} \, n}^+, {}^2A = \mathrm{PSU}_n, {}^2D = \mathrm{P}\Omega_{\mathrm{even} \, n}^-”$ and how to prove the **Conjecture 3** of **Page 9** by announcing the two follow-up papers “The Strong Sylow Theorem for the Prime p in the Locally Finite Classical Groups” and “The Strong Sylow Theorem for the Prime p in Locally Finite and p -Soluble Groups” which we hope to finalise in 2025. They are the first two parts of **The Second Trilogy about Sylow Theory in Locally Finite Groups** whose third part will be our forthcoming research paper “Augustin-Louis Cauchy's and Évariste Galois' Contributions to Sylow Theory in Finite Groups”. **The First Trilogy** are [15] on p -uniqueness subgroups and [this paper] on \mathbb{A}^n and $A = \mathrm{PSL}_n$ (see the **Postscript** on **Page 15**).

Part 1 of **The Second Trilogy** considers the locally finite classical groups which are the linear, symplectic, unitary and orthogonal groups over locally finite fields. The linear groups are dealt with in this paper and the others are subgroups of the linear groups which are defined through a non-singular bilinear form (or a scalar product) which is either skew-symmetric (or alternate) or Hermitian or symmetric (defining a quadratic form) as the group of isometries of the form. They were introduced in the classical books [1] and [60] and are further studied in [6], [24] and [52]. We do not refer to the groups of Lie type resp. the Chevalley groups and the twisted Chevalley groups being defined through a Dynkin diagram automorphism followed by a field automorphism, which correspond to the classical groups (see [24], pp. 151-152) and whose fine introductory references are the “Lecture Notes on Chevalley Groups” by Robert Steinberg (1967 and 2016) together with the book “Simple Groups of Lie type” by Roger W. Carter (1972 and 1989). Thus we study $\mathrm{P}\Omega_{\mathrm{odd} \, n}, \mathrm{P}\mathrm{Sp}_n, \mathrm{P}\Omega_{\mathrm{even} \, n}^-, \mathrm{PSU}_n$ and $\mathrm{P}\Omega_{\mathrm{even} \, n}^-$ and not $B, C, D, {}^2A$ and 2D . Hence the proofs of **Part 1** for the further five types of Classical Groups can and will also eventually be based on our very beautiful **Theorem 2** about the **General Linear Groups**.

Part 2 of **The Second Trilogy** considers (locally) finite and p -soluble groups. It summarises the work by B. Hartley and A. Rae regarding λ_p and p^{2p} (see **Page 38** of [15] and the **References** of [44]) and the foregoing work on the classical Hall-Higman theory regarding λ_p and p^{2p} , c_p, d_p, p^{2p} and r_p by P. Hall, G. Higman, A.H.M. Hoare, T.R. Berger, F. Gross, E.G. Bryukhanova and

last but not least by **A. Turell** [47] as indicated on **Page 8** and **Page 9**. It then proves **Conjecture 3** not only in **English** but partly in **Portuguese** for well-founded historical reasons.

Part 3 of **The Second Trilogy** pays tribute to **Augustin-Louis Cauchy's** and **Évariste Galois's** contributions to Sylow theory in finite groups. It proves in a unified way **Lagrange's theorem** and **Cauchy's concealed second and third group theorems** by exploring and using the following three rectangles a.k.a. tableaux which we show here for the first time though with only minor comments in order to raise inquisitiveness:

complete right transversal for G in H	the first row consists of all elements z_k of G ($1 \leq k \leq M$) acting on H in the following rows via multiplication from the left by their inverses	correspondence	$\text{set}_{H:G} \text{Orbi}(G) := G \setminus H$ of all orbits of H under G acting by left translation
$h_1 := 1 =: z_1$	$z_2 \quad z_3 \quad \dots \quad z_M$	\leftrightarrow	$G = {}_1\text{Orb}(G)$
h_2	$z_2 h_2 \quad z_3 h_2 \quad \dots \quad z_M h_2$	\leftrightarrow	$G h_2 = {}_2\text{Orb}(G)$
h_3	$z_2 h_3 \quad z_3 h_3 \quad \dots \quad z_M h_3$	\leftrightarrow	$G h_3 = {}_3\text{Orb}(G)$
...
h_n	$z_2 h_n \quad z_3 h_n \quad \dots \quad z_M h_n$	\leftrightarrow	$G h_n = {}_n\text{Orb}(G)$

rectangle $|G| \times [H:G]$ of elements

set of certain orbits of H under G acting by left translation	the first row consists of all right cosets Gx_i^k of G in H ($0 \leq k \leq p-1$) with the powers of some p -blank x_1 of G in H ; the following rows consist of right cosets of G in H with the powers of left conjugates of x_1	correspondence	$X = \langle x_i \rangle$: set of all orbits of H under $G \cup X$; the simultaneous actions of G by left translation and of X by right translation
$Gx_1^0 h_1 = G$	$Gx_1 \quad Gx_1^2 \quad \dots \quad Gx_1^{p-1}$	\leftrightarrow	cosets $G \langle x_1 \rangle = GX = \text{double coset } G \uparrow X$
$Gx_2^0 h_2 = G h_2$	$Gx_2 \quad Gx_2^2 \quad \dots \quad Gx_2^{p-1}$	\leftrightarrow	cosets $G \langle x_2 \rangle = G h_2 X = \text{double coset } G h_2 X$
$Gx_3^0 h_3 = G h_3$	$Gx_3 \quad Gx_3^2 \quad \dots \quad Gx_3^{p-1}$	\leftrightarrow	cosets $G \langle x_3 \rangle = G h_3 X = \text{double coset } G h_3 X$
...
$Gx_p^0 h_p = G h_p$	$Gx_p \quad Gx_p^2 \quad \dots \quad Gx_p^{p-1}$	\leftrightarrow	cosets $G \langle x_p \rangle = G h_p X = \text{double coset } G h_p X$

tableau $p \times [H:G]/p$ of cosets

set of certain orbits of H under G acting by left translation	the first row consists of all right cosets Gx_i of G in H ($0 \leq i \leq H -1$) with the elements of some Sylow p -subgroup X of H , all of whose elements of order p are p -blanks of G in H ; the following rows consist of right cosets of G in H with the elements of left conjugates of X	correspondence	$ X = H _p = p^f$: set of all orbits of H under $G \cup X$; the simultaneous actions of G by left translation and of X by right translation
$Gx_{10} h_1 = G$	$Gx_{11} \quad Gx_{12} \quad \dots \quad Gx_{1p^f-1}$	\leftrightarrow	cosets $G[x_i 0 \leq i \leq p^f-1] = GX = \text{double coset } G \uparrow X$
$Gx_{20} h_2 = G h_2$	$Gx_{21} \quad Gx_{22} \quad \dots \quad Gx_{2p^f-1}$	\leftrightarrow	cosets $G[x_i 0 \leq i \leq p^f-1] h_2 = \text{double coset } G h_2 X$
$Gx_{30} h_3 = G h_3$	$Gx_{31} \quad Gx_{32} \quad \dots \quad Gx_{3p^f-1}$	\leftrightarrow	cosets $G[x_i 0 \leq i \leq p^f-1] h_3 = \text{double coset } G h_3 X$
...
$Gx_{p0} h_p = G h_p$	$Gx_{p1} \quad Gx_{p2} \quad \dots \quad Gx_{pp^f-1}$	\leftrightarrow	cosets $G[x_i 0 \leq i \leq p^f-1] h_p = \text{double coset } G h_p X$

rectangle $|H|_p \times [H:G]/|H|_p$ of cosets

Subsequently it first corrects a great misunderstanding of Cauchy's work of 1845/1846 in the quite renowned literature and then presents Cauchy's work of 1812/1815 in the sincere succession of the earlier work of **Joseph-Louis de Lagrange** (Giuseppe Luigi Lagrangia), of **Alexandre-Théophile Vandermonde** and of pioneer **Paolo Ruffini**, as indicated by Cauchy himself, thereby identifying and explaining the crucial parts of Cauchy's first publication of 1812/1815 on group theory.

It then presents what **Évariste Galois** surely knew about **Cauchy's group theorems** and even already about **Sylow's theorems** by referring to his published papers and with utmost care to his posthumously published papers and to his manuscripts.

Afterwards it summarises a large number of papers on **Early group theory and early Sylow theory in finite groups** centred around both Cauchy's and Galois' work and completes this résumé with quite exciting own excavations. It then closes with grateful **Acknowledgements** and a sizeable list of **References** which is and must be chronologically ordered and not by the names of the authors or institutions as usual.

In the following we describe **Part 3** in more detail.


We are planning to revise thoroughly Sylow theory starting with a **really new proof** for **Cauchy's** known as fundamental theorem in group theory (look at [https://en.wikipedia.org/wiki/Cauchy%27s_theorem_\(group_theory\)](https://en.wikipedia.org/wiki/Cauchy%27s_theorem_(group_theory))) based on **beautiful** ideas by **Galois**. In the forthcoming (third) follow-up **Research Article** "Augustin-Louis Cauchy's and Évariste Galois' Contributions to Sylow Theory in Finite Groups" beyond our **First Trilogy** (look at **Page 15**) we first describe and then provide new but classical and rather unified proofs for the very fundamental theorems by **Lagrange** and by **Cauchy** on finite groups being of – in our modest opinion – considerable historical relevance.

We can describe consequences of the **absence of group elements of prime order p** , in spite of their ready availability in overgroups, thereby providing a considerably unified and also heretofore undiscovered approach to the theorems of Lagrange and of Cauchy and their implications for p -groups. Since this approach uses only ideas from a very well-known paper by **Augustin-Louis Cauchy** presented first in 1812 and then published later in 1815, this bears considerable historic relevance. While it is widely acknowledged that Cauchy had **published** his fundamental group theorem not until 1845/1846 and had there based it on double cosets of the finite permutation group and some Sylow p -subgroup of its symmetric overgroup, one could henceforth well argue that he had presented his theorem in a truly concealed way already a good thirty years earlier. **Évariste Galois** knew both Cauchy's paper of 1815 and – based on his own rather perceptive considerations – Cauchy's group theorem and even already Sylow's existence theorem. Cauchy's and Galois' ideas are particularly lucid in the embryonic case of permutation groups of prime degree p (≥ 5) when Sylow p -subgroups of the symmetric overgroup obviously exist. If $G \subseteq H$ with H being finite, then the **unified method of proof** consists in arranging the elements of H in a **rectangle** with $|G|$ columns and $[H:G]$ rows resp. the (right) cosets of G in H in a **rectangle** with p resp. with $|H|_p$ columns and $[H:G]/p$ resp. $[H:G]/|H|_p$ rows to obtain information about $[H:G]$ (see the three rectangles above).

Cauchy's theorem of 1812/1815 is a direct consequence of $[H:\langle x \rangle] \geq |G|$ if x is an element of H of order p with $x \notin G$ which we call a **p -blank of G in H** 😊. We find that Lagrange's theorem and Cauchy's theorem are just like two sides of a coin where "Lagrange" is representing the case $p^0 = 1$ and "Cauchy"

represents the case $p^1 = p$ thereby offering a unified approach to both theorems. Therefore, “Cauchy” is not only a partial converse of “Lagrange” but it is in fact a smart “swapping” of p for 1 as well: $p^0 = 1 \circlearrowleft p = p^1$.

Cauchy depicts 1815 a p -cycle for some prime p as a regular

 p -gon and studies p -cycles in considerable detail.

We present Cauchy’s *classical proof* of Lagrange’s theorem and supplement it with a *beautiful modern proof*. Afterwards we present Cauchy’s *classical proofs* of his **published first theorem**, of his **concealed second theorem** and of his **concealed third theorem**. Subsequently we introduce double cosets and show how they lead to a *modern proof* of Cauchy’s second and third theorems what Cauchy did as well but not until 1845/1846 after very thoroughly reconsidering, sustainably impressed by a research paper of **Joseph Bertrand**, his work of 1812/1815, that is, after – believe it or not – 30 years.

We continue with **first correcting** a great misunderstanding of Cauchy’s work of 1845/1846 in the literature and **then presenting** Cauchy’s work of 1812/1815 in the very sincere succession of the earlier work of **Joseph-Louis de Lagrange** (Giuseppe Luigi Lagrangia), **Alexandre-Théophile Vandermonde** and **Paolo Ruffini**, as indicated by Cauchy himself, and identify, explain and comment the crucial parts of Cauchy’s first publication on group theory. **Finally** we proudly present what **Évariste Galois** knew already about **Cauchy’s group theorems** and about **Sylow’s famous theorems** by referring to his published papers and also to his posthumously published papers. However, this will require quite considerable further (historical) research. We would be inestimably delighted if several group theory researcher would help us with this tedious but very suspenseful work and are ready to cöordinate all the work. We are then closing with fairly comprehensive **Acknowledgements** and a greatly sizeable list of **References**.



Augustin-Louis Cauchy
(21 August 1789 until 23 May 1857)



Évariste Galois
(25 October 1811 until 31 May 1832)

9. The First Trilogy and The Second Trilogy and their reviews

The **First Trilogy** are the papers

- 1a) [Characterising Locally Finite Groups satisfying the Strong Sylow Theorem for the Prime \$p\$ – Part 1 of a Trilogy](#) (see [16]),
- 1b) [Characterising Locally Finite Groups satisfying the Strong Sylow Theorem for the Prime \$p\$ – Part 1 of a Trilogy. Second edition](#) (see [17]),
- 2) [About the Strong Sylow Theorem for the Prime \$p\$ in Simple Locally Finite Groups – Part 2 of a Trilogy](#) (see [18]), and
- 3) [The Strong Sylow Theorem for the Prime \$p\$ in Projective Special Linear Locally Finite Groups – Part 3 of a Trilogy](#) (see [19]),

and **The Second Trilogy** are the papers

- 1) [The Strong Sylow Theorem for the Prime \$p\$ in the Locally Finite Classical Groups](#),
- 2) [The Strong Sylow Theorem for the Prime \$p\$ in Locally Finite and \$p\$ -Soluble Groups](#), and
- 3) [Augustin-Louis Cauchy’s and Évariste Galois’ Contributions to Sylow Theory in Finite Groups](#).


The mathematical subject matter of **The First Trilogy** is described in its review in **Contemporary Mathematics**, Volume 4, Issue 3, pp. 484-487 (see [20]). 1a) and 1b) of the Trilogy were subsequently submitted to **Advances in Group Theory and Applications (AGTA)** and peer reviewed and published there (see [15] and **Appendix 1**) and received a review by **Mathematical Reviews** (see MR4441631) and also a review by **Zentralblatt für Mathematik** (see Zbl 1496.20065). The **Postscript on Page 15** describes briefly the contents of **The First Trilogy**.

The review in **Contemporary Mathematics** was enlarged to a **much more detailed review** in the **Journal of Mathematical & Computer Applications (JMCA)** (see [21]).

The Second Trilogy is not yet published (and even not yet finally developed) and therefore cannot be reviewed, but a review along the pattern of [21] is planned and its contents is already summarised in great detail in **Chapter 8** above. This summary will be the basis of the planned review. It is well-expected that the published papers will receive a review by **Mathematical Reviews** and a review by **Zentralblatt für Mathematik**, at least when being published by **AGTA** or by **Contemporary Mathematics** or by **JMCA** including references to the previous publications.


However, with these two trilogies the development of Sylow theory in (locally) finite groups cannot be finished. In particular, it is a major challenge to determine **all (minimal) p -uniqueness subgroups** for the known finite simple groups and their natural overgroups, the symmetric and the linear groups, and for the (locally) p -soluble groups, distinguishing $p \geq 5$, $p = 3$ and $p = 2$.


Acknowledgments

The author is sincerely very grateful to the known and unknown referees for her/his corrections, suggestions and such friendly adjuvant advice which improved the manuscript quite considerably. He wishes to thank also so very heartfully his truly most fabulous wife **Helga** . Without her tenderest and unconditional support and her love and greatest patience over so many years, this publication would never have been born. Most importantly, he is forever and ever grateful to **Prof. Brian Hartley** and to his teacher **Prof. Otto H. Kegel** (see [15] and [44], p. 25) for their beautiful papers which provide really incredible insights and give marvellous pleasure in reading and understanding the magnificent Sylow theory of both (locally) p -soluble (locally) finite groups and simple (locally) finite groups.

Postscript

The research paper [15] (see MR4441631 and Zbl 1496.20065) has as many “actual” pages as there are “known” sporadic finite simple groups. As the overwhelming majority of group theorists (including the author) believe, these 26 groups are now really all and never in the future further “sporadics” will appear (not counting the Tits group ${}^2F_4(2)$ [as some do] because it did in fact not appear sporadically at the stage). A central question of Sylow theory in locally finite groups is, as pointed out by Prof. Otto H. Kegel (see [44]), how the rank of these altogether seven rank-unbounded families of finite simple groups $\{A^n, A = \text{PSL}_n, B = \text{P}\Omega_{\text{odd } n}, C = \text{PSp}_n, D = \text{P}\Omega_{\text{even } n}, {}^2A = \text{PSU}_n, {}^2D = \text{P}\Omega_{\text{even } n}^-\}$ is bounded, say, *someway* (“in terms of”) by any p -uniqueness subgroup P . More precisely, let us discover a nice function f_p of the order of P or (much more challenging) of the p -uniqueness of each of these (classical) groups G , that bounds the rank: $n \leq f_p(|P|)$ or $n \leq f_p(a_p(G))$. The author answered Kegel’s question in the affirmative already for all the beautiful alternating groups A^n in his Diplomarbeit [14] and he is now publishing the answer as Theorem 1 b): $n \leq f_p(|P|) := (p+2) \cdot |P| \cdot 2^{|P|-1} - 1$. This is, although it is similar, much worse than the result obtained for all the beautiful linear groups $\text{GL}(n, \mathbb{F})$ (see Lemma 2 c) on Page 11). We could optimise our answer if we would come to know $a_p(A^n)$, that is, the minimal p -unique subgroups of the alternating groups. Let us look for them!

In the paper at hand we answered the question as Theorem 4 for the PSL groups $A = \text{PSL}_n$ thereby completing for the time being our (in our modest opinion) beautiful (First) Trilogy – [15] on p -uniqueness subgroups and [this paper] on A^n and $A = \text{PSL}_n$ – about Sylow Theory in Locally Finite Groups which provides a number of good suggestions to stimulate and encourage future research. All of these should become rather very challenging beautiful open problems for the international community of (locally finite) group theory researchers. We are ready to coordinate related research work (see also Page 14).  A detailed overview of the 19 families of “known” finite simple groups is given by the figure “The Periodic Table Of Finite Simple

Groups” (© 2012 by the great Iván Andrus [see <https://irandrus.files.wordpress.com/2012/06/periodic-table-of-groups.pdf> and <https://irandrus.wordpress.com/2012/06/17/the-periodic-table-of-finite-simple-groups/>]) on Page 3 and by the beautiful figure on Page 8 which shows the 19 families of finite simple groups as 13 sporadic groups above 18 infinite families around another “sporadic” group (the Tits group ${}^2F_4(2)$) and 13 sporadic groups below .

Siamo angeli con un’ala soltanto
e possiamo volare solo restando abbracciati.
We are angels who have but a single wing
and we can only fly if we cling to one another.
Wir sind Engel mit nur einem Flügel,
um fliegen zu können müssen wir uns umarmen.
Nous sommes des anges à une seule aile,
nous ne pouvons voler qu’en restant enlacés.
Somos ángeles con una única ala y sólo podemos volar abrazados.
Nós somos anjos com apenas uma asa
e só podemos voar quando nos abraçamos.

Luciano De Crescenzo


(* 18 August 1928 in Naples until ☞ 18 July 2019 in Rome).
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8057092011027 (DVD). ISBN 0-330-30576-X. ISBN 978-3-257-21670-7.
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Long live Group Theory!

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Felix F. Flemisch received his first degree Bacc.Math. in 1974 from the Albert-Ludwigs-Universität in Freiburg in Freiburg, his postgraduate degree M.Sc. in 1975 from the University of London, UK, and finally his degree Dipl.-Math. in Freiburg City, in 1982. Since May 1983 he was working for the software industry. On April 11, 1982, he married his beloved Helga in Florence in Tuscany in Italy. Since October 2019 he is retired and is still happy to spend living to work mathematics, in particular on the very beautiful Group Theory 

Conflict of Interest

The author declares gently that there are no competing personal or organisational or financial conflicts of interest with this original work or other conflicts of interest regarding the publication of this meticulous Research Article.

Pablo Picasso's *La Joie de vivre*



Pablo Picasso – *La Joie de vivre* which shepherds the **Research Article** as a flock along all abysses (see <https://www.pablocassio.org/joie-de-vivre.jsp>)

About the author

Felix F. Flemisch was born on 17 May 1951 in **Munich** in Bavaria in Germany. In **June 1971** he received his **Abitur** ☺ whose subject Mathematics was taught in a pioneering spirit by **Dr. Helmut Bergold**. Afterwards he received his first-ever degree **Baccalaureus der Mathematik (Bacc.Math.)** in **July 1974** with the alas unpublished **beautiful bachelor's thesis** "Über einfache Punkte affiner Varietäten" from the venerable **Albert-Ludwigs-Universität** at **beautiful Freiburg im Breisgau** in **green** Baden-Württemberg in Germany under the such thorough supervision of esteemed **Akadem. Rat Dr. Herbert Götz**, and then his degree **Master of Science (M.Sc.)** from the Faculty of Science of the highly recognised **University of London**, United Kingdom, in **August 1975** at its grand **Bedford College** under the supervision of greatly adored **Prof. Paul Moritz Cohn** (8 January 1924 until 20 April 2006). From October 1975 until – very regrettably – only July 1976 he was employed as **a fairly diligent Teaching Assistant with two graduations** by the hoar **Mathematische Fakultät** of **Freiburg im Breisgau's** **Albert-Ludwigs-Universität**. Subsequently he quite enthusiastically continued his postgraduate mathematical studies in such marvellous and such fabulous **Freiburg i.Br.** – with decent interruptions as **a teacher** and as **a tutor** – and then received his degree **Diplom-Mathematiker (Dipl.-Math.)** in **April 1985** under the impressive supervision of adored **Prof. Otto Helmut Kegel** (20 July 1934 until today). The **Research Paper [15]** publishes the most essential and partly well corrected portions of his German **Diplomarbeit [14]** of **October 1984** and a said scattered "sprinkling" of fairly new considerations and results which truly try to propose coming directions of research for the Sylow theory in (locally) finite groups. The publication at hand continues [15] with theorems about simple locally finite groups "of alternating type" and "of projective special linear type" and makes quite a number of suggestions for future research ☺. From February 1981 until April 1985 the author was enormous happily affiliated to the **Institut für Medizinische Biometrie und Statistik (IMBI)** at lovely **Freiburg im Breisgau** as **a considered Wissenschaftlicher Mitarbeiter**. Since **May 1985** he was based dahoam in **Munich** and devotedly working with greatest joy for the telecom industry first as **an eager System Software Developer**, then as **a fastidious Systems Engineer**, and finally as **a Director for International Standardisation of telecom software and concepts**. On the very

11 April 1992 (see also **Page 2**) he so blissful happily married the most fabulous and wonderful-ever woman **Helga** in **beautiful Florence** in **Tuscany** in **Italy**, which was a memorable marriage



celebrated along with about twenty friends and uniting the most venerable city **Weiden** in **Upper Palatinat** (i.d.OPf.) (**Helga**) with the huge cosmopolitan city **Munich** in **Upper Bavaria** (**Felix**). That was built really for eternity: **Helga** and **Felix** were meant to last forever ♡. Since **October 2016** the author is retired and is still resp. is again loving to work for mathematics, in particular for the **very beautiful Group Theory** 🧮 😊.

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References

1. E. ARTIN: "Geometric Algebra". Interscience Tracts in Pure and Applied Mathematics, Number 3, *Interscience Publishers Inc.*, New York & *Interscience Publishers Ltd.*, London (January 1957). Wiley Classics Library, *John Wiley & Sons Ltd.*, Hoboken, NJ (April 1988). ISBN 978-0-471-60839-4. Dover Books on Mathematics, *Dover Publications Inc.*, Mineola (February 2016). ISBN 978-0-486-80155-1.
2. R. BAER: "Abzählbar erkennbare gruppentheoretische Eigenschaften", *Math. Z.* **79** (1962), 344-363.
3. T.R. BERGER: "Representation theory and solvable groups: Length type problems", in: *The Santa Cruz Conference on Finite Groups, Proc. Symposia Pure Math.* **37**, *Amer. Math. Soc.*, Providence, RI (1980), 431-441. ISBN 978-0-8218-1440-6.
4. J.V. BRAWLEY – G.E. SCHNIBBEN: "Infinite Algebraic Extensions of Finite Fields", *Contemporary Mathematics* **95**, *Amer. Math. Soc.*, Providence, RI (July 1989). ISBN 978-0-8218-5101-2. <https://www.ams.org/books/conm/095/>
5. B. BREWSTER: "Sylow Intersections and Fitting Functors", In memory of Professor Hans Zassenhaus, in: *Group Theory, Proceedings of the Biennial Ohio State–Denison Conference*, 14-16 May 1992, Granville, Ohio, Editors S.K. SEHGAL & R.M. SOLOMON, *World Scientific*, Singapore (December 1993), 62-69. ISBN 978-981-02-1419-7.
6. P.J. CAMERON: "Notes on Classical Groups", M.Sc. course at the University of London, UK, School of Mathematical Sciences, Queen Mary and Westfield College, January to March 2000. https://webpace.maths.qmul.ac.uk/p.j.cameron/class_gps/
7. R. CARTER – P. FONG: "The Sylow 2-subgroups of the finite classical groups", *J. Algebra* **1** (July 1964), 139-151.

8. J.H. CONWAY – R.T. CURTIS – S.P. NORTON – R.A. PARKER – R.A. WILSON: “ATLAS of Finite Groups”, Clarendon Press, Oxford (1985, reprinted 2005 [with corrections], 2006, 2007, 2009, 2013, 2015, 2017, 2018, 2019, 2021). ISBN 978-0-19-853199-9.

9. C.W. CURTIS – I. REINER: “Representation Theory of Finite Groups and Associative Algebras”, *Pure and Applied Mathematics* 11, Wiley-Interscience, New York-London (June 1962). ISBN 0-470-18975-4. Reprinted by the AMS, Chelsea Publishing 356, Amer. Math. Soc., Providence, RI (March 2006). ISBN 0-8218-4066-5.

10. S. DELCROIX – U. MEIERFRANKENFELD: “Locally Finite Simple Groups of 1-Type”, *J. Algebra* 247 (January 2002), 728–746.

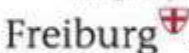
11. J.D. DIXON: “The Structure of Linear Groups”, Mathematical Series 37, Van Nostrand Reinhold, London (September 1971). ISBN 0-442-02149-6.

12. M.R. DIXON: “Sylow Theory, Formations and Fitting Classes in Locally Finite Groups”, *World Scientific*, Singapore (December 1994). ISBN 978-981-02-1795-2.

13. M.R. DIXON – L.A. KURDACHENKO – I. Ya SUBBOTIN: “Ranks of Groups: The Tools, Characteristics, and Restrictions”. *John Wiley & Sons Ltd.*, Hoboken, NJ (September 2017). ISBN 978-1-119-08027-5.

14. F.F. FLEMISCH: “Lokal endliche Gruppen mit Sylow p -Satz oder mit $\min-p$. I: Grundbegriffe, ein Charakterisierungssatz und lokale Prinzipien”, *Diplomarbeit an der verdient ehrenwerten Mathematischen Fakultät der Albert-Ludwigs-Universität zu*

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Deutschland



. Abgabetag: 8.10.1984.



15 May 1939 – 8 October 1994

Submitted on October 8, 1984, exactly ten years before the very tragic death of Brian Hartley – whose splendid contributions to locally finite group theory ([21] and many about locally finite and p -soluble groups [see [15]]) the author had studied in great detail and with the deepest admiration and affection – while mountain hiking. Brian Hartley was very well known to be such a keen and so passionate hill walker, and it happened freshishly while descending from rather steep Helvellyn (see <https://en.wikipedia.org/wiki/Helvellyn>) in the known as beautiful English Lake District's fells (see https://en.wikipedia.org/wiki/Lake_District), gay nearby his homeland, on October 8, 1994, that he collapsed with a grim heart attack and died (lack of any help) very very tragically (see https://en.wikipedia.org/wiki/Brian_Hartley) and the references cited there, in particular MacTutor [see <https://mathshistory.st-andrews.ac.uk/Bioographies/Hartley/>]). It is only with the heart that one can see rightly. What is essential is invisible to the eyes. On ne voit bien qu'avec le cœur. L'essentiel est invisible pour les yeux. Gut sehen kann man nur mit dem Herzen. Worauf es wirklich ankommt, das sehen die Augen nicht. Antoine de Saint-Exupéry (29 June 1900 until 31 July 1944). *Le Petit Prince* (April 6, 1945) (see https://en.wikipedia.org/wiki/The_Little_Prince).



15. F.F. FLEMISCH: “Characterising Locally Finite Groups Satisfying the Strong Sylow Theorem for the Prime p ”, *Adv. Group Theory Appl.* 13 (June 2022), 13–39 (see MR4441631 and Zbl 1496.20065). <https://www.advgrouptheory.com/journal/index.php#vol13>. We have included this such very beautiful research paper as **Appendix 1** for very good reasons 😊.

16. F.F. FLEMISCH: “Characterising Locally Finite groups satisfying the Strong Sylow theorem for the Prime p – Part 1 of a Trilogy”, Norderstedt, Germany: Books on Demand (March 2023). ISBN 978-3-7543-6087-3. <https://buchshop.bod.de/characterising-locally-finite-groups-satisfying-the-strong-sylow-theorem-for-the-prime-p-part-1-of-a-trilogy-dipl-math-felix-flemisch-9783754360873>

17. F.F. FLEMISCH: “Characterising Locally Finite groups satisfying the Strong Sylow theorem for the Prime p – Part 1 of a Trilogy. Second edition”, Norderstedt, Germany: Books on Demand (November 2023). ISBN 978-3-7568-0801-4. <https://buchshop.bod.de/characterising-locally-finite-groups-satisfying-the-strong-sylow-theorem-for-the-prime-p-part-1-of-a-trilogy-felix-f-flemisch-9783756808014>

18. F.F. FLEMISCH: “About the Strong Sylow theorem for the Prime p in Simple Locally Finite Groups – Part 2 of a Trilogy”, Norderstedt, Germany: Books on Demand (November 2023). ISBN 978-3-7543-3642-7. <https://buchshop.bod.de/about-the-strong-sylow-p-theorem-in-simple-locally-finite-groups-part-2-of-a-trilogy-dipl-math-felix-f-flemisch-9783754336427>

19. F.F. FLEMISCH: “The Strong Sylow theorem for the Prime p in Projective Special Linear Locally Finite Groups – Part 3 of a Trilogy”, Norderstedt, Germany: Books on Demand (April 2023). ISBN 978-3-7568-9853-4. <https://buchshop.bod.de/the-strong-sylow-theorem-for-the-prime-p-in-projective-special-linear-locally-finite-groups-part-3-of-a-trilogy-dipl-math-felix-f-flemi-9783756898534>

20. F.F. FLEMISCH: “Review of a Trilogy about Sylow Theory in Locally Finite Groups”, *Contemporary Mathematics*, Volume 4, Issue 3 (April 2023), 484–487. <https://ojs.wiserpub.com/index.php/CM/article/view/2669> and <https://ojs.wiserpub.com/index.php/CM/issue/view/cm.v4i32023.379-619>

21. F.F. FLEMISCH: “Three Beautiful Books about Sylow Theory in Locally Finite Groups”, *Journal of Mathematical & Computer Applications*, Volume 2, Issue 3 (September 2023), 1–3. <https://www.onlinescientificresearch.com/articles/three-beautiful-books-about-sylow-theory-in-locally-finite-groups.html> and <https://www.onlinescientificresearch.com/journal-of-mathematical-computer-applications-old-articles.php?journal=jmca&&v=2&&i=3&&y=2023&&m=September>

22. D.E. GORENSTEIN: “Finite Groups”, *Harper and Row*, New York (November 1968). ISBN 0-06-042413-3. Second edition, *Chelsea Publishing*, New York (June 1980). ISBN 0-8284-0301-5. Second edition reprinted by the AMS, *Chelsea Publishing* 301, *Amer. Math. Soc.*, Providence, RI (July 2007). ISBN 0-8218-4342-7.

23. D.E. GORENSTEIN – R.N. LYONS – R.M. SOLOMON: “The Classification of the Finite Simple Groups, Number 1”. *Mathematical Surveys and Monographs*, Volume 40.1. *Amer. Math. Soc.*, Providence, RI (1994). Reprinted with corrections, 2000 (see also [61.2] and [61.7]). ISBN 978-0-8218-0334-9 (Hardcover). ISBN 978-0-8218-0960-0 (Softcover). <https://www.ams.org/journals/bull/1979-01-01/S0273-0979-1979-14551-8/S0273-0979-1979-14551-8.pdf>

24. L.C. GROVE: “Classical Groups and Geometric Algebra”, *Graduate Studies in Mathematics* 39, *Amer. Math. Soc.*, Providence, RI (October 2001). ISBN 978-0-8218-2019-3. <https://www.ams.org/books/gsm/039/> and <https://www.amazon.de/Classical-Geometric-Algebra-Graduate-Mathematics/dp/0821820192/>

25. J.I. HALL: “Periodic simple groups of finitary linear transformations”, *Ann. of Math.* (2) 163 (2006), no. 2, 445–498.

26. P. HALL: "Some constructions for locally finite groups", *J. London Math. Soc.* **34** (1959), 305-319. Philip Hall (11 April 1904 [see [Page 16](#)] until 30 December 1982)
27. B. HARTLEY: "Simple locally finite groups", in: *Finite and Locally Finite Groups*, Wolters Kluwer & Springer Science + Business Media, Dordrecht (1995), 1-44. ISBN 978-94-010-4145-4.
28. G. HIGMAN: "Enumerating p -groups. I: Inequalities", *Proc. London Math. Soc.* (3) **10** (1960), 24-30.
29. O. HÖLDER: "Die Gruppen der Ordnungen p^2 , pq^2 , pqr , p^4 ", *Math. Ann.* **43** (1893), 301-412.
30. B. HUPPERT – N. BLACKBURN: "Finite Groups II", *Grundlehren der math. Wissenschaften in Einzeldarstellungen* **242**, Springer, Berlin-Heidelberg (March 1982). ISBN 978-3-642-67994-0. <https://link.springer.com/book/10.1007/978-3-642-67994-0>
31. G. JANUSH – J. ROTMAN: "Outer Automorphisms of S^6 ", *Amer. Math. Monthly* **89** (1982), 407-410.
32. LÉO KALOUNINE: "Sur les p -groupes de Sylow du groupe symétrique du degré p^m ", *C.R. Acad. Sci. Paris* **221** (20 Août 1945), 222-224. Errata. *C.R. Acad. Sci. Paris* **223** (13 Novembre 1946), 829.
33. LÉO KALOUNINE: "La structure du p -groupe de Sylow du groupe symétrique du degré p^2 ", *C.R. Acad. Sci. Paris* **222** (17 Juin 1946), 1424-1425.
34. LÉO KALOUNINE: "Sur les p -groupes de Sylow du groupe symétrique du degré p^m . (Suite centrale ascendante et descendante.)", *C.R. Acad. Sci. Paris* **223** (4 Novembre 1946), 703-705.
35. LÉO KALOUNINE: "Sur les p -groupes de Sylow du groupe symétrique du degré p^m . (Sous-groupes caractéristiques, sous-groupes parallélotopiques.)", *C.R. Acad. Sci. Paris* **224** (27 Janvier 1947), 253-255.
36. LÉO KALOUNINE: "Sur le groupe P_∞ des tableaux infinis", *C.R. Acad. Sci. Paris* **224** (14 Avril 1947), 1097-1099.
37. LÉO KALOUNINE: "La structure des p -groupes de Sylow des groupes symétriques finis", *Ann. Sci. de l'École Norm. Sup.* (3) **LXV** (Année 1948), Fasc. 3, 239-276.
38. LÉO KALOUNINE: "Sur la structure des p -groupes de Sylow des groupes symétriques finis et de quelques généralisations infinies de ces groupes", *Séminaire N. Bourbaki*, Vol. 1, 1948-1951, Exp. 5 (Décembre 1948), 29-31. <https://eudml.org/doc/109417>
39. Л.А. Калужнин (L.A. KALUŽNIN): "Об Одном Обобщении Силловских p -Подгрупп Симметрических Групп (Über eine Verallgemeinerung der p -Sylowgruppen symmetrischer Gruppen)", 197-219 (russisch mit deutschem Resumé, 220-221), *Acta Math. Acad. Sci. Hungar.* **2** (September 1951).
40. MARC KRASNER – LÉO KALOUNINE: "Produit complet des groupes de permutations et problème d'extension de groupes", *Acta Scientiarum Mathematicarum*, University of Szeged, Bolyai Institute, Szeged. Reçu le 20 Janvier 1949. I, Vol. 13, Numbers 3-4 (1949-50), 208-230. II, Vol. 14, Numbers 1-1 (1951-52), 39-66. III, Vol. 14, Numbers 2-2 (1951-52), 69-82. <http://pub.acta.hu/acta/showCustomerVolume.action?noDataSet=true>
41. O.H. KEGEL (20 July 1934 until ): "Über einfache, lokal endliche Gruppen", *Math. Z.* **95** (1967), 169-195.
42. O.H. KEGEL: "Lectures on Locally Finite Groups". Notes prepared by M.D. ATKINSON and Susan McKAY, iv+79 pages. Mathematical Institute Oxford, UK, Hilary Term (June, 1969).
43. O.H. KEGEL – B.A.F. WEHRFRITZ: "Locally Finite Groups", North-Holland Mathematical Library, Volume 3, North-Holland Publishing Company [Ltd., Inc.], Amsterdam & London & New York (1973). ISBN 0-7204-2454-2. 
44. O.H. KEGEL: "Four lectures on Sylow theory in locally finite groups", in: *Group Theory*, edited by K.N. CHENG and Y.K. LEONG, Walter de Gruyter, Berlin & New York (January 1989, reprinted November 2016), 3-27 (see [MR0981832](#) [MR 90c:20037 by Brian Hartley (March 1990)]; [Zbl 0659.20024](#) [by tough Bernhard Amberg]). ISBN 978-3-11-011366-2. ISBN 978-0-89925-406-7. <https://www.degruyter.com/view/book/9783110848397/10.1515/9783110848397-004.xml>
45. O.H. KEGEL: "Remarks on uncountable simple groups", in: *Proceedings of Ischia Group Theory 2016* (see <https://www.dipmat2.unisa.it/ischiagrouptheory/IGT2016/home2016.html> and https://www.dipmat2.unisa.it/ischiagrouptheory/IGT2016/abstracts_2016.pdf [p. 13]), *Int. J. Group Theory* **7** (2018) (see https://ijgt.ui.ac.ir/issue_4091_4092.html & [issue_4091_4093.html](https://ijgt.ui.ac.ir/issue_4091_4093.html) & [issue_4091_4094.html](https://ijgt.ui.ac.ir/issue_4091_4094.html))
46. R.D. KOPPERMAN – A.R.D. MATHIAS: "Some Problems in Group Theory", in: *The syntax and semantics of infinitary languages* (ed. Jon Barwise), *Lecture Notes in Mathematics* **72**, 131-138. Springer, Berlin-Heidelberg (November 14, 1968). ISBN 978-3-540-04242-6.
47. A.I. MALCEV: "On isomorphic matrix representations of infinite groups (Об изоморфном представлении бесконечных групп матрицами)", *Mat. Sbornik* **8** (1940), 405-422 = "On the faithful representation of infinite groups by matrices", *Amer. Math. Soc. Transl.* (2) **45** (1965), 1-18. <https://www.ams.org/books/trans2/045/>
48. U. MEIERFRANKENFELD: "Non-finitary locally finite simple groups", in: *Finite and Locally Finite Groups*, Wolters Kluwer & Springer Science + Business Media, Dordrecht (1995), 189-212. ISBN 978-94-010-4145-4.
49. U. MEIERFRANKENFELD: "Locally finite, simple groups", *Class Notes*, March 30, 2011 (see <https://users.math.msu.edu/users/meierfra/Classnotes/classnotes.html> and also <https://users.math.msu.edu/users/meierfra/Classnotes/LFG/LFG.pdf>), Department of Mathematics, Michigan State University, East Lansing, MI 48824, U.S.A.  MICHIGAN STATE UNIVERSITY (see <https://math.msu.edu/> and <https://math.msu.edu/People/directory-profile.aspx?personId=101558>).
50. P.M. NEUMANN – G.A. STOY – E.C. THOMPSON: "Groups and Geometry", *Oxford University Press*, Oxford OX2 6DP (February 1994, reprinted 2007). ISBN 978-0-19-853451-8.
51. A. SACHTJE: "Automorphismen und Untergruppen von S^n und A^n ", *Seminar zur Gruppentheorie im WS 2015/16* von Dr. David Dursthoff am Lehrstuhl für Algebra und Zahlentheorie von Prof. Gabriele Nebe an der RWTH Aachen (siehe <https://www.math.rwth-aachen.de/homes/David.Dursthoff/> und <https://www.math.rwth-aachen.de/homes/David.Dursthoff/SemWilson/sem.html>), erster Vortrag am 5.10.2015 (17 Seiten) (siehe <https://www.math.rwth-aachen.de/homes/David.Dursthoff/SemWilson/Vortrag1.pdf>)

52. D.E. TAYLOR: "The Geometry of the Classical Groups", Sigma Series in Pure Mathematics, Volume 9, *Heldermann Verlag*, Berlin (December 1992). ISBN 978-3-88538-009-2. <https://www.heldermann.de/SSPM/SSPM09/sspm09.htm> and <https://www.amazon.de/Geometry-Classical-Groups-Sigma-Mathematics/dp/388538009/>

53. A. TURELL: "Character theory and length type problems", in: *Finite and Locally Finite Groups*, *Wolters Kluwer & Springer Science + Business Media*, Dordrecht (1995), 377-400. ISBN 978-94-010-4145-4.

54. B.A.F. WEHRFRITZ: "Sylow theorems for periodic linear groups", *Proc. London Math. Soc.* (3) 18 (1968), 125-140.

55. A.J. WEIR: "The Sylow Subgroups of the Symmetric and of the Classical Groups", Ph.D. Thesis 2384, Cambridge, Jesus College 1953, vi+147 pages. Advisor P. HALL. Index to Theses Accepted for Higher Degrees in the Universities of Great Britain and Ireland, Volume IV (1953-54), *Aslib*, London (June 1957), p. 42, N^o 771.

56. A.J. WEIR: "Sylow p -Subgroups of the General Linear Group over Finite Fields of Characteristic p ", *Proc. AMS*, Volume 6, Number 3 (June 1955), 454-464.


57. A.J. WEIR: "Sylow p -Subgroups of the Classical Groups over Finite Fields with Characteristic Prime to p ", *Proc. AMS*, Volume 6, Number 4 (August 1955), 529-533.

58. A.J. WEIR: "The Sylow Subgroups of the Symmetric Groups", *Proc. AMS*, Volume 6, Number 4 (August 1955), 534-541.

59. L. WEISNER: "On the Sylow subgroups of the symmetric and alternating groups", *Amer. J. Math.* 47 (1925), 121-124.

60. H. WEYL: "The Classical Groups – Their Invariants and Representations", Princeton Landmarks in Mathematics and Physics series, Copyright 1939 and 1946 by *Princeton University Press*. Second edition, with supplement, published 1953. Reprint Edition (Fifteenth printing, November 1997). ISBN 978-0-691-05756-9.

61. WIKIPEDIA: "Classification of finite simple groups". https://en.wikipedia.org/wiki/Classification_of_finite_simple_groups. This page was last edited on 3 January 2025, at 20:20 (UTC).

 **WIKIPEDIA** The Free Encyclopedia In the following we cite, referring to [61.xy], a number of all sorts of very interesting articles which are related to the Classification of Finite Simple Groups (CFSG).

61.1 **Aschbacher, Michael G.:** *The Status of the Classification of the Finite Simple Groups*. Notices of the AMS, Volume 51, Number 7 (August, 2004), 736-740. **MR2072045** (in the online edition no review but MSC and References; in the print edition of 2004 no proper entry but entries in Author index with MSC and in Subject index under 20D05 but each without reference of type 2004x:20abc); **Zbl 1113.20302** (no review but two lines Summary and MSC). <https://www.ams.org/notices/200407/fea-aschbacher.pdf>

61.2 **M.G. Aschbacher – R.N. Lyons – S.D. Smith – R.M. Solomon:** *The classification of Finite Simple Groups. Groups of Characteristic 2 Type*. Mathematical Surveys and Monographs 172. *Amer. Math. Soc.*, Providence, RI (March 9, 2011) (see also [61.6]). **MR2778190**; **Zbl 1218.20007** (by Anatoli Kondrat'ev)

61.3 **Elwes, Richard:** *An enormous theorem: the classification of finite simple groups*. Plus Magazine, Issue 41 (December, 2006). <https://plus.maths.org/content/os/issue41/features/elwes/index>

61.4 **Solomon, Ronald M.:** *A brief history of the classification of the finite simple groups*. *Bull. Amer. Math. Soc.*, New Ser. 38, No. 3 (July, 2001), 315-352. **MR1824893** (**MR 2002k:20002** by Gernot Stroth [November, 2002]); **Zbl 0983.20001** (by Ulrich Dempwolff). <https://www.ams.org/journals/bull/2001-38-03/S0273-0979-01-00909-0/S0273-0979-01-00909-0.pdf>

61.5 **Guralnick, Robert M.:** *Commentary on "A brief history of the classification of the finite simple groups" by Ronald Solomon*. *Bull. Amer. Math. Soc.*, New Ser. 55, No. 4 (October, 2018), 451-452. **MR3854073** (inadequate Summary only [eMRS 1F January, 2019, p. 8]); **Zbl 1395.20009** (no review but MSC and References). <https://www.ams.org/journals/bull/2018-55-04/S0273-0979-2018-01638-8/S0273-0979-2018-01638-8.pdf>

61.6 **Scientific American** (July 1, 2015): *Researchers Race to Rescue the Enormous Theorem before Its Giant Proof Vanishes*. Before they die, aging mathematicians are racing to save the Enormous Theorem's proof, all 15,000 pages of it, which divides existence four ways. <https://www.scientificamerican.com/article/researchers-race-to-rescue-the-enormous-theorem-before-its-giant-proof-vanishes/> and <https://www.spektrum.de/magazin/die-rettung-des-riesentheorems/1378756> (February 24, 2016). Written by **Stephen Ornes** (see <https://stephenornes.com> and <https://stephenornes.com/?p=791> [The Whole Universe Catalog. October 24, 2015]).

61.7 **S.D. Smith:** *Applying the Classification of Finite Simple Groups: A User's Guide*. Mathematical Surveys and Monographs 230. *Amer. Math. Soc.*, Providence, RI (March 30, 2018). **MR 3753581** (by V.D. Mazurov [eMRS 1F October, 2018, pp. 5-6]); **Zbl 1415.20004** (by Robert Wilson)

61.8 **Solomon, Ronald M.:** *Afterword to the article "A brief history of the classification of the finite simple groups"*. *Bull. Amer. Math. Soc.*, New Ser. 55, No. 4 (October, 2018), 453-457. **MR3854074** (insufficient Summary only [eMRS 1F January, 2019, p. 9]); **Zbl 1395.20012** (no review but MSC and References). <https://www.ams.org/journals/bull/2018-55-04/S0273-0979-2018-01639-X/S0273-0979-2018-01639-X.pdf>

61.9 **Solomon, Ronald M.:** *The Classification of Finite Simple Groups: A Progress Report*. Notices of the AMS, Volume 65, Number 6 (June/July, 2018), 646-651. **MR3792856** (by Jürgen Müller [eMRS 1F June, 2019, p. 25]); **Zbl 1398.20001** (no review but MSC). <https://www.ams.org/journals/notices/201806/rnoti-p646.pdf>




61.10 **Steingart, Alma:** *A group theory of group theory: Collaborative mathematics and the 'uninvention' of a 1000-page proof*. *Social Studies of Science*, Volume 42, Issue 2 (April, 2012 [February 23, 2012]), 185-213. <https://journals.sagepub.com/toc/sss/42/2>


61.11 **Wolffe, Julia:** *Michael Aschbacher and the sociology of mathematical proof w.r.t. the classification theorem for finite simple groups.* Julia Wolffe's Notes (January 28, 2020). <https://juliawolffnotes.home.blog/2020/01/28/michael-aschbacher-and-the-sociology-of-mathematical-proof-w-r-t-the-classification-theorem-for-finite-simple-groups/>





61.12 **Aschbacher, Michael G.:** *Daniel Gorenstein (1923-1992).* Notices of the AMS, Volume 39, Number 10 (December, 1992), 1190-1191. [Danny passed away four months ago.] MR1193435 (MR 93m:01053 no review but a two-lines Hint [December, 1993]); Zbl 0821.01027 (no review but MSC). <https://www.ams.org/journals/notices/199212/199212FullIssue.pdf>

61.13 **Aschbacher, Michael G.:** *Daniel Gorenstein, 1923-1992 – A Biographical Memoir by Michael Aschbacher.* Biographical Memoirs by the National Academy of Sciences (NAS) (March, 2016), 1-17. [Danny passed away 23½ years ago.] <https://authors.library.caltech.edu/records/1dh85-bt711> and <https://authors.library.caltech.edu/records/1dh85-bt711/files/gorenstein-daniel.pdf> and as well <https://www.nasonline.org/publications/biographical-memoirs/memoir-pdfs/gorenstein-daniel.pdf>

61.14 Here are further interesting hyperlinks regarding CFSG: <https://math.mit.edu/research/highschool/primes/circle/documents/2022/Gracie.pdf>; <https://mathworld.wolfram.com/ClassificationTheoremofFiniteGroups.html>; <https://e.math.cornell.edu/people/mam/classes/chicago/Classification.pdf>; https://encyclopediaofmath.org/wiki/Simple_finite_group; https://mathshistory.st-andrews.ac.uk/Extras/Simple_groups_classification/

MR – AMS Mathematical Reviews  considered together with **MathSciNet**  and with **MR Lookup**  (see <https://www.ams.org/mr-database> and <https://mathscinet.ams.org/mathscinet> and <https://mathscinet.ams.org/mrlookup>)


Zbl – Zentralblatt MATH  (see <https://www.zbmath.org/>)

Note – The MR number in brackets refers to the print edition of **Mathematical Reviews** , which was printed very regrettably only until 2012 , and includes the reviewer and the month of publication. Since 2013 references are to the online edition of **MathSciNet**  and the **electronic Mathematical Reviews**  (**eMR**) **Sections** (see <https://www.ams.org/publications/mrsections/mrsections>).

Appendix 1

Reference [15] with MR Review and Zbl Review

15. F.F. FLEMISCH: “Characterising Locally Finite Groups Satisfying the Strong Sylow Theorem for the Prime p ”, *Adv. Group Theory Appl.* 13 (June 2022), 13-39 (see MR4441631 and Zbl 1496.20065). <https://www.advgrouptheory.com/journal/index.php#vol13> and <https://www.advgrouptheory.com/journal/Volumes/13/Flemisch.pdf>

While the MR Review is very disgracefully simply stating only the main result and is telling nothing at all about the *new ideas*, the Zbl Review states at least the Abstract as a Summary and all References but also very regrettably states nothing about the *beautiful new ideas* .

For example, we are rightly a little very proud of two discoveries: 1) **Theorem 3.6** on **Page 28** of [15], which found a symmetry between non-conjugated Sylow p -subgroups, and then also 2) that the **minimal** members of the set **Unique_p \bar{U}** from **Page 35** of [15] should play for a finite U a very similar important rôle as its **maximal** members which are the Sylow p -subgroups. It then becomes a challenge to determine the **minimal** members for sufficiently “known” (locally) finite groups, in particular for all the “known” finite simple groups and the finite p -soluble groups, and their core properties, in particular conjugacy and minimal w.r.t. order vs. minimal w.r.t. inclusion. These are mathematical ideas which propose exciting new directions for (timeless and eternal) Sylow theory in (locally) finite groups during the coming years where we intend to join in, to support, to coordinate and to try to shape. They could not have been included in The First Trilogy and are as well because of their complexity not scheduled to become part of The Second Trilogy. Hence, they will be fascinating topics of very hopefully joint research for the time after publication of The Second Trilogy.



The MR Review is available at MR Lookup under <https://mathscinet.ams.org/mathscinet/relay-station?mr=4441631> and in detail on **Page 16** of the **eMR Section 1F for January 2023** at <https://www.ams.org/mrlisting/2023/1F/2023-1F-01.pdf>.

The Zbl Review is available at Zentralblatt MATH under <https://zbmath.org/1496.20065> and its PDF at <https://zbmath.org/pdf/07554056.pdf>.

- For the complete **Appendix 1**, having 33 pages, see **Page 21** to **Page 53**.

Appendix 2

Introduction to the Talk by Felix F. Flemisch at IGT 2024 on April 11th, the 120th birthday of Philip Hall

My name is **Felix Flemisch**. I come from **Munich** in Bavaria in Germany. In the 1970ties and 1980ties I was a considerably busy and faithful student of **Prof. Otto H. Kegel**  in such **beautiful** Freiburg i.Br. in Germany. In 2021 I luckily came again in contact with my adored teacher and met him in person and in good shape during June and July of 2022 in Freiburg. I present at IGT 2024 a **POSTER** about a new paper on **Sylow theory in simple locally finite groups** which is based on the famous **Kegel covers** and a **beautiful** paper of mine about rounding off the **general Sylow theory in locally finite groups**, friendly published by AGTA, under the rigid supervision of esteemed **Prof. Francesco de Giovanni** . **Prof. Kegel** gave me kindly the hint to submit the paper to AGTA whose review process improved the paper substantially so that it now can be the basis for further work on the subject.

Both papers have a quite strong relationship to **Prof. Kegel's work on Sylow theory**, each one proving a conjecture of him and centred around the gay concept of a **p -uniqueness subgroup** which is a finite p -subgroup being friendly contained in such a unique Sylow p -subgroup. The **POSTER** shows the **twelve slides** of my talk as a PowerPoint presentation which include as well tough suggestions to stimulate and encourage future research. I much hope to enthuse group theorists with them and I am ready to coordinate related research work. This is my main interest why I present the **POSTER**. However, I am sadly aware that locally finite groups, and their Sylow theory in particular, seem not (yet) to be current topics of group theory research except some special questions presented on Tuesday. A limited number of nicely printed copies of the paper's **abstract**, its **POSTER** in DIN A3, and its **preprint** are available. I will deposit them tomorrow morning in SALA CARTAROMANA. An underlying **research paper** to this Talk will be published.

- For the complete **Appendix 2**, having 18 pages and including the **beautiful twelve slides** of the presentation, some **beautiful** photographs of Freiburg i.Br., two **beautiful** photographs of **Prof. Otto H. Kegel** and four photographs of the wonderfully **beautiful** Lake Ammersee in Bavaria, see **Page 54** to **Page 71**.


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Appendix 1

Reference [15] with MR Review and Zbl Review

15. F.F. FLEMISCH: “Characterising Locally Finite Groups Satisfying the Strong Sylow Theorem for the Prime p ”, *Adv. Group Theory Appl.* **13** (June 2022), 13-39 (see MR4441631 and Zbl 1496.20065).

<https://www.advgrouptheory.com/journal/index.php#vol13> and
<https://www.advgrouptheory.com/journal/Volumes/13/Flemisch.pdf>

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For example, we are rightly a little very proud of two discoveries: 1) **Theorem 3.6** on **Page 28** of [15], which found a symmetry between non-conjugated Sylow p -subgroups, and then also 2) that the **minimal** members of the set **Unique_pU** from **Page 35** of [15] should play for a finite U a very similar important rôle as its **maximal** members which are the Sylow p -subgroups. It then becomes a challenge to determine the **minimal** members for sufficiently “known” (locally) finite groups, in particular for all the “known” finite simple groups and the finite p -soluble groups, and their core properties, in particular conjugacy and minimal w.r.t. order vs. minimal w.r.t. inclusion. These are mathematical ideas which propose exciting new directions for (timeless and eternal) Sylow theory in (locally) finite groups during the coming years where we intend to join in, to support, to coordinate and to try to shape. They could not have been included in The First Trilogy and are as well because of their complexity not scheduled to become part of The Second Trilogy. Hence, they will be fascinating topics of very hopefully joint research for the time after publication of The Second Trilogy.

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MR4441631 20D20 20D15 20F50

Flemisch, Felix F.

Characterising locally finite groups satisfying the strong Sylow theorem for the prime p . (English summary)

Adv. Group Theory Appl. **13** (2022), 13-39.

Let p be a prime and let G be a locally finite group. Then, G is said to satisfy the *Sylow Theorem for the prime p* if all maximal p -subgroups of G are conjugate. The group G is said to satisfy the *strong Sylow Theorem for the prime p* if every subgroup of G satisfies the Sylow Theorem for the prime p . Further, a finite p -subgroup P of G is said to be *singular* in G if for every finite subgroup F of G containing P there is a unique Sylow p -subgroup of F containing P . In this paper, it is shown that G satisfies the strong Sylow Theorem for the prime p if and only if every subgroup S of G contains a finite p -subgroup which is singular in S . This answers a question posed by Otto H. Kegel in 1987. The paper is based on the author's thesis from the year 1984 [*Lokal endliche Gruppen mit Sylow p -Satz oder mit min- p . I: Grundbegriffe, ein Charakterisierungssatz und lokale Prinzipien*, Diplomarbeit, Univ. Freiburg, 1984; per bibliography]. Stefan Kohl

(see Theorem 2).

Marco Trombetti

MR4482310 20D15 08A35

Ghumashyan, Heghine (AR-EU; Yerevan);
Guričan, Jaroslav (SK-KMSK-NDM; Bratislava)

Endomorphism kernel property for finite groups. (English summary)

Math. Bohem. **147** (2022), no. 3, 347–358.

Summary: "A group G has the endomorphism kernel property (EKP) if every congruence relation θ on G is the kernel of an endomorphism on G . In this note we show that all finite abelian groups have EKP and we show infinite series of finite non-abelian groups which have EKP."

MR4440439 20D15 20J99

Kalteh, O. (IR-IAUMS-M; Mashhad); Jafari, S. Hadi (IR-IAUMS-M; Mashhad)

Capable groups of order p^3q . (English summary)

Algebra Discrete Math. **33** (2022), no. 1, 104–115.

A group G is called capable if there exists a group E such that $G \cong E/Z(E)$. The epicenter $Z^*(G)$ of G is the smallest central subgroup such that $G/Z^*(G)$ is capable. Obviously, G is capable if and only if $Z^*(G) = 1$.

This paper studies the capability of groups of order p^3q , where p and q are distinct prime numbers and $p > 2$. More specifically, by calculating the non-abelian exterior square $G \wedge G$, the authors determine the epicenter for groups of order p^3q in Theorem 2. As a corollary, they identify the capability of groups of order p^3q . Junqiang Zhang

MR4441631 20D20 20D15 20F50

Flemisch, Felix F.

Characterising locally finite groups satisfying the strong Sylow theorem for the prime p . (English summary)

Adv. Group Theory Appl. **13** (2022), 13–39.

Let p be a prime and let G be a locally finite group. Then, G is said to satisfy the *Sylow Theorem for the prime p* if all maximal p -subgroups of G are conjugate. The group G is said to satisfy the *strong Sylow Theorem for the prime p* if every subgroup of G satisfies the Sylow Theorem for the prime p . Further, a finite p -subgroup P of G is said to be *singular* in G if for every finite subgroup F of G containing P there is a unique Sylow p -subgroup of F containing P . In this paper, it is shown that G satisfies the strong Sylow Theorem for the prime p if and only if every subgroup S of G contains a finite p -subgroup which is singular in S . This answers a question posed by Otto H. Kegel in 1987. The paper is based on the author's thesis from the year 1984 [*Lokal endliche Gruppen mit Sylow p -Satz oder mit \min - p . I: Grundbegriffe, ein Charakterisierungssatz und lokale Prinzipien*, Diplomarbeit, Univ. Freiburg, 1984; per bibliography]. Stefan Kohl

MR4346240 20D25 20D15 20D20

Allcock, Daniel (1-TX; Austin, TX)

Variations on Glauberman's ZJ theorem. (English summary)

Int. J. Group Theory **11** (2022), no. 2, 43–52.

It is well known that the classical ZJ theorem by G. Glauberman has been proved in various versions, depending on the various possible definitions of the Thompson subgroup. In this paper the author presents an "axiomatic" version of the ZJ theorem, and proposes new choices for the family of abelian subgroups of the Sylow p -subgroup S of the finite group G that can generate a sort of generalized Thompson subgroup for which a ZJ-type theorem holds. Furthermore, I believe that the paper could be very

Flemisch, F. F.

Characterising locally finite groups satisfying the strong Sylow theorem for the prime p . (English) Zbl 1496.20065

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Summary:

During his lectures to the 1987 Singapore Group Theory Conference [10] Otto H. Kegel proposed the following question: "If every subgroup S of the locally finite group G contains a finite p -subgroup which is singular in S , does G then satisfy the strong Sylow Theorem for the prime p ?" In this paper we answer the question in the affirmative. The paper formed an essential part of the author's German Diplomarbeit of 1984 (the "Charakterisierungssatz") written before he left academia [4]. We present the *Charakterisierungssatz* as Theorem 3.9, and summarise then the result as Theorem 3.10, stating that if G is a locally finite group and p is a prime, then G satisfies the strong Sylow theorem for the prime p if and only if every subgroup S of G contains a finite p -subgroup which is singular in S . Subsequently we present a few novel concepts for Sylow theory in (locally) finite groups to encourage future research. The paper is divided in four sections: Introduction; Good Sylow p -subgroups and p -uniqueness subgroups; Basic theorems of Sylow theory in locally finite groups and our *Charakterisierungssatz*; Novel concepts for Sylow theory in (locally) finite groups.

MSC:

- 20D20 Sylow subgroups, Sylow properties, π -groups, π -structure
- 20F50 Periodic groups, locally finite groups
- 20D15 Finite nilpotent groups, p -groups
- 20E25 Local properties of groups
- 20E34 General structure theorems for groups

Keywords:

singular (Sylow) p -subgroup; (very) good Sylow p -subgroup; p -uniqueness subgroup; minimal p -unique subgroup; (numerical) Sylow p -invariant a_p

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References:

- [1] A.O. Asar: "A conjugacy theorem for locally finite groups", *J. London Math. Soc.* (2) 6, No. 2 (1973), 358-360. · Zbl 0253.20042
- [2] R. Baer: "Abzählbar erkennbare gruppentheoretische Eigenschaften", *Math. Z.* 79 (1962), 344- 363. · Zbl 0105.25901
- [3] M.R. Dixon: "Sylow Theory, Formations and Fitting Classes in Locally Finite Groups", *World Scientific*, Singapore (1994). · Zbl 0866.20029
- [4] F.F. Flemisch: "Lokal endliche Gruppen mit Sylow p -Satz oder mit $\min-p$. I: Grundbegriffe, ein Charakterisierungssatz und lokale Prinzipien", *Diplomarbeit*, University of Freiburg, Germany (1984).
- [5] D. Gorenstein – R. Lyons – R. Solomon: "The Classification of the Finite Simple Groups, Part 1", *American Mathematical Society*, Providence, RI (2000). · Zbl 0816.20016

- [6] B. Hartley: "Sylow subgroups of locally finite groups", *Proc. London Math. Soc.* (3) 23 (1971), 159-192. · Zbl 0221.20040
- [7] B. Hartley: "Sylow p -subgroups and local p -solubility", *J. Algebra* 23 (1972), 347-369. · Zbl 0246.20022
- [8] B. Hartley: "Sylow theory in locally finite groups", *Comp. Math.* 25 (1972), 263-280. · Zbl 0248.20036
- [9] B. Hartley: "Simple locally finite groups", in: *Finite and Locally Finite Groups*, Kluwer, Dordrecht (1995), 1-44. · Zbl 0855.20030
- [10] O.H. Kegel: "Four lectures on Sylow theory in locally finite groups", in: *Group Theory, de Gruyter*, Berlin (1989), 3-28. · Zbl 0659.20024
- [11] L.G. Kovács – B.H. Neumann – H. de Vries: "Some Sylow subgroups", *Proc. Royal Soc. London, Series A* 260 (1961), 304-316. · Zbl 0102.26205
- [12] A. Rae: "Local systems and Sylow subgroups in locally finite groups. I", *Proc. Cambridge Philos. Soc.* 72 (1972), 141-160. · Zbl 0247.20024
- [13] A. Rae: "Local systems and Sylow subgroups in locally finite groups. II", *Proc. Cambridge Philos. Soc.* 75 (1974), 1-22. · Zbl 0275.20060

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CHARACTERISING LOCALLY FINITE GROUPS SATISFYING THE STRONG SYLOW THEOREM FOR THE PRIME P

Felix F. Flemisch



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Characterising Locally Finite Groups Satisfying the Strong Sylow Theorem for the Prime p

FELIX F. FLEMISCH

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Abstract

During his lectures to the 1987 Singapore Group Theory Conference Otto H. Kegel proposed the following question: “If every subgroup S of the locally finite group G contains a finite p -subgroup which is singular in S , does G then satisfy the strong Sylow Theorem for the prime p ?” In this paper we answer the question in the affirmative. The paper formed an essential part of the author’s German Diplomarbeit of 1984 (the “Charakterisierungssatz”) written before he left academia [4]. We present the *Charakterisierungssatz* as Theorem 3.9, and summarise then the result as Theorem 3.10, stating that if G is a locally finite group and p is a prime, then G satisfies the strong Sylow theorem for the prime p if and only if every subgroup S of G contains a finite p -subgroup which is singular in S . Subsequently we present a few novel concepts for Sylow theory in (locally) finite groups to encourage future research. The paper is divided in four sections: Introduction; Good Sylow p -subgroups and p -uniqueness subgroups; Basic theorems of Sylow theory in locally finite groups and our *Charakterisierungssatz*; Novel concepts for Sylow theory in (locally) finite groups.

Mathematics Subject Classification (2020): 20D20, 20F50, 20D15

Keywords: singular p -subgroup; good Sylow p -subgroup;
minimal p -unique subgroup

1 Introduction

In his four workshop lectures on Sylow theory in locally finite groups at the famed Singapore Group Theory Conference of June 1987 [10], Otto H. Kegel stated that he could not settle the following question: *if*

every subgroup S of the locally finite group G contains a finite p -subgroup which is singular in S , does G then satisfy the strong Sylow Theorem for the prime p ? Recall that the group G of arbitrary cardinality is defined to be *locally finite* if every finite subset of G is contained in a finite subgroup of G and the finite p -subgroup P of the locally finite group G is said to be *singular in G* if for every finite subgroup F of G containing P there is just a unique Sylow p -subgroup of F containing P . Here a p -group for the prime p is a group of arbitrary cardinality each of whose elements has order a finite power of p . Then a p -group is finite if and only if its order is a finite power of p . The locally finite group G is said to satisfy the *Sylow Theorem for the prime p* (or the *Sylow p -Theorem*) if the maximal p -subgroups of G are all conjugate in G and G satisfies the *strong Sylow Theorem for the prime p* if every subgroup of G satisfies the Sylow Theorem for the prime p . Kegel's lectures present the basics of Sylow theory in locally finite groups, give an overview of the work of Brian Hartley and Andrew Rae on Sylow theory in locally p -soluble groups, and reveal in great detail the normal structure for groups satisfying the strong Sylow Theorem for the prime p in the general case (for $p \geq 5$). Chapters 2 and 4 of [3] give a good overview as well but without appreciating Hartley's, Rae's and Kegel's fundamental papers properly and avoiding all their beautiful details.

In this publication we turn Kegel's question into a theorem: *If every subgroup S of the locally finite group G contains a finite p -subgroup which is singular in S , then G satisfies the strong Sylow Theorem for the prime p .* Since the converse is also true (see [4] and [10]), this characterises the locally finite groups which satisfy the strong Sylow Theorem for the prime p . The proof of our *Charakterisierungssatz* is not presented in its original form since it was written in German as the main result of the author's Diplomarbeit during 1978–1984 (see [4]). We decided against a presentation (for historical reasons) as an amalgam of English and German and translated all employed parts into English, thereby introducing a large number of corrections and embellishments, in particular Theorem 3.6.

The central discovery that enabled in those days the proof was the relationship of p -subgroups which are singular to the *good p -subgroups* (see [12]) and the *strongly local p -subgroups* (see [13]) of Andrew Rae. Let G be any locally finite group and let P be a p -subgroup of G . A *local system for G* is a family Σ of finite subgroups such that every element of G lies in a Σ -group and for every two Σ -groups there exists another Σ -group which contains both, for example, the local

system of all finite subgroups of G . The p -group P is said to *reduce into a local system* Σ for G if for every Σ -group U we have that $P \cap U$ is a Sylow p -subgroup of U , and then P is a maximal p -subgroup of G (see below), P is said to be *good* if there exists a local system for G into which P reduces, and P is said to be *strongly local* or, as we prefer to say, *very good* if given any local system Σ for G there exists a subsystem of Σ into which P reduces. A very good p -subgroup is of course good, and, as we show below, any singular p -subgroup P of a locally finite group G is contained in a unique maximal p -subgroup of G which is very good and the existence of P enforces the conjugacy of the good Sylow p -subgroups in countable locally finite groups

We have the ambition to present not only our own results but also important known results to offer some context and a unified depiction. So when we refer to [4] it does not always mean (although it almost always means) that we present research results of ourselves.

2 Good Sylow p -subgroups and p -uniqueness subgroups

A maximal p -subgroup of a locally finite group G is called here a *Sylow p -subgroup* of G and we denote the set of all Sylow p -subgroups of G by $\text{Syl}_p G$. If a p -subgroup of a locally finite group G reduces into a local system for G , it is a maximal p -subgroup.

Lemma 2.1 (see [4]) *Let p be a prime and let P be a p -subgroup of a locally finite group G . If there exists a local system Σ for G into which P reduces, then P is a Sylow p -subgroup of G .*

PROOF — Let $S \in \text{Syl}_p G$ with $P \leq S$. Suppose, $P \neq S$. Then there exists an element $x \in S \setminus P$. Let $U \in \Sigma$ with $x \in U$. It follows that $\langle P \cap U, x \rangle$ is a p -subgroup of U with $P \cap U < \langle P \cap U, x \rangle \leq S$. This contradicts the prerequisite $P \cap U \in \text{Syl}_p U$. \square

Notice that the above result is proved in [3], Lemma 2.2.10, only for nested local systems and in a more complicated way. The local system Σ for the locally finite group G is said to be *nested* (in German *geschachtelt*) if there is a sequence $\{U_n \mid n \in \mathbb{N}\}$ of finite subgroups of G such that $U_n \leq U_{n+1}$ for all $n \in \mathbb{N}$ and $\Sigma = \{U_n \mid n \in \mathbb{N}\}$. If G is a countable locally finite group and $\{x_n \mid n \in \mathbb{N}\}$ an enumeration of G , let $U_n := \langle x_1, x_2, \dots, x_n \rangle$ ($n \in \mathbb{N}$). Then $\{U_n \mid n \in \mathbb{N}\}$ is a nested

local system for G . If the locally finite group G has a nested local system, then G is countable. We can identify all the good Sylow p -subgroups of countable locally finite groups by means of nested local systems for them.

Lemma 2.2 (see [4]) *Let G be a countable locally finite group.*

- a) *If Σ is a local system for G , then Σ contains a local subsystem Σ_1 which is nested.*
- b) *Let $\Sigma = \{U_n \mid n \in \mathbb{N}\}$ be a nested local system for G . Then there exist with respect to (w.r.t.) Σ good Sylow p -subgroups of G . In particular, G contains at least one good Sylow p -subgroup.*

PROOF — a) Let Σ be a local system for G and $\{x_n \mid n \in \mathbb{N}\}$ an enumeration of G . For $x, y \in G$, we define $U_x \in \Sigma$ with $x \in U_x$ and $\langle U_x, U_y \rangle \leq U_{xy}$ as follows: let $U_{x_1} \in \Sigma$ with $x_1 \in U_{x_1}$; if subgroups $U_{x_1 x_2 x_3 \dots x_n} \in \Sigma$ are already defined with

$$x_1, x_2, x_3, \dots, x_n \in U_{x_1 x_2 x_3 \dots x_n} \quad (n \in \mathbb{N}),$$

let $U_{x_{n+1}} \in \Sigma$ with $x_{n+1} \in U_{x_{n+1}}$ and $U_{x_1 x_2 x_3 \dots x_n x_{n+1}} \in \Sigma$ with

$$\langle U_{x_1 x_2 x_3 \dots x_n}, U_{x_{n+1}} \rangle \leq U_{x_1 x_2 x_3 \dots x_n x_{n+1}} \quad (n \in \mathbb{N}).$$

Then the countable subset $\Sigma_1 := \{U_{x_1 x_2 x_3 \dots x_n} \mid n \in \mathbb{N}\}$ of Σ is a nested local system for G .

- b) Let $P_1 \in \text{Syl}_p U_1$. If

$$P_1 \leq P_2 \leq \dots \leq P_n$$

are already finite p -subgroups of G with $P_i \in \text{Syl}_p U_i$ ($1 \leq i \leq n$), let $P_{n+1} \in \text{Syl}_p U_{n+1}$ with $P_n \leq P_{n+1}$ ($n \in \mathbb{N}$). Define $S := \bigcup_n P_n$. Then S is a p -subgroup of G , which reduces into Σ , and so is good with $S \in \text{Syl}_p G$ by Lemma 2.1. \square

Another argument for proving Lemma 2.2 b) comes from Kegel's Lemma 1.1 of [10] and is very similar to that of Lemma 2.1. Note also that Lemmata 2.1 and 2.2 a) are (and were) well-known but we presented slick improved proofs and did not find Lemma 2.2 a) in the literature. For Lemma 2.2 b) see also [12], 1.11.

We can now introduce the p -uniqueness subgroups and present the close relationship between them and the good Sylow p -subgroups.

In [4] we call p -dominant a p -subgroup of the locally finite group G if it is finite and is contained in a unique Sylow p -subgroup S of G , and call then S *singular* (in German *einzigartig* or *einmalig* or *singulär*, in a double sense). Although “dominant” in German is “dominant” in English we now find it smarter to define such a p -subgroup of G as a p -uniqueness subgroup (in German, quite a bit unwieldy, p -Einzigartigkeitsuntergruppe or p -Einmaligkeitsuntergruppe) of G for S or w.r.t. S . We observe that there is no danger of confounding our p -uniqueness subgroups with the p -uniqueness subgroups which play a major role in the classification of the finite simple groups (see page 82 of [5]).

Proposition 2.3 *Let G be a locally finite group and let p be a prime. Let P be a finite p -subgroup of G . The following properties are equivalent:*

- 1) P is a p -uniqueness subgroup of G .
- 2) P is singular in G .
- 3) Whenever P_1 and P_2 are finite p -subgroups of G with $P \leq P_1 \cap P_2$ then $\langle P_1, P_2 \rangle$ is a p -group.

PROOF — 1) \Rightarrow 2) Suppose P is not singular in G . Then we have a finite subgroup F of G such that P is contained in at least two Sylow p -subgroups P_1 and P_2 of F . Let S_i be a Sylow p -subgroup of G which contains P_i ($i = 1, 2$). If $S_1 = S_2$ then $\langle P_1, P_2 \rangle \leq \langle S_1, S_2 \rangle \cap F$ is a p -group which contradicts $P_1 \in \text{Syl}_p F$ and $P_2 \in \text{Syl}_p F$. Thus $S_1 \neq S_2$. Therefore P is not a p -uniqueness subgroup of G .

2) \Rightarrow 3) Let $P \leq P_1 \cap P_2$ where P_1 and P_2 are finite p -subgroups of G and suppose that $F := \langle P_1, P_2 \rangle$ is not a p -group. Then $P \leq F$ and since $\langle P_1, P_2 \rangle$ is not a p -group there are two distinct Sylow p -subgroups Q_1 and Q_2 of F containing P_1 and P_2 , respectively. But then $P \leq Q_1 \cap Q_2$ and so P is not singular in G .

3) \Rightarrow 1) Suppose that 3) holds and that P is not a p -uniqueness subgroup of G . Then there are distinct Sylow p -subgroups Q_1 and Q_2 of G such that $P \leq Q_1 \cap Q_2$. Let $x \in Q_1 \setminus Q_2$ and $y \in Q_2 \setminus Q_1$. It follows that $P_1 := \langle P, x \rangle$ and $P_2 := \langle P, y \rangle$ are finite p -groups and that $\langle P_1, P_2 \rangle$ is not a p -group, contradicting 3). \square

Kegel discovered insight gaining equivalent conditions for the conjugacy of good Sylow p -subgroups in countable locally finite groups. We expandedly restate and improvedly reprove his result in our terminology thereby adding the property of the existence of a p -uniqueness subgroup. We also notice that Kegel’s argument for 2) \Rightarrow 4) on page 6 and following of [10] is really not fully convincing.

Theorem 2.4 (see [10], Theorem 1.2) *For the countable locally finite group G and the prime p the following properties are equivalent:*

- 1) *There exists a nested local system $\{G_i \mid i \in \mathbb{N}\}$ for G and an index i_0 such that for every pair $j \geq i \geq i_0$ of indices every Sylow p -subgroup P_i of G_i lies in a unique Sylow p -subgroup P_j of G_j .*
- 2) *There exists a finite p -subgroup P_0 of G which is singular in G .*
- 3) *There exists a p -uniqueness subgroup P_0 of G .*
- 4) *Any two good Sylow p -subgroups of G are conjugate in G .*

PROOF — 1) \Rightarrow 2) Choose $P_{i_0} \in \text{Syl}_p G_{i_0}$ and put $P_0 := P_{i_0}$. Let F be any finite subgroup of G containing P_0 . For every index j such that $F \leq G_j$, the unique Sylow p -subgroup of G_j containing P_0 must contain a Sylow p -subgroup of F , and no other Sylow p -subgroup of F can contain P_0 . Clearly 2) \Rightarrow 1). From Proposition 2.3 follow 2) \Rightarrow 3) and 3) \Rightarrow 2). To show 4) \Rightarrow 1) assume that for any nested local system $\{G_i \mid i \in \mathbb{N}\}$ for G and any index i_0 , there are infinitely many pairs $j \geq i \geq i_0$ of indices for which some (and hence any by conjugation) Sylow p -subgroup of G_i is contained in at least two Sylow p -subgroups of G_j . We then can construct, similar to Theorem 3.2 or Theorem 3.8 below, 2^{\aleph_0} maximal p -subgroups of G which are good by Lemma 2.2 and cannot all be conjugate in G . Thus 4) entails 1), and hence 2). It remains to show 3) \Rightarrow 4). Let P and Q be good Sylow p -subgroups of G obtained as two unions of Sylow p -subgroups of nested local systems $\{G_i \mid i \in \mathbb{N}\}$ and $\{H_i \mid i \in \mathbb{N}\}$ for G (see Lemma 2.2) and let S_0 be the unique Sylow p -subgroup of G containing P_0 ; we show that P is conjugate to S_0 and S_0 is conjugate to Q , and therefore P is conjugate to Q ; if P and S_0 are not conjugate then one of them must have property (\star) of Theorem 3.1 (see below) which means in particular that it is not singular; so P has property (\star) ; now P reduces into $\{G_i \mid i \in \mathbb{N}\}$, that is, $P \cap G_i \in \text{Syl}_p G_i$ for all $i \in \mathbb{N}$; there exists an index i_0 such that $P_0 \leq G_{i_0}$; then $P_0 \leq P_{i_0}$ for some unique $P_{i_0} \in \text{Syl}_p G_{i_0}$; now, by Sylow's classical theorem, let x be an element of G_{i_0} such that $P_{i_0}^x = P \cap G_{i_0}$; then $P_{i_0}^x$ is a finite p -subgroup of P which is contained in just only one Sylow p -subgroup of G thereby contradicting property (\star) of P ; for exactly the same reasons S_0 is conjugate to Q ; therefore P must be conjugate to Q . \square

Let S be a Sylow p -subgroup of the locally finite group G . A finite subgroup F of G is called S -dominant if S reduces into every

subgroup U of G which contains F , that is, $S \cap U \in \text{Syl}_p U$ for all subgroups U of G such that $F \leq U$.

Lemma 2.5 (see [4]) *Let G be a locally finite group, p a prime, $S \in \text{Syl}_p G$ and F a finite subgroup of G . The following properties are equivalent:*

- 1) F is S -dominant.
- 2) For each finite subgroup U of G with $F \leq U$ we have $S \cap U \in \text{Syl}_p U$.

PROOF — 1) \Rightarrow 2) is clear, so we only need to prove that 2) implies 1). Since F is finite, there exists a local system Σ for G such that for each Σ -group U we have $F \leq U$. Let V be a subgroup of G with $F \leq V$. Then $\Sigma_1 := \{V \cap U \mid U \in \Sigma\}$ is a local system for V into which $S \cap V$ reduces. Therefore from Lemma 2.1 follows $S \cap V \in \text{Syl}_p V$. \square

Lemma 2.6 (see [4]) *Let G be a locally finite group and $S \in \text{Syl}_p G$. The following properties are equivalent:*

- 1) S is very good.
- 2) There exists an S -dominant subgroup of G .

PROOF — 1) \Rightarrow 2) Suppose no S -dominant subgroup of G exists. Then, according to Lemma 2.5, to every finite subgroup F of G there exists one finite subgroup U_F of G with $F \leq U_F$ and $S \cap U_F \notin \text{Syl}_p U_F$. Then $\Sigma := \{U_F \mid F \text{ finite subgroup of } G\}$ is a local system for G that possesses no local subsystem into which S reduces.

2) \Rightarrow 1) Let F be an S -dominant subgroup of G and Σ a local system for G . Let $\Sigma_1 := \{U \mid U \in \Sigma \text{ and } F \leq U\}$. Then Σ_1 is, because of the S -dominance of F , a local subsystem of Σ into which S reduces. \square

Lemma 2.7 (see [4]) *Let G be a locally finite group and let p be a prime.*

- a) If F is a p -uniqueness subgroup of G and S is the singular Sylow p -subgroup of G with $F \leq S$, then F is an S -dominant subgroup of G .
- b) Every singular Sylow p -subgroup of G is very good.

PROOF — Since b) follows from a) and Lemma 2.6 we only need to prove a). Let U be a subgroup of G with $F \leq U$. Let $P \in \text{Syl}_p U$ and $T \in \text{Syl}_p G$ with $F \leq S \cap U \leq P \leq T$. From $F \leq S$ and the p -uniqueness of F follows $T = S$. Therefore $S \cap U \geq S \cap P = P$. \square

The following consequence of this lemma is a relevant insight.

Theorem 2.8 (see [4]) *Let p be a prime and P be a p -uniqueness subgroup of the locally finite group G (or, equivalently by Proposition 2.3, let P be a singular p -subgroup of G). Then the singular Sylow p -subgroup S of G containing P is very good.*

We can now summarise the relationship between good Sylow p -subgroups and p -uniqueness subgroups together with the Sylow p -subgroups containing them as follows:

- singular Sylow p -subgroups are very good;
- p -uniqueness subgroups are singular, and conversely;
- in countable locally finite groups good Sylow p -subgroups are identified by nested local systems;
- in countable locally finite groups the existence of a p -uniqueness subgroup compels the conjugacy of all good Sylow p -subgroups.

We end the discussion of good Sylow p -subgroups by pointing out that there exist 1) countable locally finite groups with Sylow p -subgroups which are not good (see the note at page 5 of [10]: “It may be worthwhile to point out that a countable infinite locally finite group may have maximal p -subgroups which” are not good) and 2) locally finite groups of cardinality 2^{\aleph_0} without good Sylow p -subgroups.

First, we let G be a finite group with $|\text{Syl}_p G| \geq 2$, e.g. the symmetric group \underline{S}^{2p} of degree $2p$ for the prime p for which we know surely that

$$|\text{Syl}_p \underline{S}^{2p}| \geq 2p - 2 \geq 2.$$

Consider the \mathbb{N} -fold cartesian power

$$\begin{aligned} G^{[\mathbb{N}]} &:= \prod \{G_i \mid G_i := G \text{ for all } i \in \mathbb{N}\} \\ &= \{(x_1, x_2, \dots) \mid x_i \in G_i \text{ for all } i \in \mathbb{N}\} \end{aligned}$$

of G and notice that *it satisfies the Sylow p -Theorem.*

PROOF — For $S, T \in \text{Syl}_p G^{[\mathbb{N}]}$ there are $S_i, T_i \in \text{Syl}_p G_i = \text{Syl}_p G$ ($i \in \mathbb{N}$) such that S , resp. T , is the cartesian product of the S_i 's, resp. the T_i 's. If $x_i \in G_i = G$ with $S_i^{x_i} = T_i$ ($i \in \mathbb{N}$) and $x := (x_i)_{i \in \mathbb{N}}$, then $S^x = T$. \square

The group $G^{[\mathbb{N}]}$ contains the \mathbb{N} -fold direct power

$$G^{(\mathbb{N})} := \prod^0 \{(x_i)_{i \in \mathbb{N}} \in G^{[\mathbb{N}]} \mid x_i = 1 \text{ for almost all } i \in \mathbb{N}\},$$

which does not satisfy the Sylow p -Theorem.

PROOF — Let $S, T \in \text{Syl}_p G^{(\mathbb{N})}$. If there is an $x \in G^{(\mathbb{N})}$ with $S^x = T$, then $S^{x\pi_i} = T^{\pi_i}$ for almost all $i \in \mathbb{N}$. Thus for $P, Q \in \text{Syl}_p G$ with $P \neq Q$, the groups $P^{(\mathbb{N})}$ and $Q^{(\mathbb{N})}$ are not in $G^{(\mathbb{N})}$ — but in $G^{[\mathbb{N}]}$ — conjugate Sylow p -subgroups of $G^{(\mathbb{N})}$. Alternatively, it follows from $|G^{(\mathbb{N})}| = \aleph_0$ and $|\text{Syl}_p G^{(\mathbb{N})}| = 2^{\aleph_0}$ — since $|\text{Syl}_p G| \geq 2$ we can refer to Theorems 3.1 and 3.2 (see below) — that not all Sylow p -subgroups of $G^{(\mathbb{N})}$ can be conjugate. □

The example $G^{(\mathbb{N})} \leq G^{[\mathbb{N}]}$ shows that in uncountable locally finite groups the Sylow p -Theorem is not inherited by normal subgroups.

Moreover, $G^{[\mathbb{N}]}$ contains the diagonal subgroup

$$D := \{(x_i)_{i \in \mathbb{N}} \in G^{[\mathbb{N}]} \mid (\exists x \in G)(\forall i \in \mathbb{N}) x_i = x\} \simeq G$$

via the isomorphism

$$\delta : D \longrightarrow G, \quad ((x_i)_{i \in \mathbb{N}})^\delta := x,$$

from D onto G with $D \cap G^{(\mathbb{N})} = \langle 1 \rangle$. Since $G^{(\mathbb{N})}$ is a normal subgroup of $G^{[\mathbb{N}]}$, we have $\langle G^{(\mathbb{N})}, D \rangle = DG^{(\mathbb{N})}$; this is a countable subgroup of $G^{[\mathbb{N}]}$. The Sylow p -subgroups of $G^{[\mathbb{N}]}$ (resp. of $G^{(\mathbb{N})}$) are cartesian (resp. direct) products of the Sylow p -subgroups of the G_i 's ($i \in \mathbb{N}$), namely $\prod \{S_n^{\pi_i} \mid i \in \mathbb{N}\}$ (resp. $\prod^0 \{S_n^{\pi_i} \mid i \in \mathbb{N}\}$) for $S_n \in \text{Syl}_p G$ ($n \in \mathbb{N}$), where $\pi_i: G^{[\mathbb{N}]} \rightarrow G_i$ is the projection $\pi_i((x_k)_{k \in \mathbb{N}}) := x_i$ on the factor G_i ($i \in \mathbb{N}$). Any $P \in \text{Syl}_p D$ normalises exactly one Sylow p -subgroup $S(P)$ of $G^{[\mathbb{N}]}$ (resp. exactly one Sylow p -subgroup $S^0(P)$ of $G^{(\mathbb{N})}$), namely $S(P) = \prod \{P^{\pi_i} \mid i \in \mathbb{N}\}$ (resp. $S^0(P) = \prod^0 \{P^{\pi_i} \mid i \in \mathbb{N}\}$). Therefore every Sylow p -subgroup of D is a p -uniqueness subgroup of $DG^{(\mathbb{N})}$, and $PS^0(P)$, for $P \in \text{Syl}_p D \simeq \text{Syl}_p G$, is a singular Sylow p -subgroup of $DG^{(\mathbb{N})}$ and so is good, even very good, by Theorem 2.8; these Sylow p -subgroups are conjugate: if $P_1, P_2 \in \text{Syl}_p D$ and $P_1^x = P_2$ with $x \in D$, then

$$\begin{aligned} (P_1 S^0(P_1))^x &= (P_1 \prod^0 \{P_1^{\pi_i} \mid i \in \mathbb{N}\})^x \\ &= P_2 \prod^0 \{P_2^{\pi_i} \mid i \in \mathbb{N}\} = P_2 S^0(P_2) \end{aligned}$$

(see also Theorem 2.4). The countable group $G^{(\mathbb{N})}$ also has by Lemma 2.2 good Sylow p -subgroups, which are not conjugate, and we

are able to designate some distinguished of them explicitly: let

$$U_i := G_1 \times G_2 \times \dots \times G_i \quad (i \in \mathbb{N});$$

then $\Sigma := \{U_i \mid i \in \mathbb{N}\} \cap G^{(\mathbb{N})}$ is a nested local system for $G^{(\mathbb{N})}$; if $P_i \in \text{Syl}_p G_i = \text{Syl}_p G$ ($i \in \mathbb{N}$), then

$$P^0 := (P_1 \times P_2 \times \dots) \cap G^{(\mathbb{N})}$$

is a p -subgroup of $G^{(\mathbb{N})}$ which reduces into Σ and thus is a good Sylow p -subgroup of $G^{(\mathbb{N})}$ by Lemma 2.1.

The group $DG^{(\mathbb{N})}$ has indeed also (many) Sylow p -subgroups, which are not good: since $|\text{Syl}_p G| \geq 2$ we can construct using the method employed in the proof of Theorem 3.2 or that employed in the proof of Theorem 3.8 an infinitely (\aleph_0) high tree of finite p -subgroups of $DG^{(\mathbb{N})}$ with $\langle 1 \rangle$ as the root which branches properly at each location with proper inclusions and where two immediate successors of each point do not generate a p -group; this tree has 2^{\aleph_0} branches which constitute 2^{\aleph_0} many ascending unions of finite p -subgroups and thus 2^{\aleph_0} many p -subgroups P_ι where any two of them do not generate a p -group; choosing for each P_ι a Sylow p -subgroup S_ι of $DG^{(\mathbb{N})}$ containing P_ι now gives 2^{\aleph_0} Sylow p -subgroups of $DG^{(\mathbb{N})}$ ($1 \leq \iota \leq 2^{\aleph_0}$) on the treetop; since the good Sylow p -subgroups of the countable group $DG^{(\mathbb{N})}$ are conjugate (Theorem 2.4), at most \aleph_0 of these 2^{\aleph_0} Sylow p -subgroups can be good; there remain (with or without the continuum hypothesis) at least $2^{\aleph_0} - \aleph_0$ many Sylow p -subgroups in the treetop which are not good and too many to be conjugate in $DG^{(\mathbb{N})}$. We note that Rae [12] constructs, by introducing the unwieldy concept of “weakly goodness” and by referring to another group he constructed (see [12], 5.11), a countable locally soluble group possessing a Sylow p -subgroup which is not good (see [12], 5.31). This example is much more complicated than ours.

Second, let p and q be primes with $q \equiv 1 \pmod{p}$ and

$$A := \langle a, b \mid a^p = b^q = (ab)^p = 1 \rangle.$$

Then $|A| = pq$ and A has q Sylow p -subgroups and a normal Sylow q -subgroup, so is metabelian. If $(p, q) = (2, 3)$, then $A = \underline{S}^3$ is the symmetric group of degree 3. The group A contains the elements a and $a' := ab$ of order p which are not p -consonant, that is, they do

not generate a p -group. The \mathbb{N} -fold cartesian power $A^{[\mathbb{N}]}$ of A is locally finite and metabelian of exponent pq . László G. Kovács, Bernhard H. Neumann and Hugo de Vries constructed, based on the elements a and a' (and exemplarily for $(p, q) = (2, 3)$), an \mathbb{N} -fold interdirect power H of A , that is, $A^{(\mathbb{N})} \leq H \leq A^{[\mathbb{N}]}$, with the following properties (see [11], Theorem 3.7): H is metabelian of exponent q and order 2^{\aleph_0} with a countable Sylow p -subgroup and a Sylow p -subgroup of order 2^{\aleph_0} (hence without Sylow Theorem for the prime p). They also constructed, using again a and a' , an \mathbb{N} -fold interdirect power H of A with the following amazing properties (see [11], Theorem 4.4, and also [12], 1.13): H has order 2^{\aleph_0} , each Sylow p -subgroup of H is countable, H has a countable normal (hence unique) Sylow q -subgroup, which has no complement in H , and each Sylow p -subgroup has a complement in H , which is normal in H and contains elements of order p . No Sylow p -subgroup of H can be good: suppose a Sylow p -subgroup S of H reduces into a local system Σ for H ; we then choose a Σ -group U containing an element x of order p of a complement of S , and a $P \in \text{Syl}_p U$ containing x ; since $S \cap U \in \text{Syl}_p U$ there is a $y \in U$ with $P^y = S \cap U$; then $\langle x \rangle^y \leq S$ whereas $\langle x \rangle^y$ belongs to the normal complement of S , which is a contradiction.

In the following section we shall point out that there exist countable locally finite groups 3) without singular Sylow p -subgroups, 4) with good Sylow p -subgroups which are not very good, and 5) with very good Sylow p -subgroups which are not singular.

3 Basic theorems of Sylow theory in locally finite groups and our “Charakterisierungssatz”

In this section we first present — with quite considerably improved proofs — the basics of Sylow theory in locally finite groups (Theorem 3.1 to Theorem 3.5) and subsequently prepare and carry out the proof of our *Charakterisierungssatz* (Theorem 3.6 to Theorem 3.9) which, in turns, allows us to prove very easily our main theorem (Theorem 3.10).

In the following statement, the property (\star) means that S is not singular; see the same property (\star) on page 8 of [10]. This property was for the first time discovered by Ali O. Asar [1].

Theorem 3.1 (see [4], and Theorem 3.6 below for a generalisation) *Any locally finite group G which does not satisfy the Sylow Theorem for the prime p contains a Sylow p -subgroup S with the following property:*

- (\star) *Every finite subgroup of S lies in at least two Sylow p -subgroups of G .*

PROOF — Let S and T be two Sylow p -subgroups of G which are not conjugate (in G). If T is not singular, that is, T does have property (\star), the result is immediate, so suppose that T is singular and let Y be a p -uniqueness subgroup for T . We show that then S has property (\star), that is, S is not singular. To this end let X be an arbitrary finite subgroup of S . Then $\langle X, Y \rangle$ is a finite group. According to the Sylow p -Theorem for finite groups there is an $x \in G$ such that X and Y^x lie in the same Sylow p -subgroup of $\langle X, Y \rangle$. Then $\langle X, Y^x \rangle$ is a p -group. From the assumption on Y it now follows that $\langle Y^x, X \rangle \leq T^x$. Hence X lies in at least the two Sylow p -subgroups S and T^x of G . Therefore X is not a p -uniqueness subgroup for S . \square

We now prepare an alternative proof of the basic theorem of Sylow theory known as the “Asar-Hartley theorem” (see [1] and [3], Theorem 2.3.11, for the original proof). Our proofs of Theorem 3.2 a) and b) with reference to a) are much clearer and more detailed than the original proof by Asar, which may be considered rather cumbersome. Note also that in [10], Theorem 1.3, Kegel sagely combines Theorem 3.1 with Theorem 3.2 c).

Theorem 3.2 (see [4]) *Let G be a locally finite group and let P be a p -subgroup of G for the prime p .*

- a) *Suppose P has the following property: (\dagger) To every finite subgroup F of P there exists an $x = x(F) \in G$ with $F^x \leq P$ such that $\langle P, P^x \rangle$ is not a p -group. Then there are 2^{\aleph_0} infinite ascending chains*

$$X_{i_1} < X_{i_1 i_2} < \dots < X_{i_1 i_2 \dots i_n} < \dots$$

of finite p -subgroups of G with indices $i_k \in \{0, 1\}$ ($k \in \mathbb{N}$) such that for all $n \in \mathbb{N}$ and each choice of indices i_k ($1 \leq k \leq n$), the group $\langle X_{i_1 i_2 \dots i_n 0}, X_{i_1 i_2 \dots i_n 1} \rangle$ is not a p -group.

- b) *Let $P \in \text{Syl}_p G$ with the property (\star). Then P has property (\dagger).*

c) Let $P \in \text{Syl}_p G$ with the property (\star) and let X be a finite subgroup of P . Then there are 2^{\aleph_0} many infinite ascending chains

$$X < X_{i_1} < X_{i_1 i_2} < \dots < X_{i_1 i_2 \dots i_n} < \dots$$

with the properties from point a).

PROOF — a) Let X be a finite subgroup of P and y an element of G such that: 1) $\langle P, P^y \rangle$ is not a p -group, and 2) $X^y \leq P$. Because of the first property there exists a finite subgroup X_0 of P with $X \leq X_0$ such that $\langle X_0, X_0^y \rangle$ is not a p -group, and because of the second property we have $\langle X, X^y \rangle \leq P$, hence $X_0 \neq X \neq X_0^y$. If we substitute in the last two sentences X by X_0 , we get two finite p -subgroups X_{00} and X_{01} of G with $X_0 < X_{00}$ and $X_0 < X_{01}$ such that $\langle X_{00}, X_{01} \rangle$ is not a p -group. Since P^y has the property (\dagger) , too, we can quite analogously substitute X by $X_1 := X_0^y$ and so get two finite p -subgroups X_{10} and X_{11} of G with $X_1 < X_{10}$ and $X_1 < X_{11}$ such that the subgroup $\langle X_{10}, X_{11} \rangle$ is not a p -group. We now have constructed four ascending chains

$$X < X_0 < X_{00}, \quad X < X_0 < X_{01}, \quad X_1 < X_{10} \quad \text{and} \quad X_1 < X_{11}$$

of finite p -subgroups of G such that the subgroups $\langle X_0, X_1 \rangle$, $\langle X_{00}, X_{01} \rangle$ and $\langle X_{10}, X_{11} \rangle$ are not p -groups. Now let $n \in \mathbb{N}$ with $n \geq 2$ and let already be constructed 2^n ascending chains

$$X_{i_1} < X_{i_1 i_2} < \dots < X_{i_1 i_2 \dots i_n}$$

of finite p -subgroups of G with indices $i_k \in \{0, 1\}$ ($1 \leq k \leq n$) such that for each $m \in \mathbb{N}$ with $m \leq n - 1$ and each choice of indices i_k ($1 \leq k \leq m$) the subgroup $\langle X_{i_1 i_2 \dots i_m 0}, X_{i_1 i_2 \dots i_m 1} \rangle$ of G is not a p -group. Whilst repeating the construction of the first two sentences successively with the 2^n groups $X_{i_1 i_2 \dots i_n}$ in place of X , we get, because each conjugate of P possesses the property (\dagger) , in each case two p -subgroups $X_{i_1 i_2 \dots i_n 0}$ and $X_{i_1 i_2 \dots i_n 1}$ of G such that

$$X_{i_1 i_2 \dots i_n} < X_{i_1 i_2 \dots i_n 0} \cap X_{i_1 i_2 \dots i_n 1}$$

and

$$\langle X_{i_1 i_2 \dots i_n 0}, X_{i_1 i_2 \dots i_n 1} \rangle$$

is not a p -group. Therewith we now have constructed 2^{n+1} ascend-

ing chains

$$X_{i_1} < X_{i_1 i_2} < \dots < X_{i_1 i_2 \dots i_n} < X_{i_1 i_2 \dots i_n i_{n+1}}$$

having the requested properties. Therefore we can w.r.t. inclusion recursively construct a tree of height \aleph_0 of finite p -subgroups of G , which branches properly at each location with proper inclusions, hence must contain 2^{\aleph_0} infinite branches. Also any two immediate successors of an arbitrary point do not generate a p -group. These branches are just the required chains.

b) Let F be a finite subgroup of P and R be a Sylow p -subgroup of G with $F \leq R \neq P$. Then there is an element x in R with $x \notin P$ and the group $\langle F, x \rangle$ is a finite p -group. Let $Y := \langle F, x \rangle \cap P$. Then we have $Y \neq \langle F, x \rangle$. It is well-known that as a finite p -group $\langle F, x \rangle$ satisfies the normaliser condition. Therefore Y is a proper subgroup of $N_{\langle F, x \rangle}(Y)$. Let y be an element in $\langle F, x \rangle$, but not in Y , which normalises Y . Then $y \notin P$. Since y is a p -element and P by assumption a Sylow p -subgroup of G , it follows that $y \notin N_G(P)$ and that $\langle P, P^y \rangle$ is not a p -group. This is the property (\dagger) from point a) for P .*

c) We combine the proofs of point a) and point b). Let $R \in \text{Syl}_p G$ with $X \leq R \neq P$, $x \in R \setminus P$ and $T := P \cap \langle X, x \rangle$. Being a finite p -group, $\langle X, x \rangle$ satisfies the normaliser condition. Hence there exists a $t \in \langle X, x \rangle \setminus T$ with $t \in N_{\langle X, x \rangle}(T)$. Then $\langle P, P^t \rangle$ is not a p -group, since else $t \in P$, and so there exists a finite subgroup X_0 of P with $X \leq X_0$ such that with $X_1 := X_0^t$ the group $\langle X_0, X_1 \rangle$ is not a p -group. Thus, we have $X_0 \neq X \neq X_1$ since $\langle X, X^t \rangle \leq T$ is a p -group. Of course, $X \leq X_0$, but also $X \leq X_1$ because of $t \in X$. We can repeat this construction whilst replacing X by X_0 and also by its conjugate X_1 . Thereby we construct subgroups $X_{00}, X_{01}, X_{10}, X_{11}$ and four ascending chains

$$X < X_0 < X_{00}, X < X_0 < X_{01}, X < X_1 < X_{10} \text{ and } X < X_1 < X_{11}$$

of finite p -subgroups of G . We subsequently repeat this construction with each of the $X_{i_1 i_2}$'s and whilst doing this infinitely often we construct 2^{\aleph_0} many chains

$$X < X_{i_1} < X_{i_1 i_2} < \dots < X_{i_1 i_2 \dots i_n} < \dots$$

* Asar [1, Lemma 1] (unwieldy) considers instead of R a p -subgroup Y of G such that $Y < U (= P)$, chooses $y \in Y \setminus U$, defines $F^* := U \cap \langle F, y \rangle$ with $F \leq U \cap Y$, finds $F \leq F^*$ and $N_{\langle F, y \rangle}(F^*) > F^*$, and finally concludes $N_G(F^*) < N_G(U)$, since U is the unique maximal p -subgroup of $N_G(U)$ and $N_{\langle F, y \rangle}(F^*) < U$.

of finite p -subgroups of G with the properties from point a). So we can, starting from an arbitrary subgroup X of P as a “minimal point” or a “root”, recursively w.r.t. inclusion construct a tree of height \aleph_0 of finite p -subgroups of G , which branches properly at each location, hence must contain 2^{\aleph_0} infinite branches. Also any two immediate successors of an arbitrary point do not generate a p -group. These branches are just the required chains. \square

Theorem 3.2 enables us to prove very easily the “Asar-Hartley theorem” which characterises locally finite groups satisfying the strong Sylow Theorem for the prime p by a cardinality result without the need to endeavour the continuum hypothesis (for a proof closer to the original one of Asar, the reader can consult [10], pp. 8–9).

Theorem 3.3 (see Asar [1], Hartley [6],[8]*) *Let G be a locally finite group and p be a prime. Suppose that for every countable subgroup H of G we have $|\text{Syl}_p H| < 2^{\aleph_0}$. Then G satisfies the strong Sylow p -Theorem.*

PROOF — Suppose G does not satisfy the strong Sylow Theorem for the prime p . Then there is a subgroup U of G which does not satisfy the Sylow Theorem for the prime p . Thus according to Theorems 3.1 and 3.2 there are 2^{\aleph_0} many infinite ascending chains

$$X_{i_1} < X_{i_1 i_2} < \dots < X_{i_1 i_2 \dots i_n} < \dots$$

of finite p -subgroups of U with the properties from point a) of Theorem 3.2. Let \mathcal{M} be the set of all p -subgroups of U which are an ascending union of one of these chains. Then it follows $|\mathcal{M}| = 2^{\aleph_0}$ and that any two \mathcal{M} -groups cannot generate a p -group. Now let

$$H_n := \langle X_{i_1 i_2 \dots i_n} \mid i_k \in \{0, 1\}, 1 \leq k \leq n \rangle \quad (n \in \mathbb{N})$$

and

$$H := \bigcup_{n \in \mathbb{N}} H_n.$$

Then H is a countable subgroup of U and so of G . Since H contains every \mathcal{M} -group it follows that $|\text{Syl}_p H| = 2^{\aleph_0}$. This contradicts the assumption on the countable subgroups of G . \square

* The result for countable locally finite groups was obtained independently by Brian Hartley using a quite different method which allowed him to generalise it from the prime p to a set of primes π when the finite groups of a nested local system have each a nilpotent Hall π -subgroup (see [6]). However, Hartley has extended his proof in [8] to uncountable locally finite groups by another beautiful method.

The cardinality statement of Theorem 3.3 has an immediate first corollary for countable locally finite groups.

Theorem 3.4 *Let G be a countable locally finite group. The following properties are equivalent:*

- 1) *For every (countable) subgroup H of G we have $|\text{Syl}_p H| < 2^{\aleph_0}$.*
- 2) *G satisfies the strong Sylow Theorem for the prime p .*
- 3) *G satisfies the Sylow Theorem for the prime p .*
- 4) *$|\text{Syl}_p G| < 2^{\aleph_0}$.*
- 5) *Every (countable) subset of G is contained in a subgroup U of G with $|\text{Syl}_p U| < 2^{\aleph_0}$.*

The second corollary of Theorem 3.3 would certainly as a conjugacy assertion be very difficult to be proved but is as a cardinality statement trivial. Recall first that a class of groups \mathfrak{X} is *countably recognisable* if, whenever all countable subgroups of a group G belong to \mathfrak{X} , then G itself is an \mathfrak{X} -group (see Baer [2]).

Theorem 3.5 *The locally finite group G satisfies the strong Sylow Theorem for the prime p if and only if every countable subgroup of G satisfies the strong Sylow Theorem for the prime p . In particular, the class $\text{Syl-}p$ of all locally finite groups satisfying the strong Sylow Theorem for the prime p is countably recognisable.*

We now can prove our key discovery whenever the Sylow Theorem for the prime p is not valid in a countable locally finite group which shows a symmetry between not conjugate Sylow p -subgroups.

Theorem 3.6 *Let G be a countable locally finite group and p be a prime. If two Sylow p -subgroups of G are not conjugate, then neither is singular.*

PROOF — Let S and T be Sylow p -subgroups of G which are not conjugate. We saw in Theorem 3.1 that one of S or T is not singular. Without loss of generality (w.l.o.g.) we may suppose that S is not singular. To prove the result we must show that T is not singular either. If T is not good, it cannot be singular, since by Theorem 2.8 singular Sylow p -subgroups are very good. So let T be good w.r.t. the nested local system $\{G_n \mid n \in \mathbb{N}\}$ for G and let F be an arbitrary finite

subgroup of T . We show that F cannot be a p -uniqueness subgroup for T and so T is not singular since F is chosen arbitrarily. Since S and T are not conjugate, we have $S \neq T$.

There exists an $m = m(F) \in \mathbb{N}$ with $F \leq G_m$. After the reenumeration $\{n \mapsto n + m - 1 \mid n \in \mathbb{N}\}$, it is possible to assume $F \leq G_1$. Then $F \leq T \cap G_1 \in \text{Syl}_p G_1$. If $T \cap G_n$ is the unique Sylow p -subgroup of G_n for all $n \in \mathbb{N}$ then T is the unique Sylow p -subgroup of G and we obtain the contradiction that $S = T$. Hence there is an $n \in \mathbb{N}$ such that G_n has a Sylow p -subgroup R with $R \neq T \cap G_n$. Renumbering again if needed we may assume that $R \in \text{Syl}_p G_1$ with $R \neq T \cap G_1$. Choose $y \in R \setminus (T \cap G_1)$, so in particular $y \notin T$. By the Sylow p -Theorem for finite groups there is an $x \in G_1$ such that $(T \cap G_1)^x = R$ and so $F^x \leq R$ since $F \leq T \cap G_1$. From $\langle F^x, y \rangle \leq R$ follows that $\langle F^x, y \rangle$ is a finite p -group. Let $Y := \langle F^x, y \rangle \cap T$. Then $Y \neq \langle F^x, y \rangle$ since $y \notin T$.

But Y satisfies, as is well-known, the normaliser condition and so we can choose $z \in N_{\langle F^x, y \rangle}(Y) \setminus Y$. Then $z \notin T$ since otherwise z belongs to $T \cap \langle F^x, y \rangle = Y$. But z is a p -element outside of T and $T \in \text{Syl}_p G$, and so $z \notin N_G(T)$. Therefore $\langle T, T^z \rangle$ is not a p -group. In particular, $T \neq T^z$ and $F \leq T \cap T^z$. Therefore the arbitrarily chosen F is not a p -uniqueness subgroup for T . \square

Whenever a countable locally finite group contains a singular Sylow p -subgroup then all good Sylow p -subgroups will be conjugate by Theorem 2.4. Whenever every countable subgroup of a (countable) locally finite group contains a singular Sylow p -subgroup then all Sylow p -subgroups are conjugate. This core insight is spelled out by the following theorem.

Theorem 3.7 (see [4]) *Let G be a locally finite group and let p be a prime. Suppose that every countable subgroup of G contains a singular Sylow p -subgroup. Then G satisfies the strong Sylow Theorem for the prime p .*

PROOF — According to Theorem 3.5 we can assume that G is countable, and according to Theorem 3.4 it suffices to show that G satisfies the Sylow Theorem for the prime p . However, this is now immediate since by assumption G has a singular Sylow p -subgroup S . Let T be any Sylow p -subgroup of G . If S and T are not conjugate, then by Theorem 3.6 neither is singular. With this contradiction S and T are conjugate and the result follows. \square

Since the above result is very significant, we provide an alternative proof by proving the contrapositive.

PROOF — Suppose G does not satisfy the Sylow Theorem for the prime p . Then, according to Theorem 3.1, Theorem 3.2 b), and Theorem 3.2 a), there are 2^{\aleph_0} infinite ascending chains

$$X_{i_1} < X_{i_1 i_2} < \dots < X_{i_1 i_2 \dots i_n} < \dots$$

of finite p -subgroups of G with the properties from Theorem 3.2 a). Let

$$U_n := \langle X_{i_1 i_2 \dots i_n} \mid i_k \in \{0, 1\}, 1 \leq k \leq n \rangle \quad (n \in \mathbb{N})$$

and

$$U := \bigcup_{n \in \mathbb{N}} U_n = \langle X_{i_1 i_2 \dots i_n} \mid i_k \in \{0, 1\}, 1 \leq k \leq n \in \mathbb{N} \rangle.$$

Then U is a (countable) subgroup of G and $\{U_n \mid n \in \mathbb{N}\}$ is a nested local system for U . We show that U does not contain any singular Sylow p -subgroup. Let F^* be a finite p -subgroup of U . There exists an $m = m(F^*) \in \mathbb{N}$ with $F^* \leq U_m$. By definition of U_m there are indices $j_1, j_2, \dots, j_m, \dots, k_1, k_2, \dots, k_m, \dots, l_1, l_2, \dots, l_m$ with

$$F^* \leq \langle X_{j_1 j_2 \dots j_m}, X_{k_1 k_2 \dots k_m}, \dots, X_{l_1 l_2 \dots l_m} \rangle.$$

Then

$$P_1 := \langle X_{j_1 j_2 \dots j_m 0}, X_{k_1 k_2 \dots k_m 0}, \dots, X_{l_1 l_2 \dots l_m 0} \rangle$$

and

$$P_2 := \langle X_{j_1 j_2 \dots j_m 1}, X_{k_1 k_2 \dots k_m 1}, \dots, X_{l_1 l_2 \dots l_m 1} \rangle$$

are finite p -subgroups of U with $F^* \leq P_1 \cap P_2$ such that $\langle P_1, P_2 \rangle$ is not a p -group. We now choose $Q_{1,0}, Q_{2,0} \in \text{Syl}_p U_m$ with $P_1 \leq Q_{1,0}$ and $P_2 \leq Q_{2,0}$. If

$$Q_{1,0} \leq Q_{1,1} \leq \dots \leq Q_{1,n} \quad \text{and} \quad Q_{2,0} \leq Q_{2,1} \leq \dots \leq Q_{2,n}$$

are already p -subgroups of U with $Q_{1,i}, Q_{2,i} \in \text{Syl}_p U_{m+i}$ ($0 \leq i \leq n$), let $Q_{1,n+1}, Q_{2,n+1} \in \text{Syl}_p U_{m+n+1}$ such that $Q_{1,n} \leq Q_{1,n+1}$ and $Q_{2,n} \leq Q_{2,n+1}$ ($n \in \mathbb{N}_0$). Let

$$Q_1 := \bigcup_{n \in \mathbb{N}_0} Q_{1,n} \quad \text{and} \quad Q_2 := \bigcup_{n \in \mathbb{N}_0} Q_{2,n}.$$

Then Q_1 and Q_2 are both p -subgroups of U with $F^* \leq Q_1 \cap Q_2$ such

that $\langle Q_1, Q_2 \rangle$ is not a p -group. Per construction, Q_1 and Q_2 reduce into the nested local system $\{U_{m+n} \mid n \in \mathbb{N}_0\}$ for U . By Lemma 2.1, the groups Q_1 and Q_2 are two good Sylow p -subgroups of U containing F^* , that is, F^* is not a p -uniqueness subgroup of U . Thus U does not contain any p -uniqueness subgroup. \square

Third, we supplement Theorem 3.7 with an example of a countable locally finite group H without the (strong) Sylow Theorem for the prime p but with a (countable) subgroup U without singular Sylow p -subgroups. Let $H := DG^{(\mathbb{N})}$ be the group from p. 21, $V := G^{(\mathbb{N})}$ and F be a finite subgroup of the good Sylow p -subgroup P^0 of V from p. 21. We show that F cannot be a p -uniqueness subgroup of V . Since F is finite, there is an $m = m(F) \in \mathbb{N}$ with $F \leq U_m$. Because of $|\text{Syl}_p G| \geq 2$ there is a $Q_{m+1} \in \text{Syl}_p G_{m+1}$ with $Q_{m+1} \neq P_{m+1}$. Then

$$Q^0 := (P_1 \times P_2 \times \dots \times P_m \times Q_{m+1} \times P_{m+2} \times \dots) \cap G^{(\mathbb{N})}$$

contains the group F and we have $Q^0 \neq P^0$. So V has the distinguished good Sylow p -subgroup P^0 which is not singular (notice that by Theorem 3.1 there must be such a Sylow subgroup since V does not satisfy the Sylow p -Theorem). By the second part of the proof of Theorem 3.7, there is a (countable) subgroup U of V which does not contain any singular Sylow p -subgroup.

Fourth, let $G = \underline{S}^{(\mathbb{N})}$ be the countable locally finite group of finitary permutations on a countably infinite set (that is, which move only finitely many elements), p a prime, and $\{n_i \mid i \in \mathbb{N}\}$ a sequence in \mathbb{N} with $n_i + 2p \leq n_{i+1}$ ($i \in \mathbb{N}$). Then $\Sigma := \{\underline{S}^{n_i} \mid i \in \mathbb{N}\}$ is a nested local system for G . By Lemma 2.2 b) there exists an $S \in \text{Syl}_p G$ which is good w.r.t. Σ . We know that $|\text{Syl}_p \underline{S}^{2p}| \geq 2p - 2 \geq 2$. Let $T_1, T_2 \in \text{Syl}_p \underline{S}^{2p}$ with $T_1 \neq T_2$. Let $i \in \mathbb{N}$. Then

$$\underline{S}^{n_i} \leq \underline{S}^{n_i} \times \underline{S}^{2p} \leq \underline{S}^{n_{i+1}}.$$

We put $F_i := \underline{S}^{n_i} \times T_2$, if $S \cap \underline{S}^{2p} = T_1$, and $F_i := \underline{S}^{n_i} \times T_1$ otherwise. Then we have $S \cap F_i \notin \text{Syl}_p F_i$ and $\underline{S}^{n_i} \leq F_i \leq \underline{S}^{n_{i+1}}$. Hence $\{F_i \mid i \in \mathbb{N}\}$ is a (nested) local system for G containing no local subsystem of Σ into which S reduces. Thus S is a good Sylow p -subgroup of G which is not very good.

Fifth, the good Sylow p -subgroup P^0 of $V := G^{(\mathbb{N})}$ provides an example of a Sylow p -subgroup which is very good but not singular.

Let Σ^* be a local system for V ; by Lemma 2.2 a) there exists a nested local subsystem $\Sigma_1 = \{V_n \mid n \in \mathbb{N}\}$ of Σ and by Lemma 2.2 b) there is a Sylow p -subgroup Q of V which is good w.r.t Σ_1 . Since P^0 is good w.r.t. $\Sigma = \{U_i \mid i \in \mathbb{N}\}$, it will contain a conjugate of every finite p -subgroup P of V : there is a Σ -group $U = U(P)$ with $P \leq U$; let $R \in \text{Syl}_p U$ with $P \leq R$; by Sylow Theorem there is a $y \in U$ with $R^y = P^0 \cap U$; hence $P^y \leq P^0$. Therefore

$$(Q \cap V_n)^{x_n} \leq V_n^{x_n} \cap P^0$$

for some $x_n \in V$ ($n \in \mathbb{N}$). Thus $V_n^{x_n} \cap P^0$ is a Sylow p -subgroup of V_n and therefore $|P^0 \cap V_n| = |Q \cap V_n|$. It follows that $P^0 \cap V_n$ has the size of a Sylow p -subgroup of V_n ($n \in \mathbb{N}$), and consequently P^0 reduces into the subsystem Σ_1 of the given local system Σ^* .

The following core result may be very well-known but we can present a novel and shorter proof.

Theorem 3.8 (see [4]) *Let G be a locally finite group and let p be a prime. To any finite p -subgroup P of G shall pertain two finite p -subgroups P_1 and P_2 of G with $P \leq P_1 \cap P_2$ such that $\langle P_1, P_2 \rangle$ is not a p -group. Then there will exist a countable subgroup H of G with $|\text{Syl}_p H| = 2^{\aleph_0}$.*

PROOF — We construct recursively an infinite ascending chain

$$F_0 < F_1 < \dots < F_n < \dots$$

of finite subgroups of G and for every $n \in \mathbb{N}_0$ a set Σ_n of p -subgroups of F_n such that for every $n \in \mathbb{N}_0$ we have: (i) $|\Sigma_n| = 2^n$; (ii) every two Σ_n -groups do not generate a p -group; (iii) for $n \geq 1$ every Σ_{n-1} -group lies in at least two Σ_n -groups.

Let $F_0 := \langle 1 \rangle$ and $\Sigma_0 := \{\langle 1 \rangle\}$. Let $n \in \mathbb{N}$ and suppose

$$F_0 < F_1 < \dots < F_{n-1} \quad \text{and} \quad \{\Sigma_i \mid i < n\}$$

have already been constructed. We let Σ_n be the set of all finite p -subgroups P_1, P_2 of G such that $\langle P_1, P_2 \rangle$ is not a p -group and there exists exactly one Σ_{n-1} -group P with $P \leq P_1 \cap P_2$. From the properties (i)–(iii) of Σ_{n-1} and from the prerequisite on G then follow (i)–(iii) for Σ_n . Let F_n be the span of all Σ_n -groups. Hereafter F_n is a finite

subgroup of G with $F_{n-1} < F_n$. Let

$$H := \bigcup_{i \in \mathbb{N}_0} F_i.$$

Then H is a countable subgroup of G . Let \mathcal{M} be the set of all p -subgroups of G which are an ascending union of a chain

$$S_0 < S_1 < \dots < S_n < \dots$$

of finite p -subgroups $S_i \in \Sigma_i$ ($i \in \mathbb{N}_0$). According to (i) and (iii) we have $|\mathcal{M}| = 2^{\aleph_0}$ and according to (ii) any two \mathcal{M} -groups cannot generate a p -group. H contains every \mathcal{M} -group, so from the properties of \mathcal{M} (and the countability of H) it follows that $|\text{Syl}_p H| = 2^{\aleph_0}$. We have constructed an infinitely high (\aleph_0) tree of finite p -subgroups of G which branches properly at each location with proper inclusions and in which any two immediate successors of an arbitrary point do not generate a p -group. This tree has 2^{\aleph_0} many infinite branches. \square

We are ready to state and prove our *Charakterisierungssatz*.

Theorem 3.9 (see [4]) *Let G be a locally finite group and let p be a prime. The following properties are equivalent:*

- 1) G satisfies the strong Sylow Theorem for the prime p .
- 2) In every subgroup U of G every Sylow p -subgroup of U is singular.
- 3) Every countable subgroup H of G contains a p -uniqueness subgroup of H .
- 4) Every countable subgroup H of G contains a singular Sylow p -subgroup of H .
- 5) Every countable subgroup of G satisfies the Sylow Theorem for the prime p .
- 6) If H is a countable subgroup of G , then $|\text{Syl}_p H| < 2^{\aleph_0}$.

PROOF — 2) \Rightarrow 3) and 3) \Rightarrow 4) are clear. 4) \Rightarrow 5) is valid by Theorem 3.7, 5) \Rightarrow 6) is valid by Theorem 3.4, and 6) \Rightarrow 1) is valid by The-

orem 3.3. It remains to show $1) \Rightarrow 2)$.^{*} Assume $1)$ holds and let $U \leq G$. Then U satisfies the strong Sylow Theorem for the prime p . By Theorems 3.5 and 3.4 we have that $|\text{Syl}_p H| < 2^{\aleph_0}$ for any countable subgroup H of U . By Theorem 3.8 there is a finite p -subgroup P of U such that for all finite p -subgroups P_1 and P_2 of U with $P \leq P_1 \cap P_2$ the group $\langle P_1, P_2 \rangle$ is a p -group. By Proposition 2.3 it follows that P is a p -uniqueness subgroup of U . Let $S \in \text{Syl}_p U$ with $P \leq S$. Moreover, let $T \in \text{Syl}_p U$ and $x = x(T) \in U$ with $S = T^{x^{-1}}$. Then P^x is a p -uniqueness subgroup of U with $P^x \leq T$, and hence T is singular by means of P^x . \square

It would have been easier to show that Theorem 3.9 $1)$ implies that every Sylow p -subgroup S of an arbitrary subgroup U of G is very good. In fact, let Σ be a local system for U . By Lemma 2.2 a) there exists a nested local system Σ_1 of Σ , and by Lemma 2.2 b) there is a $T \in \text{Syl}_p U$ which reduces into Σ_1 . Since G satisfies the strong Sylow Theorem for the prime p , we find an $x \in U$ such that $S = T^x$. Let $\Sigma_2 := \{Y \mid Y \in \Sigma_1, x \in Y\}$. Then Σ_2 is a local subsystem of Σ into which S reduces: for $S \cap Y = T^x \cap Y = (T \cap Y)^x \in \text{Syl}_p Y$ when $Y \in \Sigma_2$.

Having proved our *Charakterisierungssatz*, we are now ready to prove the announced main theorem characterising the locally finite groups which satisfy the strong Sylow p -Theorem.

Theorem 3.10 *Let G be a locally finite group and let p be a prime. The following properties are equivalent:*

- 1) G satisfies the strong Sylow Theorem for the prime p .
- 2) Every subgroup S of G contains a finite p -subgroup which is singular in S .

PROOF — The result follows from a combination of Proposition 2.3 and Theorem 3.9. \square

^{*} In Theorem 1.5 of [10] (If the locally finite group G satisfies the strong Sylow Theorem for the prime p there exists a finite p -subgroup P which is singular in G), Kegel ingeniously constructs, by contradiction, an infinite (\aleph_0) tower of countable subgroups of G , such that none of the finite p -subgroups of a member can be singular in the upper next, whose union has 2^{\aleph_0} maximal p -subgroups and therefore contradicts Theorem 3.4.

4 Novel concepts for Sylow theory in (locally) finite groups

We end this paper with some further thoughts, a result, and some questions that could be quite useful for future researchers into Sylow theory in (locally) finite groups. The status quo of Sylow theory in locally finite groups has been beautifully summarised in [3] and [10]; here, a special place is occupied by the contributions of Brian Hartley (see [6],[7],[8]), who also contributed prodigiously to simple locally finite groups (see [9]). Concerning [9], which appeared posthumously, we notice that it does not cite [10] (not even in its list of 56 references). This is regrettable since Hartley states in his 1990 Mathematical Review of [10] the following: “If the simple locally finite group G satisfies the strong Sylow Theorem for the (even one) prime p , then G is linear. This depends on the classification of finite simple groups and an *assertion* about singular p -subgroups of classical groups. Another proof of this result has since been given by the reviewer (not yet published).” However, due to the tragic death of Brian Hartley on October 8, 1994, aged 55, this certainly very interesting proof was never prepared for publication. With someone of Hartley’s stature, there is no question that his word is good enough and that in any case he supplied a new proof with probably quite a number of new insights. It might therefore be worthwhile and even most desirable to inspect Hartley’s estate.

In every locally finite group G , for all subgroups U of G , the set $\text{Unique}_p U$ of finite p -subgroups which are p -uniqueness subgroups of U is non-empty if G satisfies the strong Sylow Theorem for the prime p , that is, if G belongs to the class $\text{Syl-}p$ of locally finite groups satisfying the strong Sylow Theorem for the prime p , and should this set be non-empty for a countable U then all the good Sylow p -subgroups of U are conjugate. Let U be finite. Then we have already $\text{Unique}_p U \neq \emptyset$ because we have $\text{Syl}_p U \leq \text{Unique}_p U$. The Sylow p -subgroups of U are of course the maximal members of $\text{Unique}_p U$, with respect to inclusion and order. It is a very very considerable challenge to try to determine the minimal members of $\text{Unique}_p U$, with respect to either inclusion or order, in case that U and $\text{Syl}_p U$ are sufficiently “known”, in particular if U is a “known” finite simple group or a p -soluble group. Note that whenever $P < Q < R$ are p -subgroups of U where Q is a minimal p -uniqueness subgroup, or will be minimal singular in U , then P is contained in at least two,

in fact in at least $p + 1$, Sylow p -subgroups of U and R will be another p -uniqueness subgroup of U . The author is much hoping that some progress be made to this challenge in the future. For example, the question of whether (resp. when) the minimal p -uniqueness subgroups are conjugate, quite similar to the maximal ones, is surely of some interest, or, whether minimal w.r.t. inclusion implies minimal w.r.t. order, the converse being clearly obvious. We would then also come to better know the p -uniqueness subgroups of locally finite groups, in particular the simple and the locally p -soluble ones, and, many thanks to Kegel's Theorem 4.4, of locally finite groups in general belonging to the lovely class $Syl\text{-}p$. A good starting point would be to study minimal p -uniqueness subgroups of the finite symmetric and alternating groups where a Sylow 2-subgroup of an alternating group is a next to maximal 2-uniqueness subgroup of the symmetric overgroup so that we have to study only the symmetric groups and to show at least that their ranks are "somehow" bounded in terms of a p -uniqueness subgroup and in ideal circumstances to determine all the minimal ones (see what follows).

Let G be a locally finite group, $S \in Syl_p G$ and $F \leq G$. We call F *minimal p -unique w.r.t. S* , if F is a minimal p -uniqueness subgroup of G w.r.t. order such that $F \leq S$, that is, F is p -unique with $F \leq S$ and each (finite) subgroup P of S with $|P| < |F|$ lies in at least two Sylow p -subgroups of G . If there exists an $S \in Syl_p G$, such that F is, w.r.t. S , minimal p -unique, then F is called *minimal p -unique (in G)*. Obviously, G is p -closed if and only if $\langle 1 \rangle$ is minimal p -unique (in G).

Theorem 4.1 (see [4]) *Let G be a locally finite group satisfying the strong Sylow Theorem for the prime p .*

- a) *Each Sylow p -subgroup of G contains at least one minimal p -unique subgroup of G .*
- b) *Each two minimal p -unique subgroups of G have the same order.*

PROOF — a) Let $S \in Syl_p G$ and let $U(G, S)$ be the set of all p -uniqueness subgroups F of G such that $F \leq S$. According to Theorem 3.9 we have $U(G, S) \neq \emptyset$ and of course each $U(G, S)$ -group has finite order. Thus $U(G, S)$ contains (w.r.t. S) a minimal p -unique subgroup due to the well ordering of \mathbb{N} .

b) Let F_1 and F_2 be two minimal p -unique subgroups of G . For symmetry reasons it suffices to show $|F_1| \leq |F_2|$. Let $S_1, S_2 \in Syl_p G$

with $F_1 \leq S_1$ and $F_2 \leq S_2$. Since $G \in \text{Syl-}p$ there is an $x \in G$ such that $S_1 = S_2^x$. Then F_2^x is a p -uniqueness subgroup of G with $F_2^x \leq S_1$. Thus $|F_1| \leq |F_2^x| = |F_2|$ since F_1 is minimal p -unique w.r.t. S_1 . \square

Let G be a locally finite group satisfying the strong Sylow p -Theorem and let $S \in \text{Syl}_p G$. According to Theorem 4.1 a) S contains (w.r.t. S) a minimal p -unique subgroup F . We define $a_p = a_p(G) \in \mathbb{N}_0$ by $|F| =: p^{a_p}$, that is, we let a_p be the composition length of F . According to Theorem 4.1 b) this definition is independent of the special choice of the Sylow p -subgroup S of G . Whereby consequently a_p is a (numeric) Sylow p -invariant of G . We call a_p the p -uniqueness of G . This Sylow p -invariant enqueues into the list — even is in the vanguard — of other Sylow p -invariants which play a major role in (locally) finite group theory, e.g. the order p^{b_p} of a Sylow p -subgroup, its nilpotency class c_p , its solubility length d_p , its exponent p^{e_p} , the composition length $i_p - 1$ of a proper maximal (w.r.t. order) Sylow p -intersection and further. The real peculiarity of a_p is that it is not determined by a Sylow p -subgroup as abstract p -group alone but depends on its embedding into the whole group and the respective relationships to the other Sylow p -subgroups. Then (w.r.t. inclusion or order maximal) intersections of two or several Sylow p -subgroups are of interest and deserve further study. For example, two core questions for Sylow theory in (locally) finite groups are how the p -length of a finite p -soluble group and the rank of a (known) finite simple group are bounded in terms of a p -uniqueness subgroup.

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REFERENCES

- [1] A.O. ASAR: "A conjugacy theorem for locally finite groups", *J. London Math. Soc.* (2) 6, No. 2 (1973), 358–360.
- [2] R. BAER: "Abzählbar erkennbare gruppentheoretische Eigenschaften", *Math. Z.* 79 (1962), 344–363.
- [3] M.R. DIXON: "Sylow Theory, Formations and Fitting Classes in Locally Finite Groups", *World Scientific*, Singapore (1994).
- [4] F.F. FLEMISCH: "Lokal endliche Gruppen mit Sylow p -Satz oder mit $\min-p$. I: Grundbegriffe, ein Charakterisierungssatz und lokale Prinzipien", *Diplomarbeit*, University of Freiburg, Germany (1984).
- [5] D. GORENSTEIN – R. LYONS – R. SOLOMON: "The Classification of the Finite Simple Groups, Part 1", *American Mathematical Society*, Providence, RI (2000).
- [6] B. HARTLEY: "Sylow subgroups of locally finite groups", *Proc. London Math. Soc.* (3) 23 (1971), 159–192.
- [7] B. HARTLEY: "Sylow p -subgroups and local p -solubility", *J. Algebra* 23 (1972), 347–369.
- [8] B. HARTLEY: "Sylow theory in locally finite groups", *Comp. Math.* 25 (1972), 263–280.
- [9] B. HARTLEY: "Simple locally finite groups", in: *Finite and Locally Finite Groups*, *Kluwer*, Dordrecht (1995), 1–44.
- [10] O.H. KEGEL: "Four lectures on Sylow theory in locally finite groups", in: *Group Theory*, *de Gruyter*, Berlin (1989), 3–28.
- [11] L.G. KOVÁCS – B.H. NEUMANN – H. DE VRIES: "Some Sylow subgroups", *Proc. Royal Soc. London, Series A* 260 (1961), 304–316.
- [12] A. RAE: "Local systems and Sylow subgroups in locally finite groups. I", *Proc. Cambridge Philos. Soc.* 72 (1972), 141–160.
- [13] A. RAE: "Local systems and Sylow subgroups in locally finite groups. II", *Proc. Cambridge Philos. Soc.* 75 (1974), 1–22.

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ISCHIA GROUP THEORY 2016 (see [45])



Otto H. Kegel & Andrea Caranti (Ischia, 2016) • courtesy of F. de Giovanni
<https://www.advgrouptheory.com/GTArchivum/Pictures/gtphotos/KegelCaranti.jpg>



Mahmut Kuzucuoğlu & Otto H. Kegel (Ischia, 2016) • courtesy of N. Vavilov
<https://www.advgrouptheory.com/GTArchivum/Pictures/gtphotos/kuzuKegel.jpg>

Appendix 2

Talk by Felix F. Flemisch at Ischia Group Theory 2024

- Introduction to the Talk by Felix F. Flemisch at IGT 2024 • The Mathematical Institute in Freiburg im Breisgau on April 11th, the 120th birthday of Philip Hall
- The 12 Slides of the Talk
- Professor Otto H. Kegel that's him all over
- The Ancient University City Freiburg im Breisgau
- Thank you very much for your Patient Attention

Introduction to the Talk by Felix F. Flemisch at IGT 2024 on April 11th, the 120th birthday of Philip Hall

This Talk of **only three minutes** was originally scheduled to take place in the first session at IGT 2024 by chairman **Prof. Dieter Kießch** on Tuesday from 9.20 to 11.15. It should in no way detract from his thorough Talk about **Prof. Kegel** (9.20-9.25) and from the Talks by **Mahmut Kuzunoğlu** (9.25-10.10) [who honoured Prof. Kegel brilliantly – **I thank Mahmut for the excellent work**], by Luise-Charlotte Kappe (10.10-10.50) and by Elena Burina resp. Viji Z. Thomas (10.50-11.15). The "Lecture" therefore should take place at the very end of the session just before the coffee break (11.15-11.18) thereby stealing three minutes from the break ... 😊
However, in the meantime this schedule was cancelled and I was kindly given the time slot from 18.05 to 18.30 on Thursday in the session by chairman **Alessio Rosso**, since **Dimitry Malinin** cannot come to our conference and **Natalia Maslov** moved to Wednesday. I present **first** this Introduction and **then** the **12 slides** of my **POSTER**.

My name is **Felix Flemisch**. I come from **Munich** in Bavaria in Germany. In the 1970ties and 1980ties I was a considerably busy and faithful student of **Prof. Otto H. Kegel** ♡ in such **beautiful** Freiburg i.Br. in Germany. In 2021 I luckily came again in contact with my adored teacher and met him in person and in good shape during June and July of 2022 in Freiburg. I present at IGT 2024 a **POSTER** about a new paper on **Sylow theory in simple locally finite groups** which is based on the very famous **Kegel covers** and on a **beautiful** paper of mine about rounding off **the general Sylow theory in locally finite groups**, friendly published by AGTA, under the rigid supervision of esteemed **Prof. Francesco de Giovanni** †. **Prof. Kegel** gave me kindly the hint to submit the paper to AGTA whose review process improved the paper very substantially so that it now can be the sound basis for further work on the subject.

Both papers have a quite strong relationship to **Prof. Kegel's work on Sylow theory, each one proving a conjecture of him** and centred around the quite gay concept of a **p -uniqueness subgroup** which is a finite p -subgroup being friendly contained in a **unique** Sylow p -subgroup. The **POSTER** shows the **twelve slides** of my talk as a PowerPoint presentation which include as well rather tough suggestions to stimulate and encourage future research. I much hope to enthuse group theorists with them and I am ready to support and coordinate related research work. This is my main interest why I present the **POSTER**. However, I am sadly aware that locally finite groups, and their Sylow theory in particular, seem not (yet) to be current topics of group theory research except some special questions presented on Tuesday. A limited number of nicely printed copies of the paper's **abstract**, its **POSTER** in DIN A3, and its **preprint** are available. I will deposit them tomorrow morning in SALA CARTAROMANA. An underlying **research paper** of this Talk will be published.

The Mathematical Institute in Freiburg im Breisgau



This is the **Mathematical Institute** at Albert-Ludwigs-University in **Freiburg im Breisgau** in Germany where from 1975 until 1999 **Prof. Kegel** occupied his chair, gave **beautiful** lectures and seminars, invited researchers over researchers, and hosted students in the morning **offering a cup of coffee** (or two) thereby doing careful supervision work and suggesting fascinating research topics.

Slide 1

The Strong Sylow Theorem for the Prime p in Simple Locally Finite Groups

Dipl.-Math. **Felix F. Flemisch**, M.Sc., Bacc.Math.



Dedicated to **Prof. Otto H. Kegel**
on the occasion of his 90th birthday on July 20

Ischia Group Theory 2024 from April 8 to April 13



Talk on Thursday, **April 11**, the 120th birthday of **Prof. Philip Hall**

THE STRONG SYLOW THEOREM FOR THE PRIME p
IN SIMPLE LOCALLY FINITE GROUPS



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Dedicated to Prof. Otto H. Kegel on the occasion of his 90th birthday

<http://www.advgrouptheory.com/GT/Archivum/Pictures/g1photos/OttoKegel.jpg>

This **Research Article** continues [15]. We begin with giving a profound overview of the structure of arbitrary simple groups and in particular of the simple locally finite groups and reducing their Sylow theory for the prime p to a quite famous conjecture by Prof. Otto H. Kegel (see [44], Theorem 2.4: “Let the p -subgroup P be a p -uniqueness subgroup in the finite simple group S which belongs to one of the seven rank-unbounded families. Then the rank of S is bounded in terms of P .”) about the rank-unbounded ones of the 19 known families of finite simple groups. We introduce a new scheme to describe the 19 families, the family \mathcal{T} of types, define the rank of each type, and emphasise the rôle of Kegel covers. Prof. Kegel rediscovered from Prof. Philip Hall (see [46]) that an infinite simple group has a local system consisting of countably infinite simple subgroups (see [45], [46] and [44], Theorem 2.5) (and conversely) and if they are locally finite he discovered groundbreakingly that they have a Kegel cover (see [44], Theorem 2.6), that is, a nested local system $\{G_n\}$ with maximal normal subgroups $M_n \leq G_n$, such that $G_n \cap M_{n+1} = \langle 1 \rangle$ so that G_n embeds into G_{n+1}/M_{n+1} . This part of the Research Article presents a unified rather complete picture of known results all of whose proofs are by reference.

We then apply **new ideas** to prove the conjecture for **the Alternating Groups**.

Thereupon we are remembering Kegel covers and \star -sequences and the classification of simple locally finite groups according to their Kegel covers. Next we suggest a way 1) and a way 2) how to prove and even how to optimise Kegel’s conjecture step-by-step or peu à peu which leads to Conjecture 1, Conjecture 2 and Conjecture 3 thereby unifying Sylow theory in locally finite simple groups with Sylow theory in locally finite and p -soluble groups whose joint study directs very reliably Sylow theory in (locally) finite groups. For any unexplained terminology we refer to [15].

We then continue the program begun above to optimise along the way 1) the theorem about the first type $\Xi = \underline{A}^n$ of infinite families of finite simple groups step-by-step to further types by proving it for the second type $\Xi = \underline{A} = \text{PSL}_n$. We apply **new ideas** to prove Conjecture 2 about **the General Linear Groups** over locally finite fields, stating that their rank is bounded in terms of their p -uniqueness, and then break down this insight to **the Special Linear Groups** and **the Projective Special Linear (PSL) Groups** over locally finite fields. We close with good suggestions for future research ▶ regarding the remaining five rank-unbounded types (the “Classical Groups”) and the way 2), ▶ regarding (locally) finite and p -soluble groups, and ▶ regarding **our new perceptions** of the pioneering contributions by Cauchy and by Galois to Sylow theory in finite groups. We much hope to enthuse group theorists with these suggestions and are ready to contribute to, to support and to coördinate all related work.

It follows from our two theorems that simple locally finite groups which satisfy the Strong Sylow theorem for even one Prime p are linear and hence countable if they have a local system of countable simple subgroups each having a Kegel cover “of alternating type” or “of projective special linear type”.

References

- [15] F.F. FLEMISCH: “Characterising Locally Finite Groups Satisfying the Strong Sylow Theorem for the Prime p ”. *Adv. Group Theory Appl.* **13** (June 2022), 13–39. <https://www.advgrouptheory.com/Journal/Volumes/13/flemisch.pdf>.
- [44] O.H. KEGEL: “Four lectures on Sylow theory in locally finite groups”. In: Group Theory, Proceedings of the Singapore Group Theory Conference held at the National University of Singapore, June 8–19, 1987, *Walter de Gruyter & Co., Berlin & New York* (January 1989, reprinted November 2016), 3–27. ISBN 3-11-011366-X. <https://www.degruyter.com/view/book/9783110848397/10.1515/9783110848397-004.xml>.
- [45] O.H. KEGEL: “Remarks on uncountable simple groups”. In: Proceedings of Ischia Group Theory 2016, *Int. J. Group Theory* **7** (March/June/September 2018). http://www.dipmat2.unisa.it/ischiagroupttheory/IGT2016/home_2016.html.

Slide 3

The Strong Sylow Theorem for the Prime p in Simple Locally Finite Groups

DIPLO.-MATH. FELIX F. FLEMISCH, M.Sc., BACC.MATH.

Dedicated to Prof. Otto H. Kegel on the occasion of his 90th birthday
Ischia Group Theory 2024 from April 8 to April 13

Let p be a prime: 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97, 101, 103, 107, 109, 113, 127, 131, 137, 139, 149, 151, 157, 163, 167, 173, 179, 181, 191, 193, 197, 199, 211, 223, 227, 229, 233, 239, 241, 251, 257, 263, 269, 271, 277, 281, 283, 293, 307, 311, 313, 317, 331, 337, 347, 349, 353, 359, 367, 373, 379, 383, 389, 397, 401, 409, 419, 421, 431, 433, 439, 443, 449, 457, 461, 463, 467, 479, 487, 491, 499, 503, 509, 521, 523, 541, 547, 557, 563, 569, 571, 577, 587, 593, 599, 601, 607, 613, 617, 619, 631, 641, 643, 647, 653, 659, 661, 673, 677, 683, 691, 701, 709, 719, 727, 733, 739, 743, 751, 757, 761, 769, 773, 787, 797, 809, 811, 821, 823, 827, 829, 839, 853, 857, 863, 877, 881, 883, 887, 907, 911, 919, 929, 937, 941, 947, 953, 967, 971, 977, 983, 991, 997, 1009, 1013, 1019, ...

[44] O.H. KEGEL: "Four lectures on Sylow theory in locally finite groups", in: Group Theory,

Walter de Gruyter, Berlin & New York (1989), 3-27 (see MR0981832 [MR 90c:20037] and Zbl 0659.20024).

In his four workshop lectures on Sylow theory in locally finite groups at the Singapore Group Theory Conference of June 1987 (see [44]), Prof. Kegel stated as a theorem and proved "by inspection" what is actually a conjecture:

Theorem 2.4 "Let the p -subgroup P be a p -uniqueness subgroup in the finite simple group S which belongs to one of the seven rank-unbounded families. Then the rank of S is bounded in terms of P ."

The family \mathcal{T} of types of known finite simple groups {abelian p ,

$$\Delta^n, A = \text{PSL}_n, B = \text{P}\Omega_{\text{odd } n}, C = \text{PSp}_n, D = \text{P}\Omega_{\text{even } n}, {}^2A = \text{PSU}_n, {}^2D = \text{P}\Omega_{\text{even } n},$$

$E_6, E_7, E_8, F_4, G_2, {}^3B_2, {}^3D_4, {}^2E_6, {}^2F_4, {}^2G_2$, sporadic \star } is beautiful.

It contains 18 infinite families and one finite family:

the abelian groups, seven rank-unbounded (infinite) families,

ten infinite families with a fixed rank, and 26 sporadic groups.

In this paper we prove Kegel's conjecture for Δ^n and for $A = \text{PSL}_n$.

It continues [15] F.F. FLEMISCH: "Characterising Locally Finite Groups

Satisfying the Strong Sylow Theorem for the Prime p ", Adv. Group Theory

Appl. 13 (June 2022), 13-39 (see MR0981832 and Zbl 0659.20024).

We have included that beautiful predecessor paper as Appendix 1,

although it is open access, since this paper cannot be understood without

that predecessor paper – so one needs to have it present when reading this paper –

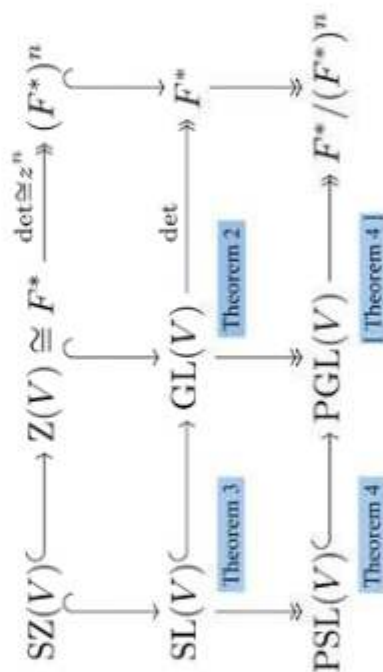
and included as well the MR Review and the Zbl Review and an important comment.

Slide 4

We sketch the proof for \underline{A}^n . Let the finite p -group P act on \underline{A}^n . Let α be a point and $P_\alpha := \{x \in P \mid \alpha^x = \alpha\} \subseteq P$ be the stabiliser of α . We denote by $\mathbf{U}(P)$ the set of all subgroups of P and for $U \in \mathbf{U}(P)$ by $\mathbf{R}(P, U) := \{Ux \mid x \in P\}$ the set of all right cosets of U in P . Then P operates by multiplication from the right for $U \in \mathbf{U}(P)$ transitively on $\mathbf{R}(P, U)$ with $\text{Cor}_P U := \cap \{U^x \mid x \in P\}$ as kernel.

The **classification of transitive P -sets** reads as follows (see [48], Chapter 6): *Every transitive P -set $\Omega \neq \emptyset$ is P -isomorphic to $\mathbf{R}(P, P_\alpha)$ for all $\alpha \in \Omega$, and for any $U, V \in \mathbf{U}(P)$ the two sets $\mathbf{R}(P, U)$ and $\mathbf{R}(P, V)$ are P -isomorphic if and only if U and V are conjugate in P .* Hence for the action of P we have a bijection between the class $\mathcal{J}(P)$ of all P -isomorphism types of transitive P -sets and the set of all conjugacy classes (in P) of subgroups of P , and so $|\mathcal{J}(P)| = g_p(|P|) :=$ the number of conjugacy classes of subgroups of P . Hence for every P -set Ω the class $\mathcal{J}(P, \Omega)$ of P -isomorphism types of P -orbits on Ω has at most $g_p(|P|)$ elements and since every subgroup of P is a subset containing 1, we can summarise $|\mathcal{J}(P, \Omega)| \leq g_p(|P|) \leq |\mathbf{U}(P)| \leq 2^{|P|-1}$. If P is a p -subgroup of \underline{S}^n , which is contained in exactly $k \in \mathbb{N}$ Sylow p -subgroups of \underline{S}^n , and if $m := k + p + 1$, then $n \leq m \cdot |P| \cdot g_p(|P|) - 1 \leq m \cdot |P| \cdot 2^{|P|-1} - 1$, and in particular $n \leq (p + 2) \cdot |P| \cdot 2^{|P|-1} - 1$ for $k = 1$ (see **Page 5** of the **Research Article**), whence, if not so, P has at least m many P -isomorphic P -orbits on $\Omega := \{1, 2, \dots, n\}$ (see **Page 5**). We deduce from this basic fact the central observation that $\{S \in \text{Syl}_p \underline{S}^n \mid S \text{ is } P\text{-invariant}\} =: \text{Syl}_p(\underline{S}^n, P) \geq |\text{Syl}_p \underline{S}^m| \geq m - 2 \geq k + 1$ by using **beautiful new ideas** (see **Page 6**). \square

We sketch the proof for $\mathbf{A} = \text{PSL}_n$. We apply a three-stage-approach whilst **first** proving the theorem for the **General Linear Groups** over (commutative) locally finite fields (**Theorem 2**), **then** for the **Special Linear Groups** over locally finite fields (**Theorem 3**) and **finally** for the **Projective Special Linear (PSL) Groups** over locally finite fields (**Theorem 4**), thereby using $\text{GL}(n, F) = \text{SL}(n, F) \bullet F^*$ and $\text{PSL}(n, F) = \text{SL}(n, F) / Z(\text{SL}(n, F))$. This approach can be presented with a **beautiful** diagram:



Slide 5

The major work is required for the **General Linear Groups** with two different and both **beautiful new ideas** for characteristic $\neq p$ and characteristic p . In characteristic $\neq p$ we use that, if for a finite p -group P operating on a finite-dimensional vector space V over a locally finite field and a direct decomposition of V into irreducible P -submodules, there are k many of the P -submodules P -isomorphic, then at least $|Syl_p \underline{S} \mid$ Sylow p -subgroups of $GL(V)$ are P -invariant (see **Proposition 7 a**). In characteristic p we use that, if k is the dimension of the P -submodule $C_V(P) := \{v \in V \mid v^x = v \text{ for all } x \in P\}$ of a non-trivial modular P -module V , then again there are at least $|Syl_p \underline{S} \mid$ many P -invariant Sylow p -subgroups of $GL(V)$ (see **Proposition 7 b**). We then argue that from **Proposition 7** follows that $n \leq (p+2) \cdot |P|^2 - 1$ for a p -uniqueness subgroup P of $GL(n, F)$ (see **Lemma 2** on **Page 11**).

For the transition from $GL(n, F)$ to $SL(n, F)$ we are using that a p -uniqueness subgroup of $SL(n, F)$ is a p -uniqueness subgroup of $GL(n, F)$ as well. From $SL(n, F)$ to $PSL(n, F)$ we use that $P := Q \cdot D(SL(n, F)) / D(SL(n, F))$ is a p -uniqueness subgroup of $PSL(n, F)$ when Q is a p -uniqueness subgroup of $SL(n, F)$, and conversely, together with **Proposition 4** and **Proposition 6**. □

Let G be a countably infinite locally finite simple group. Then there will exist a nested local system $\{R_n \mid n \in \mathbb{N}\}$ for G of finite subgroups such that for each $n \in \mathbb{N}$ the group R_n is perfect and there exists a maximal normal subgroup M_{n+1} of R_{n+1} satisfying $M_{n+1} \cap R_n = \langle 1 \rangle$, whence R_{n+1} / M_{n+1} is simple and $R_n \cong R_{n+1} / M_{n+1}$; such a nested local system is called **Kegel cover** for G . We call G to be of **type** $\Xi \in \mathcal{T}$, if it has a Kegel cover $\Sigma = \{(R_k, M_k) \mid k \in \mathbb{N}\}$ such that infinitely many R_{k+1} / M_{k+1} 's belong to Ξ (wherefore we can replace Σ by these infinitely many R_{k+1} 's), and call G to be of **alternating type** if it is of type \underline{A}^n and to be of **projective special linear type** if it is of type $A = PSL_n$.

Theorem 1 (see [14]) *Let $n \in \mathbb{N}$ and let p be a prime such that $p \leq n$. Let P be a finite p -group acting on \underline{A}^n . Let $g_p(|P|)$ be the number of conjugacy classes of subgroups of P and let k be the number of P -invariant Sylow p -subgroups of \underline{A}^n . Then $g_p(|P|) \leq 2^{|P| - 1}$.*

a) *If isomorphic subgroups of P are conjugate and $b := \log_p |P|$ (so that $|P| = p^b$), then*

$$g_p(|P|) \leq p((b-2)^4 + 2(b-2)^3 + (b-2)^2 + (b-2) + 1) / 4 - ((b-2)^2 + b - 2) / 2 - 90 + (|P|-1) / (p-1) + 25.$$

b) *Let $m := k + p + 1$. Then $n \leq m \cdot |P| \cdot g_p(|P|) - 1$. If $k = 1$, then $n \leq f_p(|P|) := (p+2) \cdot |P| \cdot 2^{|P| - 1} - 1$.*

A periodic linear group G is locally finite and satisfies the strong Sylow Theorem for every prime p , and hence $a_p(G)$ is defined (see **Slide 7** below). We first prove **Conjecture 2** (see **Slide 7**) regarding the **General Linear Groups** over locally finite fields:

Slide 6

Theorem 2 Let $n \in \mathbb{N}$ and let p be a prime. Let \mathcal{F} be a locally finite (commutative) field.

- a) If \mathcal{F} has characteristic p and $a_p = a_p(\text{GL}(n, \mathcal{F}))$ then $n \leq (p+2) \cdot p^{ap} - 1$.
- b) If \mathcal{F} has characteristic $\neq p$ and $a_p = a_p(\text{GL}(n, \mathcal{F}))$ then $n \leq (p+2) \cdot p^{2ap} - 1$.

We then break down **Theorem 2** to the **Special Linear Groups** over locally finite fields:

Theorem 3 Let $n \in \mathbb{N}$ and let p be a prime. Let \mathcal{F} be a locally finite (commutative) field.

- a) If \mathcal{F} has characteristic p and $a_p = a_p(\text{SL}(n, \mathcal{F}))$ then $n \leq (p+2) \cdot p^{ap} - 1$.
- b) If \mathcal{F} has characteristic $\neq p$ and $a_p = a_p(\text{SL}(n, \mathcal{F}))$ then $n \leq (p+2) \cdot p^{2ap} - 1$.

We continue with breaking down **Theorem 3** to the **Projective Special Linear (PSL) Groups** over locally finite fields:

Theorem 4 Let $n \in \mathbb{N}$ and let p be a prime. Let \mathcal{F} be a locally finite field and P be a minimal p -unique subgroup of $\text{PSL}(n, \mathcal{F})$ so that $|P| = p^p$.

- a) If \mathcal{F} has characteristic p and $a_p = a_p(\text{PSL}(n, \mathcal{F}))$ then $n \leq f_p(|P|) := (p+2) \cdot p^{ap} - 1$.
- b) If \mathcal{F} has characteristic $\neq p$ and $a_p = a_p(\text{PSL}(n, \mathcal{F}))$ then $n \leq f_p(|P|) := (p+2) \cdot p^{2ap} - 1$.

Let G be an infinite simple group. G has a local system consisting of countably infinite simple subgroups (see [45] O.H. KEGEL: "Remarks on uncountable simple groups". In: Proceedings of Ischia Group Theory 2016, Int. J. Group Theory 7 [March/June/September 2018]). Let each of these be locally finite of **alternating type** or of **projective special linear type**. Then **Theorem 1** and **Theorem 4** imply the following intriguing consequences of the Strong Sylow Theorem for the Prime p :

Theorem 5 Let G be a simple locally finite group of **alternating type** or of **projective special linear type**

satisfying the Strong Sylow Theorem for the even one Prime p . Then G is linear and countable. □

Planning future research

Our **Theorem 1** could be optimised in two ways:

- 1) Extend it from type \underline{A}^n **step-by-step** to further types Ξ with an appropriate (similar) function f_p , that is, the rank $r(G)$ of a group G of type Ξ is bounded by $f_p(|P|)$ for a p -uniqueness subgroup P of G .
- 2) Determine for the type \underline{A}^n and **peu à peu** for further types Ξ the minimal p -unique subgroups, that is, the p -uniqueness subgroups of the non-abelian simple groups of type \underline{A}^n and of type Ξ which are minimal with respect to order (see [15]).

Slide 7

Let G be a locally finite group satisfying the strong Sylow p -Theorem and let $S \in \text{Syl}_p G$. Then S contains some (w.r.t. S) minimal p -unique subgroup F . We define $a_p = a_p(G) \in \mathbb{N}_0$ by $|F| =: p^{a_p}$, that is, we let a_p be the composition length of F . This definition is independent of the choice of the Sylow p -subgroup, so a_p is a (numerical) Sylow p -invariant of G . We call a_p the **p -uniqueness of G** .

Conjecture 1 Let $\mathcal{T} := \{\text{abelian } p, \mathbb{A}^n, A = \text{PSL}_n, B = \text{P}\Omega_{\text{odd } n}, C = \text{PSp}_n, D = \text{P}\Omega_{\text{even } n}^-, E = \text{PSU}_n, {}^2D = \text{PSU}_n, {}^2D = \text{P}\Omega_{\text{even } n}, E_6, E_7, E_8, F_4, G_2, {}^2B_2, {}^3D_4, {}^2E_6, {}^2F_4, {}^2G_2, \text{sporadic}_*\}$ be the family of types of known finite simple groups and let G be a finite simple group of type $\Xi \in \mathcal{T}$. Then the $\text{rank}_p(G)$ of G is bounded in terms of the p -uniqueness $a_p(G)$.

Conjecture 2 Let $n \in \mathbb{N}$ and let p be a prime. Let F be a locally finite (commutative) field.

- a) If F has characteristic p and $a_p = a_p(\text{GL}(n, F))$ then $n \leq (p+2) \cdot p^{a_p} - 1$.
- b) If F has characteristic $\neq p$ and $a_p = a_p(\text{GL}(n, F))$ then $n \leq (p+2) \cdot p^{2a_p} - 1$.

We give a brief attention to (locally) p -soluble groups since it is the reliable joint study of the (locally) simple and the (locally) p -soluble groups which directs the Sylow theory in (locally) finite groups. The central observation is the following best possible claim:

Conjecture 3 Let p be a prime. Let G be a p -soluble finite group, $\lambda_p(G)$ be its p -length, and $a_p(G)$ be its p -uniqueness. Then $\lambda_p(G) \leq a_p(G) + 1$.

The **classical Hall-Higman theory**, created by P. Hall, G. Higman, A.H.M. Hoare, T.R. Berger, F. Gross and E.G. Bryukhanova, introduces for finite p -soluble groups (best possible) inequalities between their p -length λ_p and the order p^{b_p} of a Sylow p -subgroup, its nilpotency class c_p , its solubility length d_p , its exponent p^{e_p} , or the rank r_p of a maximal elementary abelian subgroup. Our true and ambitious aim is to **extend the Hall-Higman theory** to the



beautiful p -uniqueness p^{a_p} of a Sylow p -subgroup, a truly **Herculean**

endeavour 😊. The real challenge is to prove

Conjecture 3. It is much expected that the cases $p \geq 5$, $p = 3$ and $p = 2$ must be treated fairly separately and that $p = 3$ and $p = 2$ will require rather special methods as is already indicated by the available literature.

Slide 8

Our proofs of **Conjecture 1** for the types \mathbf{A}^n and $\mathbf{A} = \text{PSL}_n$, that is, to carve out **the optimising way 1**, are characterised by the fact that we *need not at all know their Sylow p -subgroups*. There is no doubt that we can extend those proofs straightforwardly to the further five classical groups $\mathbf{B} = \text{P}\Omega_{\text{odd},n}$, $\mathbf{C} = \text{PSP}_n$, $\mathbf{D} = \text{P}\Omega_{\text{even},n}$, $\mathbf{A} = \text{PSU}_n$ and $\mathbf{D} = \text{P}\Omega_{\text{even},n}$ by considering the respective bilinear form defining these groups of Lie type, resp. the vector spaces they act upon as isometries, and their resulting Sylow p -subgroups (*without knowing them*). They can well be considered proved which we shall confirm in the **follow-up paper** “**The Strong Sylow Theorem for the Prime p in the Locally Finite Classical Groups**” considering the locally finite classical groups which are *the linear, symplectic, unitary and orthogonal groups* over locally finite fields. The linear groups are dealt with in this paper and the others are subgroups of the linear groups which are defined through a non-singular bilinear form (or scalar product) being either skew-symmetric (or alternate) or Hermitian or symmetric (defining a quadratic form) as *the group of isometries of the form*. They were nicely introduced to us in the classical books [1] and [58] and are further studied in [6], [24] and [50]. We do not refer to the groups of Lie type resp. the Chevalley groups and the twisted Chevalley groups defined through a Dynkin diagram automorphism followed by a field automorphism, which correspond to the classical groups (see [24], pp. 151-152) and whose fine introductory references are the “Lecture Notes on Chevalley Groups” by **Robert Steinberg** (1967 and 2016) together with the book “Simple Groups of Lie type” by **Roger W. Carter** (1972 and 1989). Therefore we study $\text{P}\Omega_{\text{odd},n}$, PSP_n , $\text{P}\Omega_{\text{even},n}$, PSU_n and $\text{P}\Omega_{\text{even},n}$ and not \mathbf{B} , \mathbf{C} , \mathbf{D} , \mathbf{A} and \mathbf{D} . Consequently the proofs for **the further five types of Classical Groups** can and will be based successfully on our **very beautiful Theorem 2** about the **General Linear Groups**. We are preparing to publish our first follow-up paper in 2025.

Our **second follow-up paper** “**The Strong Sylow Theorem for the Prime p in Locally Finite and p -Soluble Groups**” considers (locally) finite and p -soluble groups. It summarises the work by **B. Hartley** and **A. Rae** regarding λ_p and p^{ap} (see **Page 37** of [15] and the **References** of [44]) and the foregoing work on Hall-Higman theory regarding λ_p and p^{bp} , \mathbf{c}_p , \mathbf{d}_p , p^{cp} and \mathbf{r}_p by **P. Hall**, **G. Higman**, **A.H.M. Hoare**, **T.R. Berger**, **F. Gross**, **E.G. Bryukhanova** and last but not least by **A. Turell** as indicated on **Page 8** and **Page 9**. It then proves **Conjecture 3** (see the **Slide 7** above) not only in **English** but partly in **Portuguese** for historical reasons.

Our **beautiful third follow-up paper** “**Augustin-Louis Cauchy’s and Évariste Galois’ Contributions to Sylow Theory in Finite Groups**” pays sincere tribute to **Augustin-Louis Cauchy’s** and **Évariste Galois’** pioneering contributions to Sylow theory in finite groups by working out their new perceptions. It proves in a unified way **Lagrange’s theorem** and **Cauchy’s concealed second and third group theorems** by exploring three **beautiful** rectangles/tableaux. We show the second rectangle and the third tableau to raise inquisitiveness:

Slide 9

set of certain orbits of H under G acting by left translation	the first row consists of all right cosets of G in H with the powers of some p -blank x_1 of G in H ; the following rows consist of right cosets of G in H with the powers of left conjugates of x_1	$X = \langle x_1 \rangle$; set of all orbits of H under $G \cup X$; the simultaneous actions of G by left translation and of X by right translation	correspondence	set of certain orbits of H under G acting by left translation	the first row consists of all right cosets Gx_{1c} of G in H ($0 \leq c \leq H _{p-1}$) with the elements of some Sylow p -subgroup X of H , all of whose elements of order p are p -blanks of G in H ; the following rows consist of right cosets of G in H with the elements of left conjugates of X	correspondence	$ X = H _p = p^b$; set of all orbits of H under $G \cup X$; the simultaneous actions of G by left translation and of X by right translation
$Gx_1^0 t_1 = G$	Gx_1	Gx_1^2	...	Gx_1^{p-1}	Gx_1	...	Gx_1^{p-1}
$Gx_2^0 t_2 = Gt_2$	$Gx_2 t_2$	$Gx_2^2 t_2$...	$Gx_2^{p-1} t_2$	$Gx_2 t_2$...	$Gx_2^{p-1} t_2$
$Gx_3^0 t_3 = Gt_3$	$Gx_3 t_3$	$Gx_3^2 t_3$...	$Gx_3^{p-1} t_3$	$Gx_3 t_3$...	$Gx_3^{p-1} t_3$
...
$Gx_5^0 t_5 = Gt_5$	$Gx_5 t_5$	$Gx_5^2 t_5$...	$Gx_5^{p-1} t_5$	$Gx_5 t_5$...	$Gx_5^{p-1} t_5$

For an outline of this very beautiful paper see Page 13 and Page 14 of the Research Article and Slide 11.



Siamo angeli con un'ala soltanto e possiamo volare solo restando abbracciati. [Italian]
 We are angels who have but a single wing and we can only fly if we cling to one another. [English]
 Wir sind Engel mit nur einem Flügel, um fliegen zu können müssen wir uns umarmen. [German]
 Nous sommes des anges à une seule aile, nous ne pouvons voler qu'en restant enlacés. [French]
 Somos ángeles con una única ala y sólo podemos volar abrazados. [Spanish]
 Nós somos anjos com apenas uma asa e só podemos voar quando nos abraçamos. [Portuguese]



(★ 18 August 1928 in Naples until ☞ 18 July 2019 in Rome)

Così parlò Bellavista. Napoli, amore e libertà.

XXIII Piedigrotta. 1977 e settembre 2019

Luciano De Crescenzo

Slide 10

A MATHEMATICIAN, like a painter or a poet, is a maker of patterns. If his patterns are more permanent than theirs, it is because they are made with *ideas*. ... The mathematician's patterns, like the painter's or the poet's, must be *beautiful*; the *ideas*, like the colours or the words, must fit together in a harmonious way. *Beauty* is the first test: there is no permanent place in the world for ugly mathematics.
Godfrey Harold Hardy. A Mathematician's Apology. § 10. July 18, 1940.

L'autore è appassionatamente curioso del futuro.
The author is passionately curious about the future.
Der Autor ist sehr leidenschaftlich neugierig auf die Zukunft.
L'auteur est passionnément curieux de l'avenir.
O autor é muito apaixonadamente curioso sobre o futuro.
Felix Fortunatus Flemisch. Firenze. April 11, 1992.

The Research Article has the following seventeen beautiful Chapters: Sketch of proof for $A = \text{PSL}_n$;
1 Introduction; 2 Proof of Theorem 1; 3 About Kegel covers; 4 Planning future research – Part 1; 5 Proof of Theorem 2;
6 Proof of Theorem 3; 7 Proof of Theorem 4; 8 Planning future research – Part 2;
9 The First Trilogy and The Second Trilogy and their reviews; Acknowledgements;
Postscript, Luciano De Crescenzo, Felix F. Flemisch, Conflicts of Interest, Pablo Picasso's *La Joie de vivre*;
About the author in Munich, in Freiburg i.Br., in London, in Weiden i.d.OPf., and in Florence in Tuscany in Italy;
75 References; Appendix 1 – Reference [15] with MR Review and Zbl Review;
Appendix 2 – Talk by Felix F. Flemisch at Ischia Group Theory 2024.

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Slide 11

We are planning to revise thoroughly Sylow theory starting with a **really new proof** for Cauchy's known as fundamental theorem in group theory (look at [https://en.wikipedia.org/wiki/Cauchy%27s_theorem_\(group_theory\)](https://en.wikipedia.org/wiki/Cauchy%27s_theorem_(group_theory))) based on **beautiful** ideas by Galois. In the forthcoming (third) follow-up **Research Article** "Augustin-Louis Cauchy's and Evariste Galois' Contributions to Sylow Theory in Finite Groups" beyond our **First Trilogy** (look at the **Postscript** on **Page 15**) we first describe and then provide new but classical and rather unified proofs for the fundamental theorems by Lagrange and by Cauchy on finite groups of – in our modest opinion – considerable historical relevance.

We can describe in detail consequences of **the absence of group elements of prime order p** , in spite of their availability in overgroups, thereby providing a much unified and also heretofore undiscovered approach to the theorems of Lagrange and of Cauchy and their implications for p -groups. Since this approach uses only ideas from a very well-known paper by **Augustin-Louis Cauchy** presented first in 1812 and then published in 1815, this bears considerable historic relevance. While it is widely acknowledged that Cauchy had **published** his fundamental group theorem not until 1845/1846 and had there based it on double cosets of the finite permutation group and some Sylow p -subgroup of its symmetric overgroup, one could henceforth well argue that he had **presented** his theorem **in a truly concealed way** already a good thirty years earlier. **Evariste Galois** knew both Cauchy's paper of 1815 and – based on his own rather perceptive considerations – Cauchy's group theorem and even already Sylow's existence theorem. Cauchy's and Galois' ideas are particularly lucid in the embryonic case of permutation groups of prime degree $p (\geq 5)$ when Sylow p -subgroups of the symmetric overgroup obviously exist. If $G \sqsubseteq H$ with H being finite, then **the unified method of proof** consists in arranging the elements of H in a **rectangle** with $|G|$ columns and $[H : G]$ rows resp. the (right) cosets of G in H in a rectangle with p resp. with $|H|_p$ columns and $[H : G] / p$ resp. $[H : G] / |H|_p$ rows to obtain information about $[H : G]$ (see the three rectangles above).

Cauchy's theorem of 1812/1815 is a direct consequence of $[H : \langle x \rangle] \geq |G|$ if x is an element of H of prime order p with $x \notin G$ which we call a **p -blank of G in H** . We find that Lagrange's theorem and Cauchy's theorem are just like two sides of a coin where "Lagrange" represents the case $p^0 = 1$ and "Cauchy" the case $p^1 = p$ thereby offering a unified approach to both theorems. Hence, "Cauchy" is not only a partial converse of "Lagrange" but it is in fact a smart "swapping" of p for 1 as well: $p = 1 \Leftrightarrow p = p$.

Cauchy depicts 1815 a p -cycle for some prime p as a regular p -gon and studies p -cycles in considerable detail.

We present Cauchy's **classical proof** of **Lagrange's theorem** and supplement it with a **modern proof**. We then present Cauchy's **classical proofs** of his **published first theorem**, of his **concealed second theorem** and of his **concealed third theorem**. Subsequently we introduce double cosets and show how they lead to a **modern proof** of Cauchy's second and third theorems what Cauchy did **beautifully** as well but not until 1845/1846 after reconsidering, impressed by a paper of **Joseph Bertrand**, his work of 1812/1815, that is, after – believe it or not – 30 years.

We continue with **first** correcting a great misunderstanding of Cauchy's work of 1845/1846 in the literature and **then** presenting Cauchy's work of 1812/1815 in the sincere succession of the earlier work of **Joseph-Louis de Lagrange** (Giuseppe Luigi Lagrangia), **Alexandre-Théophile Vandermonde** and the pioneer **Paolo Ruffini**, as indicated by Cauchy himself, and identify the crucial parts of Cauchy's first publication on group theory. **Finally** we present what **Evariste Galois** knew already about **Cauchy's group theorems** and about **Sylow's famous theorems** by referring to his published papers and to his posthumously published papers as well. However, this requires rather considerable further (historical) research. We would be inestimably delighted if several group theory researchers would help us with this tedious but very suspenseful work and are ready to coordinate all the work. We are closing with fairly comprehensive **Acknowledgements** and a greatly sizeable list of **References**.



Augustin-Louis Cauchy

(21 August 1789 until 23 May 1857)



Evariste Galois

(25 October 1811 until 31 May 1832)

About the author

Felix F. Flemisch was born on 17 May 1951 in wonderful **Munich** in **Bavaria** in Germany. In **June 1971** he received his **Abitur** 😊 whose subject Mathematics was taught in a pioneering spirit by **Dr. Helmut Bergold**. Afterwards he received his first-ever degree **Baccalaureus der Mathematik (Bacc.Math.)** in **July 1974** with the alias unpublished **very beautiful bachelor's thesis** "Über einfache Punkte affiner Varietäten" from the most venerable Albert-Ludwigs-Universität at **such beautiful Freiburg im Breisgau** in **green** Baden-Württemberg in Germany under the rather thorough supervision of **Akadem. Rat Dr. Herbert Götz**, and then his degree **Master of Science (M.Sc.)** from the Faculty of Science of the highly esteemed University of **London**, Bedford College, United Kingdom, in **August 1975** under the supervision of greatly adored **Prof. Paul Moritz Cohn**. From October 1975 until – very regrettably – only July 1976 he was employed as **a fairly diligent Teaching Assistant with two graduations** by the hoar Mathematische Fakultät of **Freiburg i.Br.**'s Albert-Ludwigs-Universität. Subsequently he quite enthusiastically continued his postgraduate mathematical studies in such marvellous and fabulous **Freiburg i.Br.** – with decent interruptions as **a teacher** and as **a tutor** – and received his degree **Diplom-Mathematiker (Dipl.-Math.)** in **April 1985** under the impressive supervision of **Prof. Otto Helmut Kegel**. The Research Article [15] publishes the most essential and partly well corrected portions of his German **Diplomarbeit** [14] of **October 1984** and a "sprinkling" of new considerations and results as well which try to propose coming directions of research for Sylow theory in (locally) finite groups. The publication at hand continues [15] with theorems about simple locally finite groups "**of alternating type**" and "**of projective special linear type**" and makes quite a number of thorough suggestions for future research 😊. From February 1981 until April 1985 the author was enormous happily affiliated to the **Institut für Medizinische Biometrie und Statistik (IMBI)** at lovely **Freiburg im Breisgau** as **a considered Wissenschaftlicher Mitarbeiter**. Since **May 1985** he was based in **Munich** and devotedly working with greatest joy for the telecom industry first as **a System Software Developer**, then as **a Systems Engineer**, and in closing as **a Director for the International Standardisation of telecom software and concepts**. On the very **11 April 1992** he so blissful happily married the most fabulous and wonderful-ever woman **Helga** in **beautiful Florence** in Tuscany in Italy, which was such a memorable marriage celebrated along with about twenty friends and uniting the most venerable



city **Welden** in **Upper Palatinate** (i.d.OPf.) (**Helga**) with the huge cosmopolitan city **Munich** in **Upper Bavaria** (**Felix**). That unforgettable event was built really for eternity: **Helga and Felix were meant to last forever** ❤️. Since **October 2016** the author is retired and is still resp. is again much loving to work for Mathematics, in particular for the **very beautiful** Group Theory 😊.

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Felix Flemisch



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E-Mail: felix.flemisch@hotmail.com

The Ancient University City Freiburg im Breisgau



These are some **beautiful** views of the ancient university city **Freiburg i.Br.** (since 1457) which allow you to assess why students like and even love it so much up to the present day, of course apart from the challenging research work having and being done there in mathematics and statistics.


Professor Otto H. Kegel that's him all over

This **Research Article** is dedicated to **Prof. Otto H. Kegel** on the occasion of his 90th birthday on 20 July 2024. We therefore are closing the **Research Article** with two **beautiful** photographs of him:

Prof. Otto H. Kegel
am Mathematischen Forschungsinstitut
Oberwolfach (MFO)

Prof. Otto H. Kegel
at the Oberwolfach Research Institute
for Mathematics

(see <https://mfo.de/> and
<https://opc.mfo.de/related?id=23960> and
https://opc.mfo.de/detail?photo_id=12422)

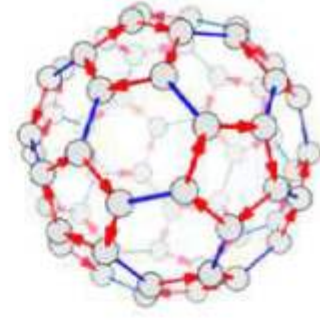
Prof. Kegel was very frequently at
the famous MFO near such **beautiful**
Freiburg im Breisgau, where he occupied
his chair from 1975 to 1999 ,
both as a guest and a speaker and as
an organiser of fascinating conferences.





Prof. Kegel,
wie er lebt und
glücklich lebt,
auf einem Spaziergang in
Freiburg im Breisgau

Prof. Kegel,
that's him all over,
his happy spitting image,
on a cosy stroll in
Freiburg im Breisgau



Long live Group Theory and in particular Sylow Theory of Locally Finite Groups!

**Am wunderschönen Ammersee in Bayern
At the wonderfully beautiful Lake Ammersee in Bavaria**

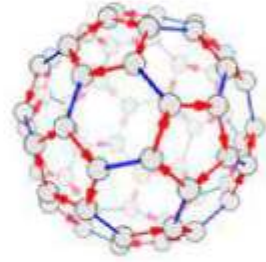


Thank you very much for your patient attention!

Are there any questions?

I would be very happy to (try to) answer them.

But I know that there are already so many by myself ...



Dedicacé à *Helga*,
qui était si précieuse à mes côtés,
avec tout mon amour et toute ma grande gratitude

Mild und leise wie sie lächelt, wie das Auge hold
sie öffnet – seht ihr's, Freunde? Sah't ihr's nicht?
Immer lichter wie sie leuchtet, Stern-umstrahlet
hoch sich hebt? Seht ihr's nicht? Wie das Herz
ihr mutig schwillt, voll und hehr im Busen
ihr quillt? Wie den Lippen, wönig mild,
süßer Atem sanft entweht: – Freunde! Seht!
Fühlt und seht ihr's nicht? – höchste Lust!

Richard Wagner: *Tristan und Isolde*;
Dritter Aufzug, Dritter Auftritt, “Liebestod”



Helga B. M. Flemisch, geb. Daubenmerkl, StD'in (F, E, DaZ) i.R. 😊

Aimer un être, c'est accepter de vieillir avec lui.
Albert Camus

Mais le pire de tout, c'est que le peintre n'a jamais terminé.
Il n'y a jamais un moment où tu peux dire:
j'ai bien travaillé et demain c'est dimanche.
Dès que tu t'arrêtes, c'est que tu recommences.
Tu peux laisser une toile de côté
en disant que tu n'y touches plus.
Mais tu ne peux jamais mettre le mot FIN.

Pablo Picasso

Les dieux avaient condamné **Sisyphé** à rouler **sans cesse** un rocher
jusqu'au sommet d'une montagne d'où la pierre retombait par
son propre poids. ... Il faut imaginer **Sisyphé** heureux.

Albert Camus



Mia zwoa g'hörn z'samm.

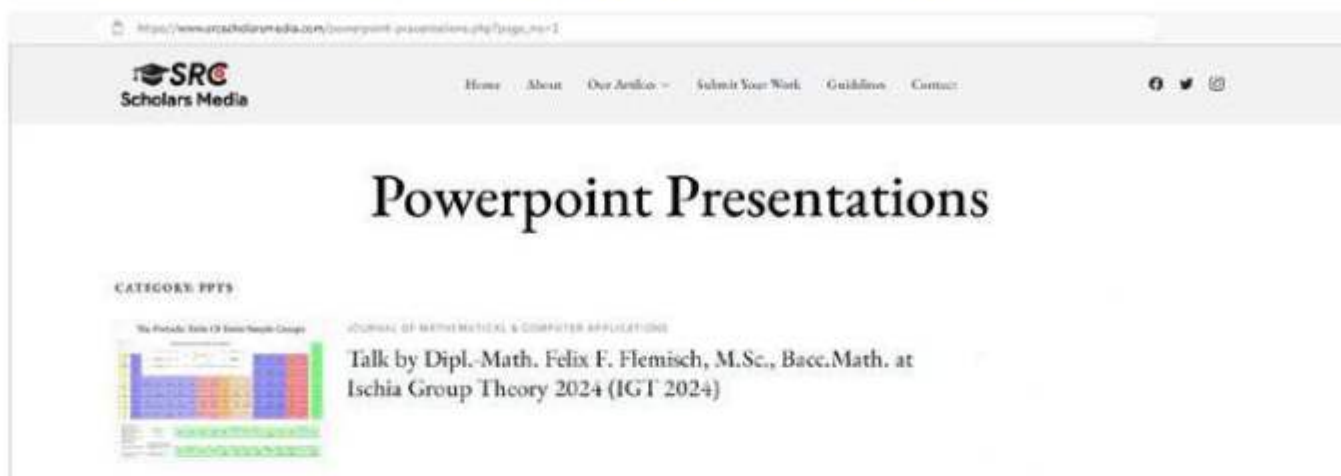
La Joie de vivre dans notre vie commune: nous deux sommes faits l'un pour l'autre.



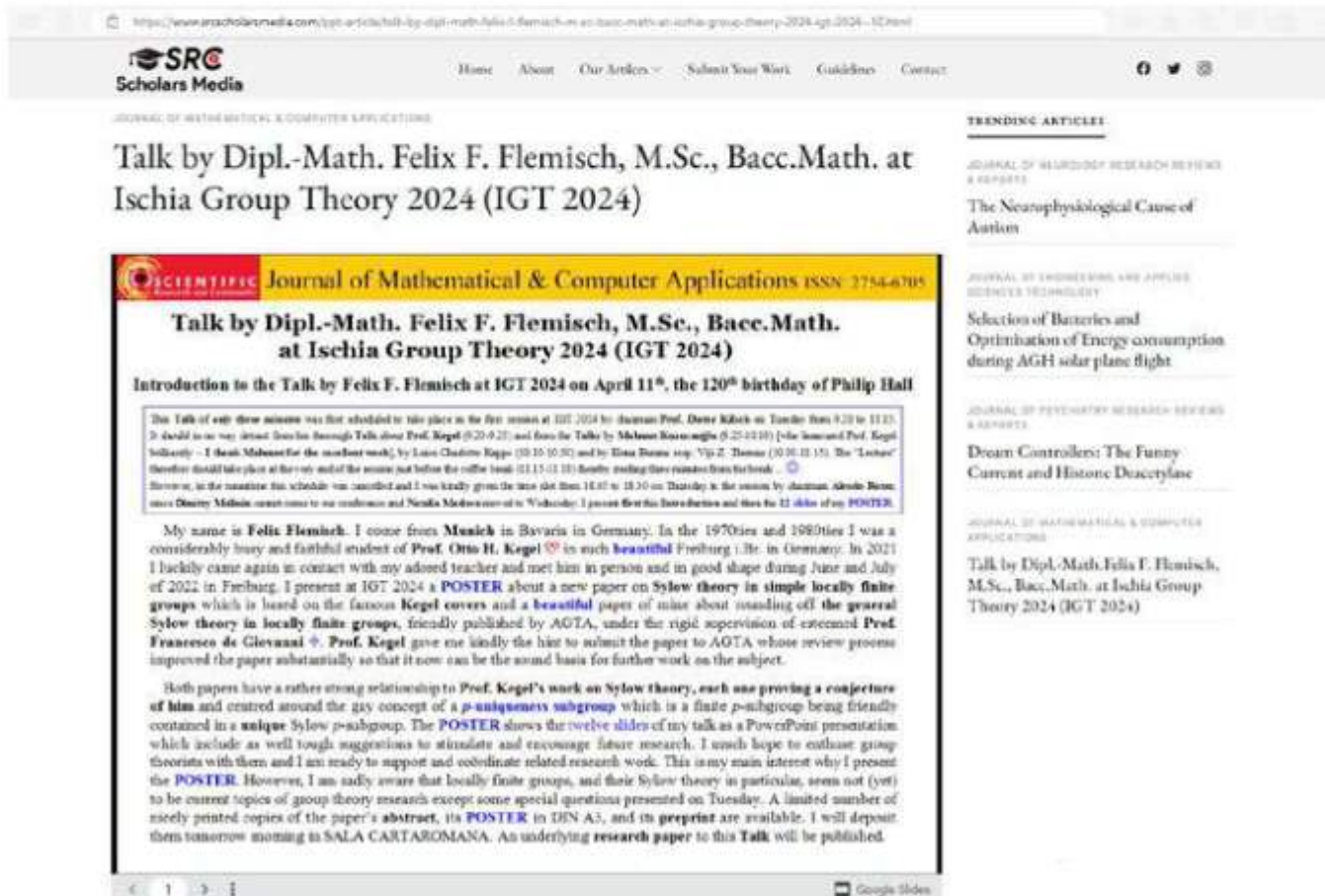
L'amour immortel, éternel et infini de l'auteur pour *Helga* et les mathématiques

Talk by Felix F. Flemisch at IGT 2024 – PowerPoint Presentation

https://www.srcscholarsmedia.com/powerpoint-presentations.php?page_no=1



<https://www.srcscholarsmedia.com/ppt-article/talk-by-dipl-math-felix-f-flemisch-m-sc-bacc-math-at-ischia-group-theory-2024-igt-2024--50.html>



Originally planned Talk of only three minutes by Felix Flemisch at IGT 2024 (see Page -4)

<https://drive.google.com/drive/folders/1UG927xau5f8vWk1SSNDX8pMK6cDjlr>

Flemisch.pdf

Freigeben

First Talk by Felix Flemisch at IGT 2024

This talk of only three minutes was first scheduled to take place in the first session at IGT 2024 by chairman Prof. Dieter Kusch (on Tuesday from 9:20 to 11:15). It should in no way detract from his talk about Prof. Kegel (9:20-9:25) and from the talks by Mahmut Kuznetsov (9:25-10:10) (who honoured Prof. Kegel brilliantly) by Larisa-Chaibote Kappe (10:10-10:50) and by Elena Bunina resp. Viji Z. Thomas (10:50-11:15). The "lecture" therefore should take place at the very end of the session just before the coffee break (11:15-11:30) thereby stealing three minutes from the break. However, in the meantime this schedule was cancelled and I was kindly given the time slot from 18:05 to 18:30 on Thursday in the session by chairman Alessio Rosas, since Dimitry Malinin cannot come to our conference and Natalia Maalava moved to Wednesday. I present first this talk and then the 12 slides of my **POSTER**.

My name is **Felix Flemisch**. I come from **Munich** in Bavaria in Germany. In the 1970ties and 1980ties I was a busy and faithful student of **Prof. Otto H. Kegel** in such beautiful **Freiburg i.Br.** in Germany. In 2021 I luckily came again in contact with my adored teacher and met him in person and in good shape during June and July of 2022 in **Freiburg**. I present at IGT 2024 a **POSTER** about a new paper on **Sylow theory in simple locally finite groups** which is based on the very famous **Kegel covers** and a paper of mine about rounding off the **general Sylow theory in locally finite groups**, friendly published by AGTA, under the rigid supervision of esteemed **Prof. Francesco de Giovanni**. Prof. Kegel gave me kindly the hint to submit the paper to AGTA whose review process improved the paper substantially.

Both papers have a quite strong relationship to Prof. Kegel's work on Sylow theory, **each one proving a conjecture of him** and centred around the gay concept of a **p -uniqueness subgroup** which is a finite p -subgroup being contained in a unique Sylow p -subgroup. The **POSTER** shows the 12 slides of a Powerpoint presentation which also include tough suggestions to stimulate and encourage future research. I much hope to enthuse group theorists with them and I am ready to coordinate related research work. This is my main interest why I present the **POSTER**. However, I am sadly aware that locally finite groups, and their Sylow theory in particular, seem not (yet) to be current topics of group theory research except some special questions presented on Tuesday. A limited number of printed copies of the paper's **abstract**, its **POSTER** in DIN A3, and its **preprint** are available from me. I will deposit them tomorrow morning in SALA CARTAROMANA.

Slide 1 / 12 - Q +



Talk by Dipl.-Math. Felix F. Flemisch, M.Sc., Bacc.Math. at Ischia Group Theory 2024 (IGT 2024)

Introduction to the Talk by Felix F. Flemisch at IGT 2024 on April 11th, the 120th birthday of Philip Hall

This Talk of only three minutes was first scheduled to take place in the first session at IGT 2024 by chairman Prof. Dieter Köhler on Tuesday from 9:20 to 11:11. It should in no way detract from his thorough Talk about Prof. Kegel (9:20-9:25) and from the Talks by Mahmut Kuzucuoglu (9:25-10:10) (who honoured Prof. Kegel brilliantly – I thank Mahmut for the excellent work), by Luise-Charlotte Kappe (10:10-10:50) and by Elena Burina resp. Viji Z. Thomas (10:50-11:15). The "Lecture" therefore should take place at the very end of the session just before the coffee break (11:15-11:18) thereby stealing three minutes from the break. 😊 However, as the meantime this schedule was cancelled and I was kindly given the time slot from 18:05 to 18:30 on Thursday in the session by chairman Alessio Russo, since Dmitry Malinin cannot come to our conference and Natalia Maslova moved to Wednesday. I present first this Introduction and then the 12 slides of my POSTER.

My name is Felix Flemisch. I come from Munich in Bavaria in Germany. In the 1970ties and 1980ties I was a considerably busy and faithful student of Prof. Otto H. Kegel ❤️ in such beautiful Freiburg i.Br. in Germany. In 2021 I luckily came again in contact with my adored teacher and met him in person and in good shape during June and July of 2022 in Freiburg. I present at IGT 2024 a POSTER about a new paper on Sylow theory in simple locally finite groups which is based on the famous Kegel covers and a beautiful paper of mine about rounding off the general Sylow theory in locally finite groups, friendly published by AGTA, under the rigid supervision of esteemed Prof. Francesco de Giovanni ♡. Prof. Kegel gave me kindly the hint to submit the paper to AGTA whose review process improved the paper substantially so that it now can be the sound basis for further work on the subject.

Both papers have a rather strong relationship to Prof. Kegel's work on Sylow theory, each one proving a conjecture of him and centered around the gay concept of a p -uniqueness subgroup which is a finite p -subgroup being friendly contained in a unique Sylow p -subgroup. The POSTER shows the twelve slides of my talk as a PowerPoint presentation which include as well tough suggestions to stimulate and encourage future research. I much hope to enthuse group theorists with them and I am ready to support and coordinate related research work. This is my main interest why I present the POSTER. However, I am sadly aware that locally finite groups, and their Sylow theory in particular, seem not (yet) to be current topics of group theory research except some special questions presented on Tuesday. A limited number of nicely printed copies of the paper's abstract, its POSTER in DIN A3, and its preprint are available. I will deposit them tomorrow morning in SALA CARTAROMANA. An underlying research paper to this Talk will be published.



The Mathematical Institute in Freiburg im Breisgau



The Mathematical Institute at Albert-Ludwigs-University in Freiburg im Breisgau in Germany where from 1975 until 1999 Prof. Kegel occupied his chair, gave beautiful lectures and seminars, invited researchers over researchers, and hosted students in the morning offering a cup of coffee (or two) thereby doing careful supervision work and suggesting fascinating research topics

The Strong Sylow Theorem for the Prime p in Simple Locally Finite Groups

Dipl.-Math. Felix F. Flemisch, M.Sc., Bacc.Math.



Dedicated to Prof. Otto H. Kegel

on the occasion of his 90th birthday on July 20



Ischia Group Theory 2024 from April 8 to April 13



Talk on Thursday, April 11, the 120th birthday of Prof. Philip Hall

THE STRONG SYLOW THEOREM FOR THE PRIME p IN SIMPLE LOCALLY FINITE GROUPS

DIPLOMATH FELIX F. FLEMISCH, M.SC., BACC.MATH.

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Dedicated to Prof. Otto H. Kegel on the occasion of his 90th birthday
<http://www.sciencedirect.com/S01676369/0000000000000000/OttoKegel.jpg>

This research paper continues [13]. We begin with giving a profound overview of the structure of arbitrary simple groups and in particular of the simple locally finite groups and reducing their Sylow theory for the prime p to a quite famous conjecture by Prof. Otto H. Kegel (see [44], Theorem 2.4: "Let the p -subgroup P be a p -uniqueness subgroup in the finite simple group S which belongs to one of the seven rank-unbounded families. Then the rank of S is bounded in terms of P ."), about the rank-unbounded ones of the 19 known families of finite simple groups. We introduce a new scheme to describe the 19 families, the family \mathcal{T} of types, define the rank of each type, and emphasise the role of Kegel covers. Prof. Kegel rediscovered from Prof. Philip Hall (see [46]) that an infinite simple group has a local system consisting of countably infinite simple subgroups (see [45], [46] and [44], Theorem 2.5) (and conversely) and if they are locally finite he discovered groundbreakingly that they have a Kegel cover (see [44], Theorem 2.6), that is, a nested local system $\{G_i\}$ with maximal normal subgroups $M_i \triangleleft G_i$ such that $G_i \cap M_{i+1} = \langle 1 \rangle$ so that G_i embeds into G_{i+1}/M_{i+1} . This part presents a unified rather complete picture of known results all of whose proofs are by reference.

We then apply new ideas to prove the conjecture for the Alternating Groups.

Thereupon we are remembering Kegel covers and \blacklozenge -sequences and the classification of simple locally finite groups according to their Kegel covers. Next we suggest a way 1) and a way 2) how to prove and even how to optimise Kegel's conjecture step-by-step or peu à peu which leads to Conjecture 1, Conjecture 2 and Conjecture 3 thereby unifying Sylow theory in locally finite simple groups with Sylow theory in locally finite and p -soluble groups whose joint study directs very reliably Sylow theory in (locally) finite groups. For any unexplained terminology we refer to [15].

We then continue the program began above to optimise along the way 1) the theorem about the first type $\Xi = \text{"}\mathfrak{A}\text{"}$ of infinite families of finite simple groups step-by-step to further types by proving it for the second type $\Xi = \text{"}\mathfrak{A} \times \text{PSL}_n\text{"}$. We apply new ideas to prove Conjecture 2 about the General Linear Groups over locally finite fields, stating that their rank is bounded in terms of their p -uniqueness, and then break down this insight to the Special Linear Groups and the Projective Special Linear (PSL) Groups over locally finite fields. We close with good suggestions for future research \blacktriangleright regarding the remaining five rank-unbounded types (the "Classical Groups") and the way 2), \blacktriangleright regarding (locally) finite and p -soluble groups, and \blacktriangleright regarding our new perceptions of the pioneering contributions by Cauchy and by Galois to Sylow theory in finite groups. We much hope to enthuse group theorists with these suggestions and are ready to contribute to, to support and to coordinate all related work.

It follows from our two theorems that simple locally finite groups which satisfy the Strong Sylow theorem for even one Prime p are linear and hence countable if they have a local system of countable simple subgroups each having a Kegel cover "of alternating type" or "of projective special linear type".

References

- [13] F.F. FLEMISCH: "Characterizing Locally Finite Groups Satisfying the Strong Sylow Theorem for the Prime p ". *Adv. Group Theory Appl.* 43 (June 2022), 43-59. <https://www.advgrouptheory.com/journal/Volumes/43/Flemisch.pdf>.
- [44] O.H. KECEL: "Four lectures on Sylow theory in locally finite groups". In: *Group Theory, Proceedings of the Singapore Group Theory Conference held at the National University of Singapore, June 8-19, 1987*, Walter de Gruyter & Co., Berlin & New York (January 1989, reprinted November 2016), 3-27. ISBN 3-11-011966-X. <https://www.degruyter.com/view/book/9783110648097/10.1515/9783110648097-004.pdf>.
- [45] O.H. KECEL: "Remarks on uncountable simple groups". In: *Proceedings of Ischia Group Theory 2016, Int. J. Group Theory 7 (March/June/September 2016)*. http://www.ijgmathe.unisa.it/ischagroupttheory/IJGT2016/home_2016.html.

The Strong Sylow Theorem for the Prime p in Simple Locally Finite Groups

DIPL.-MATH. FELIX F. FLEMSCH, M.SC., BACC.MATH.

Dedicated to Prof. Otto H. Kegel on the occasion of his 90th birthday
 Ischia Group Theory 2024 from April 8 to April 13

Let p be a prime: 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97, 101, 103, 107, 109, 113, 127, 131, 137, 139, 149, 151, 157, 163, 167, 173, 179, 181, 191, 193, 197, 199, 211, 223, 227, 229, 233, 239, 241, 251, 257, 263, 269, 271, 277, 281, 283, 293, 307, 311, 313, 317, 331, 337, 347, 349, 353, 359, 367, 373, 379, 383, 389, 397, 401, 409, 419, 421, 431, 433, 439, 443, 449, 457, 461, 463, 467, 479, 487, 491, 499, 503, 509, 521, 523, 541, 547, 557, 563, 569, 571, 577, 587, 593, 599, 601, 607, 613, 617, 619, 631, 641, 643, 647, 653, 659, 661, 673, 677, 683, 691, 701, 709, 719, 727, 733, 739, 743, 751, 757, 761, 769, 773, 787, 797, 809, 811, 821, 823, 827, 829, 839, 853, 857, 859, 863, 877, 881, 883, 887, 907, 911, 919, 929, 937, 941, 947, 953, 967, 971, 977, 983, 991, 997, ...

[44] O.H. Kegel: "Four lectures on Sylow theory in locally finite groups". in: Group Theory, Walter de Gruyter, Berlin & New York (1999), 3-27 (see MR0981832 [MR 90c:20037] and Zbl 0659.20024).

In his four workshop lectures on Sylow theory in locally finite groups at the Singapore Group Theory Conference of June 1987 (see [44]), Kegel stated as a theorem and proved "by inspection" what is actually a conjecture:

Theorem 2.4 "Let the p -subgroup P be a p -uniqueness subgroup in the finite simple group S which belongs to one of the seven *mtk* unbounded families. Then the *mtk* of S is bounded in terms of P ."

The family \mathcal{T} of types of known finite simple groups {abelian, Δ^a , $A = \text{PSL}_n$, $B = \text{PO}_{\text{alt}}$, $C = \text{PSp}_n$, $D = \text{PO}_{\text{sym}}$, $A = \text{PSU}_n$, $A D = \text{PO}_{\text{sym}}$, $E_6, E_7, E_8, F_4, G_2, {}^3B_2, {}^3D_2, {}^3F_4, {}^3G_2$, sporadic} is beautiful.

It contains 18 infinite families and one finite family: the abelian groups, seven rank-unbounded (infinite) families, ten infinite families with a fixed rank, and 26 sporadic groups.

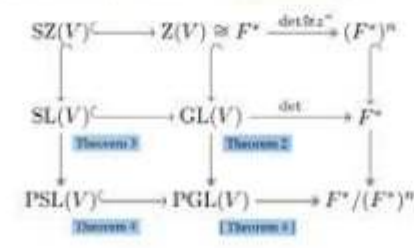
In this paper we prove Kegel's conjecture for Δ^a and for $A = \text{PSL}_n$. It continues [15] F.F. FLEMSCH: "Characterising Locally Finite Groups Satisfying the Strong Sylow Theorem for the Prime p ", Adv. Group Theory Appl. 13 (June 2022), 13-39 (see MR4441631 and Zbl 1486.20065).

We have included that beautiful predecessor paper as an Appendix, although it is open access, since this paper cannot be understood without that predecessor paper – so one needs to have it present when reading this paper – and included as well the MR Review and the Zbl Review and an important comment.

We sketch the proof for Δ^a . Let the finite p -group P act on Δ^a . Let α be a point and $P_\alpha := \{x \in P \mid \alpha^x = \alpha\} \leq P$ be the stabiliser of α . We denote by $U(P)$ the set of all subgroups of P and for $U \in U(P)$ by $R(P, U) := \{Ux \mid x \in P\}$ the set of all right cosets of U in P . Then P operates by multiplication from the right for $U \in U(P)$ transitively on $R(P, U)$ with $\text{Core } U := \bigcap \{Ux \mid x \in P\}$ as kernel.

The classification of transitive P -sets reads as follows (see [48], Chapter 6): Every transitive P -set $\Omega \neq \emptyset$ is isomorphic to $R(P, P_\alpha)$ for all $\alpha \in \Omega$, and for any $U, V \in U(P)$ the two sets $R(P, U)$ and $R(P, V)$ are P -isomorphic if and only if U and V are conjugate in P . Hence for the action of P we have a bijection between the class $\mathcal{J}(P)$ of all P -isomorphism types of transitive P -sets and the set of all conjugacy classes (in P) of subgroups of P , and so $|\mathcal{J}(P)| = g_P(|P|) =$ the number of conjugacy classes of subgroups of P . Hence for every P -set Ω the class $\mathcal{J}(P, \Omega)$ of P -isomorphism types of P -orbits on Ω has at most $g_P(|P|)$ elements and since every subgroup of P is a subset containing 1, we can summarise $|\mathcal{J}(P, \Omega)| \leq g_P(|P|) \leq |U(P)| \leq 2^{|P|-1}$. If P is a p -subgroup of S^n , which is contained in exactly $k \in \mathbb{N}$ Sylow p -subgroups of S^n , and if $m := k \cdot p + 1$, then $n \leq m \cdot |P| \leq g_P(|P|) \leq 2^{|P|-1}$, and in particular $n \leq (p+2) \cdot |P| \leq 2^{|P|-1}$ for $k=1$ (see Page 5 of the underlying research paper), whence, if not so, P has at least m many P -isomorphic P -orbits on $\Omega := \{1, 2, \dots, n\}$ (see Page 5). We deduce from this basic fact the central observation that $\{S \in \text{Syl}_p(S^n) \mid S \text{ is } P\text{-invariant}\} = \emptyset$ if $n \leq 2m - 2 \leq k + 1$ by using beautiful new ideas (see Page 6). \square

We sketch the proof for $A = \text{PSL}_n$. We apply a three-stage approach whilst first proving the theorem for the General Linear Groups over (commutative) locally finite fields (Theorem 2), then for the Special Linear Groups over locally finite fields (Theorem 3) and finally for the Projective Special Linear (PSL) Groups over locally finite fields (Theorem 4), thereby using $\text{GL}(n, F) = \text{SL}(n, F) \cdot F^*$ and $\text{PSL}(n, F) = \text{SL}(n, F) / Z(\text{SL}(n, F))$. This approach can be presented with a beautiful diagram:



Slide 5

The major work is required for the **General Linear Groups** with two different and both **beautiful new ideas** for characteristic $\neq p$ and characteristic p . In characteristic $\neq p$ we use that, if for a finite p -group P operating on a finite-dimensional vector space V over a locally finite field and a direct decomposition of V into irreducible P -submodules, there are k many of the P -submodules P -isomorphic, then at least $|\text{Syl}_p \mathbb{S}^k|$ Sylow p -subgroups of $\text{GL}(V)$ are P -invariant (see **Proposition 7 a**). In characteristic p we use that, if k is the dimension of the P -submodule $C_V(P) := \{v \in V \mid v^x = v \text{ for all } x \in P\}$ of a non-trivial modular P -module V , then again there are at least $|\text{Syl}_p \mathbb{S}^k|$ many P -invariant Sylow p -subgroups of $\text{GL}(V)$ (see **Proposition 7 b**). We then argue that from **Proposition 7** follows that $n \leq (p+2) \cdot |P|^{p-1}$ for a p -uniqueness subgroup P of $\text{GL}(n, F)$ (see **Lemma 2** on **Page 11**)

For the transition from $\text{GL}(n, F)$ to $\text{SL}(n, F)$ we are using that a p -uniqueness subgroup of $\text{SL}(n, F)$ is a p -uniqueness subgroup of $\text{GL}(n, F)$ as well. From $\text{SL}(n, F)$ to $\text{PSL}(n, F)$ we use that $P := Q \cdot \text{D}(\text{SL}(n, F)) / \text{D}(\text{SL}(n, F))$ is a p -uniqueness subgroup of $\text{PSL}(n, F)$ when Q is a p -uniqueness subgroup of $\text{SL}(n, F)$, and conversely, together with **Proposition 4** and **Proposition 6**. □

Let G be a countably infinite locally finite simple group. Then there will exist a nested local system $\{R_n \mid n \in \mathbb{N}\}$ for G of finite subgroups such that for each $n \in \mathbb{N}$ the group R_n is perfect and there exists a maximal normal subgroup M_{n+1} of R_{n+1} satisfying $M_{n+1} \cap R_n = \langle 1 \rangle$, whence R_{n+1} / M_{n+1} is simple and $R_n \leq R_{n+1} / M_{n+1}$; such a nested local system is called **Kegel cover** for G . We call G to be of **type $\Xi \in \mathcal{T}$** , if it has a Kegel cover $\Xi = \{(R_k, M_k) \mid k \in \mathbb{N}\}$ such that infinitely many R_{k+1} / M_{k+1} 's belong to Ξ (wherefrom we can replace Ξ by these infinitely many R_{k+1} / M_{k+1} 's), and call G to be of **alternating type** if it is of type A^n and to be of **projective special linear type** if it is of type $A = \text{PSL}_n$.

Theorem 1 (see [14]) Let $n \in \mathbb{N}$ and let p be a prime such that $p \leq n$. Let P be a finite p -group acting on Δ^p . Let $g_p(|P|)$ be the number of conjugacy classes of subgroups of P and let k be the number of P -invariant Sylow p -subgroups of Δ^p . Then $g_p(|P|) \leq 2^{p-1}$.

- a) If isomorphic subgroups of P are conjugate and $b := \log_2 |P|$ (so that $|P| \leq 2^b$), then

$$g_p(|P|) \leq p^b (b-2)^4 + 2(b-2)^3 + (b-2)^2 / 4 - (b-2) + b - 2 / 2 - 90 + (|P|-1) / (p-1) + 25.$$
- b) Let $m := k + p + 1$. Then $n \leq m \cdot |P| \cdot g_p(|P|) - 1$. If $k = 1$, then $n \leq f_p(|P|) := (p+2) \cdot |P| \cdot 2^{p-1} - 1$.

A periodic linear group G is locally finite and satisfies the strong Sylow Theorem for every prime p , and hence $a_p(G)$ is defined (see **Slide 7** below). We first prove **Conjecture 2** (see **Slide 7**) regarding the **General Linear Groups** over locally finite fields:

Slide 6

Theorem 2 Let $n \in \mathbb{N}$ and let p be a prime. Let \mathbb{F} be a locally finite (commutative) field.

- a) If \mathbb{F} has characteristic p and $a_p = a_p(\text{GL}(n, \mathbb{F}))$ then $n \leq (p+2) \cdot p^{n-1}$.
- b) If \mathbb{F} has characteristic $\neq p$ and $a_p = a_p(\text{GL}(n, \mathbb{F}))$ then $n \leq (p+2) \cdot p^{n-1}$.

We then break down **Theorem 2** to the **Special Linear Groups** over locally finite fields:

Theorem 3 Let $n \in \mathbb{N}$ and let p be a prime. Let \mathbb{F} be a locally finite (commutative) field

- a) If \mathbb{F} has characteristic p and $a_p = a_p(\text{SL}(n, \mathbb{F}))$ then $n \leq (p+2) \cdot p^{n-1}$.
- b) If \mathbb{F} has characteristic $\neq p$ and $a_p = a_p(\text{SL}(n, \mathbb{F}))$ then $n \leq (p+2) \cdot p^{n-1}$.

We continue with breaking down **Theorem 3** to the **Projective Special Linear (PSL) Groups** over locally finite fields:

Theorem 4 Let $n \in \mathbb{N}$ and let p be a prime. Let \mathbb{F} be a locally finite field and P be a minimal p -unique subgroup of $\text{PSL}(n, \mathbb{F})$ so that $|P| = p^n$.

- a) If \mathbb{F} has characteristic p and $a_p = a_p(\text{PSL}(n, \mathbb{F}))$ then $n \leq f_p(|P|) := (p+2) \cdot p^{n-1}$.
- b) If \mathbb{F} has characteristic $\neq p$ and $a_p = a_p(\text{PSL}(n, \mathbb{F}))$ then $n \leq f_p(|P|) := (p+2) \cdot p^{n-1}$.

Let G be an infinite simple group. G has a local system consisting of countably infinite simple subgroups (see [15] O.H. Kegel: "Remarks on uncountable simple groups", In: *Proceedings of Ischia Group Theory 2016, Int. J. Group Theory 7* (March/June/September 2018)). Let each of these be locally finite of **alternating type** or of **projective special linear type**. Then **Theorem 1** and **Theorem 4** imply the following intriguing consequences of the Strong Sylow Theorem for the Prime p :

Theorem 5 Let G be a simple locally finite group of **alternating type** or of **projective special linear type** satisfying the Strong Sylow Theorem for the even one Prime p . Then G is linear and countable. □

Planning future research

Our **Theorem 1** could be optimised in two ways:

- 1) Extend it from type A^n **step-by-step** to further types Ξ with an appropriate (similar) function f_Ξ , that is, the rank $r(G)$ of a group G of type Ξ is bounded by $f_\Xi(|P|)$ for a p -uniqueness subgroup P of G .
- 2) Determine for the type A^n and **peu à peu** for further types Ξ the minimal p -unique subgroups, that is, the p -uniqueness subgroups of the non-abelian simple groups of type A^n and of type Ξ which are minimal with respect to order (see [15]).

Slide 7

Let G be a locally finite group satisfying the strong Sylow p -Theorem and let $S \in \text{Syl}_p G$. Then S contains some (w.r.t. S) minimal p -unique subgroup F . We define $a_p = a_p(G) \in \mathbb{N}_0$ by $|F| = p^{a_p}$, that is, we let a_p be the composition length of F . This definition is independent of the choice of the Sylow p -subgroup, so a_p is a (numerical) Sylow p -invariant of G . We call a_p the p -uniqueness of G .

Conjecture 1 Let $\mathcal{T} := \{\text{abelian}, \Delta^8, A = \text{PSL}_n, B = \text{P}\Omega_{\text{odd}}, C = \text{PSp}_n, D = \text{P}\Omega_{\text{even}}, *A = \text{PSU}_n, *D = \text{P}\Omega_{\text{even}}, E_6, E_7, E_8, F_4, G_2, *E_6, *D_4, *E_6, *F_4, *G_2, \text{sporadic}\}$ be the family of types of known finite simple groups and let G be a finite simple group of type $\tau \in \mathcal{T}$. Then the rank $r(G)$ of G is bounded in terms of the p -uniqueness $a_p(G)$.

Conjecture 2 Let $n \in \mathbb{N}$ and let p be a prime. Let \mathbb{F} be a locally finite (commutative) field.

- a) If \mathbb{F} has characteristic p and $a_p = a_p(\text{GL}(n, \mathbb{F}))$ then $n \leq (p+2) \cdot p^{a_p} - 1$.
- b) If \mathbb{F} has characteristic $\neq p$ and $a_p = a_p(\text{GL}(n, \mathbb{F}))$ then $n \leq (p+2) \cdot p^{2a_p} - 1$.

We give a brief attention to (locally) p -soluble groups since it is the reliable joint study of the (locally) simple and the (locally) p -soluble groups which directs the Sylow theory in (locally) finite groups.

Conjecture 3 Let p be a prime. Let G be a p -soluble finite group, $\lambda_p(G)$ be its p -length, and $a_p(G)$ be its p -uniqueness. Then $\lambda_p(G) \leq a_p(G) + 1$.

The classical Hall-Higman theory, created by P. Hall, G. Higman, A.H.M. Hoare, T.R. Berger, F. Gross and E.G. Bryukhanova, introduces for finite p -soluble groups (best possible) inequalities between their p -length λ_p and the order p^{n_p} of a Sylow p -subgroup, its subpotency class c_p , its solubility length d_p , its exponent p^{e_p} , or the rank r_p of a maximal elementary abelian subgroup. Our true aim is to extend the Hall-Higman



theory to the beautiful p -uniqueness p^{a_p} of a Sylow p -subgroup, a truly Herculean endeavour. The challenge is to prove Conjecture 3. It is expected that the cases $p \geq 5$, $p = 3$ and $p = 2$ must be treated fairly separately and that $p = 3$ and $p = 2$ will require special methods as already indicated by the available literature.

Slide 8

Our proofs of Conjecture 1 for the types Δ^8 and $A = \text{PSL}_n$, that is, to carve out the optimising way 1), are characterised by the fact that we need not at all know their Sylow p -subgroups. There is no doubt that we can extend those proofs straightforwardly to the further five classical groups $B = \text{P}\Omega_{\text{odd}}, C = \text{PSp}_n, D = \text{P}\Omega_{\text{even}}, *A = \text{PSU}_n$ and $*D = \text{P}\Omega_{\text{even}}$ by considering the respective bilinear form defining these groups of Lie type, resp. the vector spaces they act upon as isometries, and their resulting Sylow p -subgroups (without knowing them). They can well be considered proved which we shall confirm in the follow-up paper "The Strong Sylow Theorem for the Prime p in the Locally Finite Classical Groups" considering the locally finite classical groups which are the linear, symplectic, unitary and orthogonal groups over locally finite fields. The linear groups are dealt with in this paper and the others are subgroups of the linear groups which are defined through a non-singular bilinear form (or scalar product) being either skew-symmetric (or alternate) or Hermitian or symmetric (defining a quadratic form) as the group of isometries of the form. They were nicely introduced to us in the classical books [1] and [38] and are further studied in [6], [24] and [30]. We do not refer to the groups of Lie type resp. the Chevalley groups and the twisted Chevalley groups defined through a Dynkin diagram automorphism followed by a field automorphism, which correspond to the classical groups (see [24], pp. 151-152) and whose fine introductory references are the "Lecture Notes on Chevalley Groups" by Robert Steinberg (1967 and 2016) together with the book "Simple Groups of Lie type" by Roger W. Carter (1972 and 1989). Therefore we study $\text{P}\Omega_{\text{odd}}, \text{PSp}_n, \text{P}\Omega_{\text{even}}, \text{PSU}_n$ and $\text{P}\Omega_{\text{even}}$ and not $B, C, D, *A$ and $*D$. Consequently the proofs for the further five types of Classical Groups can and will be based successfully on our very beautiful Theorem 2 about the General Linear Groups. We are preparing to publish our first follow-up paper in 2025.

Our second follow-up paper "The Strong Sylow Theorem for the Prime p in Locally Finite and p -Soluble Groups" considers (locally) finite and p -soluble groups. It summarises the work by B. Hartley and A. Rae regarding λ_p and p^{e_p} (see Page 37 of [15] and the References of [44]) and the foregoing work on Hall-Higman theory regarding λ_p and $p^{e_p}, c_p, d_p, p^{e_p}$ and r_p by P. Hall, G. Higman, A.H.M. Hoare, T.R. Berger, F. Gross, E.G. Bryukhanova and last but not least by A. Turell as indicated on Page 8 and Page 9. It then proves Conjecture 3 (see the Slide 7 above) not only in English but partly in Portuguese for historical reasons.

Our beautiful third follow-up paper "Augustin-Louis Cauchy's and Évariste Galois' Contributions to Sylow Theory in Finite Groups" pays sincere tribute to Augustin-Louis Cauchy's and Évariste Galois' pioneering contributions to Sylow theory in finite groups by working out their new perceptions. It proves in a unified way Lagrange's theorem and Cauchy's concealed second and third group theorems by exploring three beautiful rectangles/tableaux. We show the second rectangle and the third tableau to raise inquisitiveness:

Slide 9

set of certain orbits of H under G acting by left translation	the first row consists of all right cosets Gx^i of G in H ($0 \leq i < p-1$) with the powers of $\text{Sylow}(H)$ of H with the powers of left cosets of A_X	$X = \langle a_p \rangle$: set of all orbits of H under $G \cup X$. The simultaneous actions of G by left translation and of X by right translation	set of certain orbits of H under G acting by left translation	the first row consists of all right cosets Gx^i of G in H ($0 \leq i < p-1$) with the elements of $\text{Sylow}(H)$ of H with the elements of right cosets of G in H with the elements of left cosets of X	$X = \langle b_p \rangle$: set of all orbits of H under $G \cup X$. The simultaneous actions of G by left translation and of X by right translation
$Gx^0 \cdot G$	Gx_0 Gx_1 ... Gx_{p-1}	cosets $G \cdot \langle a_p \rangle = GX$ = double coset $G \cup X$	$G \langle a_p \rangle \cdot G$	Gx_0 Gx_1 ... $Gx_{p-1}a$	cosets $G \langle a_p \rangle = GX$ = double coset $G \cup X$
$Gx^1 \cdot G$	Gx_1 Gx_2 ... $Gx_{p-1}a$	cosets $G \cdot \langle a_p^2 \rangle = GX$ = double coset $G \cup X$	$G \langle a_p^2 \rangle \cdot G$	Gx_1 Gx_2 ... $Gx_{p-1}a^2$	cosets $G \langle a_p^2 \rangle = GX$ = double coset $G \cup X$
$Gx^2 \cdot G$	Gx_2 Gx_3 ... $Gx_{p-1}a^2$	cosets $G \cdot \langle a_p^3 \rangle = GX$ = double coset $G \cup X$	$G \langle a_p^3 \rangle \cdot G$	Gx_2 Gx_3 ... $Gx_{p-1}a^3$	cosets $G \langle a_p^3 \rangle = GX$ = double coset $G \cup X$
$Gx^{p-1} \cdot G$	Gx_{p-1} $Gx_{p-1}a$... $Gx_{p-1}a^{p-1}$	cosets $G \cdot \langle a_p^{p-1} \rangle = GX$ = double coset $G \cup X$	$G \langle a_p^{p-1} \rangle \cdot G$	Gx_{p-1} $Gx_{p-1}a$... $Gx_{p-1}a^{p-1}$	cosets $G \langle a_p^{p-1} \rangle = GX$ = double coset $G \cup X$

For an outline of this very beautiful paper see Page 13 and Page 14 of the underlying research paper and Slide 11.



Siamo angeli con un'ala soltanto e possiamo volare solo restando abbracciati. [Italian]
 We are angels who have but a single wing and we can only fly if we cling to one another. [English]
 Wir sind Engel mit nur einem Flügel, um fliegen zu können müssen wir uns umarmen. [German]
 Nous sommes des anges à une seule aile, nous ne pouvons voler qu'en restant enlacés. [French]
 Somos ángeles con una única ala y sólo podemos volar abrazados. [Spanish]
 Nós somos anjos com apenas uma asa e só podemos voar quando nos abraçamos. [Portuguese]

Luciano De Crescenzo

(★ 18 August 1928 in Naples until ☞ 18 July 2019 in Rome)
 Così parlò Bellavista. Napoli, amore e libertà.
 XXIII Piedigrotta. 1977 e settembre 2019



Slide 10

A MATHEMATICIAN, like a painter or a poet, is a maker of patterns. If his patterns are more permanent than theirs, it is because they are made with *ideas*. ... The mathematician's patterns, like the painter's or the poet's, must be *beautiful*; the *ideas*, like the colours or the words, must fit together in a harmonious way. *Beauty* is the first test: there is no permanent place in the world for ugly mathematics. **Godfrey Harold Hardy. A Mathematician's Apology. § 10. July 18, 1940.**

L'autore è appassionatamente curioso del futuro.
 The author is passionately curious about the future.
 Der Autor ist sehr leidenschaftlich neugierig auf die Zukunft.
 L'auteur est passionnément curieux de l'avenir.
 O autor é muito apaixonadamente curioso sobre o futuro.
Felix Fortunatus Flemisch. Firenze. April 11, 1992.

The research paper has the following seventeen beautiful Chapters: Sketch of proof for Δ^n ; Sketch of proof for $A = \text{PSL}_2$;
1 Introduction; 2 Proof of Theorem 1; 3 About Kegel covers; 4 Planning future research - Part 1; 5 Proof of Theorem 2;
6 Proof of Theorem 3; 7 Proof of Theorem 4; 8 Planning future research - Part 2;
9 The First Trilogy and The Second Trilogy and their reviews; Acknowledgements;
Postscript, Luciano De Crescenzo, Felix F. Flemisch, Conflicts of Interest, Pablo Picasso's La Joie de vivre;
About the author in Munich, in Freiburg i.Br., in London, in Weiden Ld.OPf., and in Florence in Tuscany in Italy;
75 References; Appendix 1 - Reference [15] with MR Review and Zbl Review;
Appendix 2 - Talk at IGT 2024 on Thursday, April 11, the 120th birthday of Prof. Philip Hall.

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Slide 11

We are planning to revise thoroughly Sylow theory starting with a **really new proof** for Cauchy's known as fundamental theorem in group theory (look at [https://en.wikipedia.org/wiki/Cauchy%27s_theorem_\(group_theory\)](https://en.wikipedia.org/wiki/Cauchy%27s_theorem_(group_theory))) based on **beautiful ideas** by Galois. In the forthcoming (third) follow-up **research paper** "Augustin-Louis Cauchy's and Evariste Galois' Contributions to Sylow Theory in Finite Groups" beyond our **First Trilogy** (look at the **Postscript** on **Page 15**) we first describe and then provide new but classical and rather unified proofs for the fundamental theorems by Lagrange and by Cauchy on finite groups of – in our modest opinion – considerable historical relevance.

We can describe in detail consequences of the **absence of group elements of prime order p** , in spite of their availability in overgroups, thereby providing a much unified and also heretofore undiscovered approach to the theorems of Lagrange and of Cauchy and their implications for p -groups. Since this approach uses only ideas from a very well-known paper by Augustin-Louis Cauchy presented first in 1812 and then published in 1815, this bears considerable historic relevance. While it is widely acknowledged that Cauchy had **published** his fundamental group theorem not until 1845/1846 and had there based it on double cosets of the finite permutation group and some Sylow p -subgroup of its symmetric overgroup, one could henceforth well argue that he had **presented** his theorem in a **truly concealed way** already a good thirty years earlier. Evariste Galois knew both Cauchy's paper of 1815 and – based on his own rather perceptive considerations – Cauchy's group theorem and even already Sylow's existence theorem. Cauchy's and Galois' ideas are particularly lucid in the embryonic case of permutation groups of prime degree $p (\geq 5)$ when Sylow p -subgroups of the symmetric overgroup obviously exist. If $G \supseteq H$ with H being finite, then the **unified method of proof** consists in arranging the elements of H in a **rectangle** with $|G|$ columns and $|H|$ rows resp. the (right) cosets of G in H in a rectangle with p resp. with $|H|_p$ columns and $|H : G|$ resp. $|H : G|/|H|_p$ rows to obtain information about $|H : G|$ (see the three rectangles above).

Cauchy's theorem of 1812/1815 is a direct consequence of $[H : \langle x \rangle] \geq |G|$ if x is an element of H of prime order p with $x \notin G$ which we call a **p -blank of G in H** . We find that Lagrange's theorem and Cauchy's theorem are just like two sides of a coin where "Lagrange" represents the case $p^2 = 1$ and "Cauchy" the case $p^2 = p$ thereby offering a unified approach to both theorems. Hence, "Cauchy" is not only a partial converse of "Lagrange" but it is in fact a smart "swapping" of p for 1 as well: $p^2 = 1 \Leftrightarrow p = p^2$.

Cauchy depicts 1815 a p -cycle for some prime p as a regular p -gon  and studies p -cycles in considerable detail.

We present Cauchy's **classical proof of Lagrange's theorem** and supplement it with a **modern proof**. We then present Cauchy's **classical proofs** of his **published first theorem**, of his **concealed second theorem** and of his **concealed third theorem**. Subsequently we introduce double cosets and show how they lead to a **modern proof** of Cauchy's second and third theorems what Cauchy did **beautifully** as well but not until 1845/1846 after reconsidering, impressed by a paper of Joseph Bertrand, his work of 1812/1815, that is, after – believe it or not – 30 years.

We continue with **first** correcting a great misunderstanding of Cauchy's work of 1845/1846 in the literature and **then** presenting Cauchy's work of 1812/1815 in the sincere succession of the earlier work of Joseph-Louis de Lagrange (Giuseppe Luigi Lagrangia), Alexandre-Théophile Vandermonde and the pioneer Paolo Ruffini, as indicated by Cauchy himself, and identify the crucial parts of Cauchy's first publication on group theory. Finally we present what Evariste Galois knew already about Cauchy's group theorems and about Sylow's famous theorems by referring to his published papers and to his posthumously published papers as well. However, this requires considerable further (historical) research. We would be inestimably delighted if several group theory researchers would help us with this tedious but very suspenseful work and are ready to coordinate all the work. We are closing with fairly comprehensive **Acknowledgements** and a greatly sizeable list of **References**.





Augustin-Louis Cauchy
(21 August 1798 until 23 May 1857)





Evariste Galois
(25 October 1811 until 31 May 1832)

Slide 12

About the author

Felix F. Flemisch was born on 17 May 1951 in wonderful **Munich** in Bavaria in Germany. In **June 1971** he received his **Abitur**  whose subject Mathematics was taught in a really pioneering spirit by Dr. **Helmut Bergold**. Afterwards he received his first-ever degree **Baccalaureus der Mathematik** (**Becc.Math.**) in **July 1974** with the alas unpublished very **beautiful bachelor's thesis** "Über einfache Punkte affiner Varietäten" from the most venerable **Albert-Ludwigs-Universität** at such **beautiful Freiburg im Dreisgau** in green **Baden-Württemberg** in Germany under the rather thorough supervision of **Akadem. Rat Dr. Herbert Götz**, and then his degree **Master of Science (M.Sc.)** from the Faculty of Science of the highly esteemed **University of London**, Bedford College, United Kingdom, in **August 1975** under the supervision of greatly adored **Prof. Paul Moritz Cohn**. From October 1975 until – very regrettably – only July 1976 he was employed as a **fairly diligent Teaching Assistant with two graduations** by the hoar **Mathematische Fakultät** of **Freiburg i.Br.'s Albert-Ludwigs-Universität**. Subsequently he quite enthusiastically continued his postgraduate mathematical studies in such marvelous and fabulous **Freiburg i.Br.** – with decent interruptions as a **teacher** and as a **tutor** – and received his degree **Diplom-Mathematiker (Dipl.Math.)** in **April 1985** under the impressive supervision of **Prof. Otto Helmut Kegel**. The **research paper** [15] publishes the most essential and partly well corrected portions of his **German Diplomarbeit** [14] of **October 1984** and a "sprinkling" of new considerations and results as well which try to propose coming directions of research for Sylow theory in (locally) finite groups. The publication at hand continues [15] with theorems about simple locally finite groups "of **alternating type**" and "of **projective special linear type**" and makes quite a number of thorough suggestions for future research . From February 1991 until April 1995 the author was enormous happily affiliated to the **Institut für Medizinische Diometrie und Statistik (IMDi)** at lovely **Freiburg im Dreisgau** as a **considered Wissenschaftlicher Mitarbeiter**. Since **May 1985** he was based in **Munich** and **devotedly working with greatest joy** for the telecom industry first as a **System Software Developer**, then as a **Systems Engineer**, and in closing as a **Director for the International Standardisation of telecom software and concepts**. On the very **11 April 1992** he so blissful happily married the most fabulous and wonderful-ever woman **Helga** in **beautiful Florence** in Tuscany in Italy, which was such a memorable marriage celebrated along with about twenty friends and uniting the most venerable



city **Weiden** in **Upper Palatinate** (i.d. OPf.) (**Helga**) with the huge cosmopolitan city **Munich** in **Upper Bavaria** (**Felix**). That unforgettable event was built really for eternity: **Helga and Felix were meant to last forever** . Since **October 2016** the author is retired and is still resp. is again much loving to work for Mathematics, in particular for the very **beautiful Group Theory** .

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The Ancient University City Freiburg im Breisgau



These are some **beautiful** views of the ancient university city Freiburg i.Br. (since 1457) which allow you to assess why students like and even love it so much up to the present day, of course apart from the challenging research work having and being done there in mathematics and statistics.



Professor Otto H. Kegel that's him all over

This **Talk** and the underlying **research paper** are dedicated to **Prof. Otto H. Kegel** on the occasion of his 90th birthday on 20 July 2024. We therefore are closing the **Talk** with two **beautiful** photographs of him:

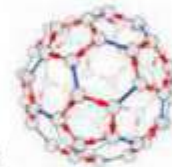
Prof. Otto H. Kegel
am Mathematischen Forschungsinstitut
Oberwolfach (MFO)
Prof. Otto H. Kegel
at the Oberwolfach Research Institute
for Mathematics
(see <https://mfo.de/> and
<https://opc.mfo.de/related?id=23960> and
https://opc.mfo.de/detail?photo_id=12422)
Prof. Kegel was very frequently at
the famous MFO near such **beautiful**
Freiburg im Breisgau, where he occupied
his chair from 1975 to 1999 🎓,
both as a guest and a speaker and as
an organiser of fascinating conferences.





Prof. Kegel,
wie er lebt und
glücklich lebt,
auf einem Spaziergang in
Freiburg im Breisgau

Prof. Kegel,
that's him all over,
his happy spitting image,
on a cosy stroll in
Freiburg im Breisgau



Long live Group Theory and in particular Sylow Theory of Locally Finite Groups!



Am wunderschönen Ammersee in Bayern
At the wonderfully beautiful Lake Ammersee in Bavaria



Thank you very much for your patient attention!

Are there any questions?

I would be very happy to (try to) answer them.

But I know that there are already so many by myself



Because of size issues the full underlying research paper "The Strong Sylow Theorem for the Prime p in Simple Locally Finite Groups", having 71 pages, could not have been included in the Conference Proceedings of IGT 2024. It therefore has been published in Volume 4 Issue 1 of the beautiful Journal of Mathematical & Computer Applications (JMCA) (see [https://doi.org/10.47363/JMCA/2025\(4\)198](https://doi.org/10.47363/JMCA/2025(4)198)). This PowerPoint Presentation shows its Appendix 2 – Talk by Felix F. Flemisch at IGT 2024 on Thursday, April 11th, the 120th birthday of Prof. Philip Hall.



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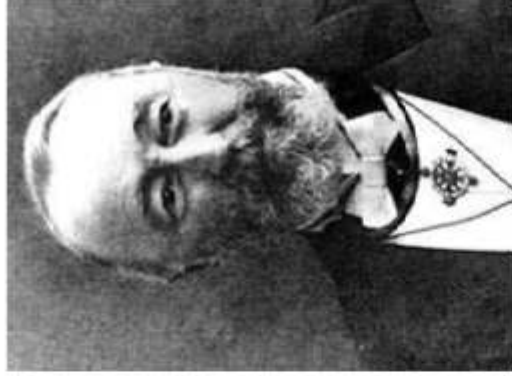
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Thank You!

The Discoverer and Explorer of Sylow Theory



Ludvig M. Sylow

(12 December 1832 until 7 September 1918)

(see https://en.wikipedia.org/wiki/Ludvig_Sylow and <https://mathshistory.st-andrews.ac.uk/Biographies/Sylow/> and https://en.wikipedia.org/wiki/Sylow_theorems)

Sylow Theorem Part 1

$$|G| = p^k m \quad H \leq G \\ |H| = p^k$$

Sylow Theorem Part 2

$$H, K \text{ Sylow } p\text{-subgroups} \\ H = gKg^{-1}$$

Sylow Theorem Part 3

$$n_p \equiv 1 \pmod{p} \quad n_p | m$$

Monsieur L. Sylow (Frederiksjald en Norge): *Théorèmes sur les groupes de substitutions.* Mathematische Annalen 5, Issue 4 (December 1872), 584-594
(see <https://link.springer.com/journal/208/volumes-and-issues/5-4>). 😊

MR1509796 (no review: see <https://mathscinet.ams.org/mathscinet-getitem?mr=1509796>);
Zbl JFM 04.0056.02 (**Eugen O. Netto**: see <https://www.zbmath.org/?q=an%3A04.0056.02> and <https://www.zbmath.org/pdf/02718820.pdf>). 😊



Ludvig Sylow

Peter Ludvig Meidell Sylow (Norwegian pronunciation: [ˈsyːlv]) (12 December 1832 – 7 September 1918) was a Norwegian mathematician who proved foundational results in group theory.^[1]

Sylow processed and further developed the innovative works of mathematicians Niels Henrik Abel and Évariste Galois in algebra. Sylow theorems and *p*-groups, known as *Sylow subgroups*, are fundamental in finite groups.^[2] By profession, Sylow was a teacher at the Frederiksborg Latin School for 40 years from 1858 to 1898, and then a professor at the University of Oslo for 20 years from 1898 to 1918.^[1] Despite the isolation in Frederiksborg, Sylow was an active member of the mathematical world. He wrote a total of approximately 25 mathematical and biographical works, corresponded with many of the leading mathematicians of the time, and was an able co-editor of *Acta Mathematica* from the journal's start in 1882.^[1] He was also elected into the Norwegian Academy of Science and Letters in 1868, a corresponding member of the Academy of Sciences in Göttingen and the University of Copenhagen awarded him an honorary doctorate in 1894.^{[1][2]}

Early life

Ludvig Sylow was born in Kristiania (now Oslo) on 12 December 1832 to later minister and customs treasurer Thomas Edvard von Westen Sylow (1792–1875) and Magdalene Cecilie Cathrine Mejdell (1806–98).^{[1][2]} His father had been an officer and a captain in the cavalry, and later he served as the head of the Ministry of the Army between 1848 and 1854.^{[1][2]}

Initially, his father was aware of his son's talent in Mathematics, so he encouraged him to work independently. From home, Sylow learned a sense of duty and hard work, but was also taught to be modest and although this was done with the best of intentions, it would become an obstacle for him later in life since it meant that he was happy to spend many years in a more lowly position

Ludvig Sylow



Ludvig Sylow

Born	Peter Ludvig Meidell Sylow 12 December 1832 Christiania, Norway
Died	7 September 1918 (aged 85) Christiania, Norway
Nationality	Norwegian
Alma mater	University of Christiania
Known for	Sylow theorems
Fields	Scientific career Mathematics

20.02.25, 10:05

than he should have had.^{[1][2]}

Ludvig Sylow - Wikipedia

Career as a mathematician

Education and first steps in mathematics

Sylow attended Christiania Cathedral School, graduating in 1850 after taking the *examen artium*.^[1] He then became a student at the University of Oslo where he began his studies of natural sciences.^[2] In 1853, the University of Oslo awarded him the Crown Prince's gold medal (*Kronprinsens guldmedalje*) for a Mathematics subject about Gnomonics.^[3] In 1856 he took the high school mathematics teacher's examination (*Realkandidat*, Norwegian to Real candidate) with excellent grades.^{[1][2][4]}

He completed his graduation in 1856, but since no university post was available, he taught for two years at Hartvig Nissen School, an independent girls' school in the Uranienborg district of Christiania, which had been founded by Hartvig Nissen and Ole Jacob Broch.^[2] His years there came during Broch's most energetic university period, and it was Broch who introduced Sylow to Carl Gustav Jacob Jacobi's fundamental work on elliptic functions, among other things.

In 1858, Sylow moved to the town of Fredrikshald (now called Halden) in Østfold county, where he taught at Frederiksborg Latin School as the Head Teacher in Mathematics and Science, a modest position that he held for a whole 40 years, from 1858 to 1898.^{[1][2][3]} Although Sylow would have made an outstanding university lecturer, he did not make a particularly good school teacher, since he was interested in the advanced areas of mathematics and had thus little enthusiasm for teaching at lower levels.^[2] Moreover, he also found it difficult to keep discipline in his classroom, so the fact that his career was largely in schools rather than universities was a poor use of his talents on two scores: Universities were the poorer for not having Sylow as a lecturer, while schools were poorer for having him as a teacher.^[2]

Abel and the theory of equations

During his studies, Sylow had become interested in the work of Niels Henrik Abel, and especially in an unfinished work on equation theory that had been left behind.^[1] However, it was only at Hartvig Nissen School (1856–58) that he began to research that work more deeply, in part thanks to Ole Jacob Broch, who was the school's pure mathematics teacher at the time.^[2] It was Broch who gave the young teacher Sylow much encouragement to continue his advanced mathematical researches.^[2] Although at first Sylow found reading Abel's papers a difficult task, he managed to struggle through them and soon found that Abel had achieved a far deeper understanding of the theory of equations than what he had managed to write in his published papers.^{[1][2]}

20.02.25, 10:05

Ludwig Sylow - Wikipedia

Some of Sylow's first attempts to publish some of Abel's unpublished results that he had found in his papers proved to be unsuccessful. For instance, he sent one of these papers to Crelle's Journal in Berlin, but the editor there, Leopold Kronecker, had already published these results having discovered them himself, and had no wish to have a paper in print which showed that Abel had proved them long before he had.^{[1][2]} Kronecker did not accept that Abel had preceded him, and therefore, he rejected Sylow's paper, but even though the article was rejected, posterity has proved Sylow right.^{[1][2]} Sylow showcased his discoveries at a Scandinavian meeting of naturalists in 1860 in Copenhagen, where he presented a solid interpretation of a strange equation-theoretic treatise by Abel, edited only in fragments.^[1]

Failure to join a university

In 1861 Sylow obtained a scholarship for studies in Paris and Berlin.^{[1][2]} In Paris he attended lectures by Michel Chasles on the theory of conics, by Joseph Liouville on rational mechanics and by Jean-Marie Duhamel on the theory of limits. He also used this scholarship to make himself acquainted with newer works, particularly in the theory of equations.^[2] In Berlin, Sylow had useful discussions with Kronecker, but was unable to attend courses by Karl Weierstrass who was ill at the time, and since there were no other courses being given in Berlin that interested him, Sylow instead decided to work in the library, studying number theory and the theory of equations.

In the following year, in 1862, Sylow lectured at the University of Christiania as a substitute for Professor Ole Jacob Broch, who had been elected to serve in the Storting, the Norwegian parliament.^{[1][2]} In his lectures Sylow explained Abel's and Galois's work on algebraic equations, and in doing so he became one of the first in Europe to lecture on Évariste Galois's works.^[2] Among his listeners was the young Sophus Lie, who would later create a strange new science on the basis of these ideas, the theory of continuous symmetry.^{[1][2]}

Lie once commented that Sylow deserved a university position because of his "broad knowledge, his sharp powers of criticism, and his outstanding mathematical work".^[2] And for a time, it seemed that the university would finally bet on him since he had received a scholarship trip to Berlin and Paris in 1861, and then spent a year doing the mathematical lectures at the Christiania University during Broch's absence abroad, during which he also began to treat and lecture Galois' group theory.^{[1][2]} But instead, his career simply stopped. When Broch again became an MP in the Storting from 1865 to 1868, he was keen to have Sylow take over the teaching of his university during this time, but the school in Fredrikshald in which Sylow was a teacher refused to give him leave to teach at the Christiania university, and they received support from the ministry.^[2] Broch left his chair as professor of pure mathematics in 1869, thus leaving a vacancy that Sylow was well qualified to have filled, and in fact, everyone expected Sylow to take over his professorship in pure mathematics. However, the University of Christiania did not rate pure mathematics very highly at that time, preferring more practical, useful, down-to-earth mathematics with more applicable topics, and Sylow was too theoretical in his approach so he was not appointed.^[2] The professor of applied mathematics, Carl Anton Bjerknes, was instead pressured to move into Broch's position, so that Cato Guldberg could take over the applied mathematics.^{[1][2]}

Sylow's theorems

Sylow was known to mathematicians of every civilized country on account of a well-known theorem that bears his name. Just as every educated person knows the Pythagorean theorem so does every mathematician speak of Abel's theorem and Sylow's theorem.

Ferdinand Georg Frobenius^[2]

Since few contemporary mathematicians were as deeply familiar with Abel's work as Sylow was, Professor Carl Anton Bjerknes advised him to study Évariste Galois's works about group theory, in which Abel had also contributed a lot.^[1] However, it was only when Sylow began to lecture about Abel's and Galois's work on algebraic equations in 1862, that he began to further develop their innovative works, especially those related to group theory, and in fact, by the end of that year, Sylow had already proved foundational results in group theory, which are now known as Sylow theorems and *p*-groups, known as *Sylow subgroups*, which are now basic terms in group theory.^[1] He was thus one of the first mathematicians to penetrate Galois' group theory. However, it was not until 1872, 10 years later, that Sylow published his most important discoveries in group theory in Alfred Clebsch's journal (*Math. Ann.*), in a small treatise of ten pages called *Théorèmes sur les groupes de substitutions*, in which Sylow generalizes his discoveries and proves what is perhaps the most profound result in the theory of finite groups.^{[1][2]} Almost all work on finite groups uses Sylow's theorems.^[2]

the most profound result in the theory of finite groups

When the famous French mathematician Camille Jordan published the standard work *Théorie des Substitutions* in 1870, Sylow was familiar with most of what was written there and more.^[2] When Jordan visited Christiania in 1872, Sophus Lie took him on an excursion to Frognerseteren with Sylow, who described to him what is now called "Sylow's theorem", which he had known since 1862.^{[1][2]} Jordan was astonished and somewhat skeptical, but shortly afterwards, he wrote enthusiastically from Sweden, and he helped Sylow to get that 10-page thesis published that same year in 1872. That thesis made Sylow a well-known European mathematician.^[2]

Written works

In 1868 he was elected into the Norwegian Academy of Science and Letters (*Det Norske Videnskaps-Akademi*).^[1] From 1870 to 1871, Sylow exchanged nine letters with Julius Petersen who, at this time, was working on his doctoral dissertation. Petersen sought Sylow's advice about the main theorem of his dissertation and all of these letters deal are about this subject. The two mathematicians exchanged another sixteen letters a few years later, in 1876 and 1877.^[2]

However, Sylow's most well-known written work rests on his 10-page thesis published in 1872 called *Théorèmes sur les groupes de substitutions* (Theorems on substitution groups), which was published in *Mathematische Annalen* Volume 5 (pages 584 to 594).^[2] This paper has the three Sylow theorems, which prove foundational results in group theory. Sylow had already proved this in 1862, but only published it in 1872, and by then, Augustin-Louis Cauchy had already proved that a group whose order is divisible by a prime.^[2] Winfried Scharlau described how Sylow was led to his discovery by his study of Galois' work, in particular of Galois' criterion for the solvability of equations of prime degree. The paper explains how Sylow used methods from Galois theory in his proofs.^[2]

Besides the thesis of 1872, Sylow's main work was the new edition of Abel's collected writings which he procured in association with former student Sophus Lie on a public basis, when in 1873, Sylow and Lie were commissioned to provide a new edition of Niels Henrik Abel's collected works, paid for by the state.^{[1][2]} The preparations for the publication of this work took eight years, from 1873 to 1881, during which he had only partial leave from his teaching work, being on leave from school for four years.^{[1][2]} Sylow and Lie prepared an edition of Abel's complete work published under the title *Oeuvres complètes de Niels Henrik Abel* (French for: Complete Works of Niels Henrik Abel).^[2] The motivation for this had come from the Norwegian Academy of Science who applied to the Norwegian Parliament for funding for the project, which was quickly granted. This funding allowed Sylow to take leave from his school in Fredrikshald for four years in order to devote himself to the project.^[2] Sylow wanted as much as possible of Abel's early works to come out, not just his great treatises with their exemplary stringency, and thus, he used this opportunity to dig up more of his early works, and in fact, there was much more additional Abel material published in the Sylow/Lie edition which appeared on 9 December 1881, than what Bjerknes used on his Abel biography of 1880.^[2]

In 1902, Sylow, in collaboration with Elling Holst, published Abel's correspondence. Further Abel documents had been discovered after the Sylow/Lie book came out in 1881 and, at the *Third Scandinavian Congress of Mathematicians*, which was held in Kristiania in 1913, Sylow discussed this new material.^[2] In addition to the Sylow theorems and the Abel material, Sylow also published a few papers on elliptic functions, particularly on complex multiplication, as well as papers on group theory.^[2]

Later career

In 1883 Sylow became an editor of *Acta Mathematica*, was elected a member of the Academy of Sciences of Göttingen and, in 1894, the University of Copenhagen awarded him an honorary doctorate.^[2]

A couple of times in his youth, Sylow briefly had the prospect of becoming a lecturer at a university, where he had absolutely belonged from the first moment, but the disfavor of the times left him unnoticed in his native land despite his name being already widely known outside Norway.^[1] As a result, Sylow spent a whole 40 years, from 1858 to 1898, holding the modest position of head teacher in mathematics and science at the Frederiksborg Latin School, a long reign that came to an end when Sylow was finally appointed as a professor of mathematics at the University of Oslo in 1898,^[3] and despite already being 65 when he obtained a university post, he was still able to hold this position for 20 years, until 1918, when he died at the age of 85.^[2] His rare talent in mathematics revealed itself immediately upon his arrival at the university, to which he brought knowledge far beyond elementary mathematics.^[3] At first, he was paid a headmaster's salary, which was approximately half the salary of a university professor, but he later received salary increases.^[2]

At the centenary of Abel's birth in 1902, Sylow gave the welcoming address at a conference to mark the centenary of his birth,^[2] giving a characterization of his great predecessor, who was hailed by all the famous mathematicians of the various countries who had met together as the one who will stand as the permanent.^[1]

Personal life

Sylow never married, but was a warm person with a nice sense of humour.^[2] He was an avid lover of being out of doors and often spent summer vacations in the mountains, usually in Kongsvoll, where he studied plants. Kongsvoll is a mountain station providing food and shelter on the route between Oslo and Trondheim, erected when the route was used by pilgrims visiting the shrine of St Olav in Trondheim.^[2]

Death

Sylow died on 7 September 1918, at the age of 85, in Christiania, Norway.^[1]

Honors

- The Crown Prince's Gold Medal (1853)
- The Norwegian Academy of Science and Letters, elected in 1868
- Corresponding member of the Academy of Sciences in Göttingen (1893)
- Editor of *Acta Mathematica* from (1893)
- Honorary doctorate at Copenhagen University (1894)

References

1. Birkeland, Bent. "Ludvig Sylow" (https://nbl.snl.no/Ludvig_Sylow). *nbl.snl.no* (in Norwegian). Retrieved 8 July 2023.
2. "Ludwig Sylow (1832 - 1918)" (<https://mathshistory.st-andrews.ac.uk/Biographies/Sylow/>). School of Mathematics and Statistics. University of St Andrews, Scotland. Retrieved 8 July 2023.
3. "885 (Salmonsens konversationsleksikon / Anden Udgave / Bind XXI: Spekulation—Søøre)" (<https://runeberg.org/salmonsens/2/22/0907.html>) [885 (Salmonsens's Conversation Lexicon / Second Edition / Volume XXI: Speculation—Søøre)]. *runeberg.org* (in Danish). Retrieved 8 July 2023.
4. Brinchmann, Christopher; Daae, Anders; Hammer, K. V., eds. (1912). "Sylow, Peter Ludvig Mejdell" (<https://runeberg.org/rvemerhvem/1912/0266.html>). *Hvem er Hvem? Haandbog over samtidige norske mænd og kvinder* (in Norwegian) (1 ed.). Kristiania: Aschehoug. Retrieved 10 July 2023.



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Peter Ludwig Mejdell Sylow

Quick Info

Born
12 December 1832
[Christiania \(now Oslo\), Norway](#)

Died
7 September 1918
[Christiania \(now Oslo\), Norway](#)



[View four larger pictures](#)

Summary

Ludwig Sylow was a high school teacher who proved what is perhaps the most profound result in the theory of finite groups.

Biography

Ludwig Sylow's parents were Thomas Edvard von Westen Sylow (1792-1875) and Magdalena Cecilie Cathrine Mejdell (1806-1898). Thomas Edvard Sylow was a captain in the cavalry and later became a government minister. From 1848 to 1854 he was Minister and Chief of the Army Ministry. He was born in Snasa, Nord-Trøndelag, Norway, the son of Peter Ludvig von Westen Sylow and Elisabeth Christine Niermeyer. Magdalena Cecilie Cathrine Mejdell was born in Septon, Land, Oppland, Norway, the daughter of Johan Ernst Mejdell and Fredrikke Caroline Sophie Kierulf. Peter Ludwig Mejdell Sylow was the eldest of his parents' ten children, having younger siblings Fredrikke Caroline Sophie Sylow (born 1834), Elisabeth Arnolda von Westen Sylow (born 1835), Johan Ernst Mejdell Sylow (born 1837), Carl Christian Weinrich Sylow (born 1838), Sverre Thomas Sylow (born 1839), Halfdan Sylow (born 1840-56), Johanne Herminie Sylow (born 1843), Stephanie Martine Sylow (born 1844), and Martin Kierulf Sylow (born 1848-58). Not all of these ten children reached adulthood: Sverre Thomas died as a baby, Halfdan died aged sixteen, and Martin Kierulf died aged ten.

Although Sylow had a good upbringing, learning how to work on his own, being taught the importance of doing one's best and working hard, nevertheless in some ways his upbringing would prove a disadvantage in his career. This was because he was taught to be modest and although this was done with the best of intentions, still it meant that he was happy to spend many years in a more lowly position than he should have had. It is interesting to note that in 1902, when Sylow gave the welcoming address at a conference to mark the centenary of [Niels Abel's](#) birth, he said (see [\[2\]](#)):-

... such a great modesty maybe does not fit this world, it may also be seen as a weakness.

Sylow attended Christiania Cathedral School, graduating in 1850. He then began his studies of natural sciences at Christiania University where he won a mathematics contest in 1853. He then took the high school teacher's examination in 1856, qualifying as a teacher of mathematics and natural science with excellent grades. As no university post was available, he taught for two years at the Hartvig Nissen school. He then moved to the town of Fredrikshald (now called Halden) in Ostfold county, where he taught at secondary school from 1858 to 1898. Sadly, although Sylow would have made an outstanding university lecturer, he did not make a particularly good school teacher. He was interested in the advanced areas of mathematics and had little enthusiasm for teaching at lower levels. Also he found it difficult to keep discipline in his classroom so the fact that his career was largely in schools rather than universities was a poor use of his talents on two scores - universities were the poorer for not having Sylow as a lecturer, while schools were poorer for having him as a teacher!

Sylow continued his mathematical studies however (see [\[5\]](#)):-

... at first working on [elliptic functions](#) in the tradition of [Abel](#) and [Jacobi](#), inspired by the professor of pure mathematics [Ole Jacob Broch](#). Finding [Abel's](#) papers on the solvability of algebraic equations by [radicals](#) more interesting, Sylow was led from them (by the professor in applied mathematics, [Carl Bjerknes](#)) to [Galois](#).

[Ole Jacob Broch](#) had studied mathematics at Christiania University after which he travelled to Paris, Berlin and Königsberg where he studied a range of mathematical topics, particularly optics and statistics. After returning to Norway, together with his friend Hartvig Nissen (1815-1874), he founded the Hartvig Nissen school, an independent girls' school in the Uranienborg district of Christiania. [Broch](#) became a lecturer in applied mathematics at Christiania University in 1848, but took over [Bernit Holmboe's](#) pure mathematics teaching in 1850 following his death. [Broch](#) retained his interest in the Hartvig Nissen school, of which he had been a co-founder, and gave the young teacher Sylow much encouragement to continue his advanced mathematical researches. Although at first Sylow found reading [Abel's](#) papers a difficult task, soon he found that [Abel](#) had achieved a far deeper understanding of the theory of equations than his published papers indicated. Sylow's first attempts to publish some of [Abel's](#) unpublished results that he had found in his papers proved a problem. This was due to the fact that he sent a paper containing these results to [Crelle's Journal](#) where the paper was considered for publication by [Leopold Kronecker](#). Now the difficulty arose since [Kronecker](#) had already published these results having discovered them himself, and had no wish to have a paper in print which showed that [Abel](#) had proved them long before he had. [Kronecker](#), therefore, rejected Sylow's paper but, of course, since Sylow had correctly portrayed [Abel's](#) unpublished results, eventually they were indeed attributed to [Abel](#) rather than [Kronecker](#).

In 1861 Sylow obtained a scholarship to travel and visited Berlin and Paris. In Paris he attended lectures by [Mittelp Cheates](#) on the [theory of conics](#), by [Joseph Liouville](#) on rational mechanics and by [Jean-Marie Duhamel](#) on the theory of limits. He says, in the report he wrote at the end of the scholarship, that he also:-

... made myself acquainted with newer works, particularly in the theory of equations.

In Berlin he had useful discussions with [Kronecker](#) but was unable to attend courses by [Karl Weierstrass](#) who was ill at the time. Bent Birkeland writes in [5] that, as there were no courses being given in Berlin that interested him:-

... he worked instead in the library, studying number theory and the theory of equations. He also got acquainted with [Carl Borchardt](#), the editor of 'Crelle's Journal' ... It is interesting to note that no lectures in algebra or the theory of equations are mentioned from his stay either in Paris or in Berlin.

In 1862 Sylow lectured at the University of Christiania, substituting for [Broch](#) who had been elected to serve in the Storting, the Norwegian parliament. In his lectures Sylow explained [Abel's](#) and [Galois's](#) work on algebraic equations. A summary of these lectures is presented in [4]. It is worth noting that although he had not proved 'Sylow's theorems' at this time (he published them 10 years later) he did pose a question about them. After proving [Cauchy's](#) theorem that a group of order divisible by a prime p has a subgroup of order p , Sylow asks whether it can be generalised to powers of p . These lectures are significant for several reasons, not least that a young student attending them was [Sophus Lie](#). Sylow's lectures were extremely valuable in giving Lie a fundamental appreciation of a topic to which he would make major contributions. [Broch](#) was again in the Storting from 1865 to 1868 and he was keen to have Sylow take over his university teaching during this time. However, the school in Fredrikshald refused to give Sylow leave to teach at the university. In 1869 [Broch](#) left his chair of pure mathematics, leaving a vacancy that Sylow was well qualified to have filled. However, the University of Christiania did not rate pure mathematics very highly at that time, preferring more practical and applicable topics. Sylow was too theoretical in his approach so he was not appointed.

In 1870-71 Sylow exchanged nine letters with [Julius Petersen](#) who, at this time, was working on his doctoral dissertation. [Petersen](#) sought Sylow's advice about the main theorem of his dissertation and these letters all deal with this. The article [8] discusses these letters. The two mathematicians exchanged another sixteen letters in 1876-77. Between 1873 and 1881 Sylow and [Lie](#) prepared an edition of [Abel's](#) complete work published under the title *Oeuvres complete de Niels Henrik Abel* ©. The motivation for this had come from the [Norwegian Academy of Science](#) who applied to the Norwegian Parliament for funding for the project which was quickly granted. This funding allowed Sylow to take leave from his school for four years in order to devote himself to the project. Arild Stubhaug writes in [3]:-

In the course of the work Sylow had an eager discussant in [C.A. Bjerknes](#), who for his part, worked on the [Abel](#) biography that was published in 1880. [Bjerknes](#) wanted as much as possible of [Abel's](#) early works to come out, not only his great treatises with their exemplary stringency, and perhaps there had been more [Abel](#) material than what [Holmboe](#) used in his edition of 1839?

Much additional [Abel](#) material was found and published in the Sylow/[Lie](#) edition which appeared on 9 December 1881. [Lie](#) said that most of the work was done by Sylow and commented that Sylow deserved a university position because of his:-

... broad knowledge, his sharp powers of criticism, and his outstanding mathematical work.

However, today Sylow's fame rests on one 10 page paper published in 1872. In this paper *Théorèmes sur les groupes de substitutions* which Sylow published in *Mathematische Annalen* Volume 5 (pages 584 to 594) appear the three Sylow theorems although we know that he had already proved his famous theorem by September 1870. [Cauchy](#) had already proved that a group whose order is divisible by a prime p has an element of order p . Sylow generalised this, proving what is perhaps the most profound result in the theory of finite groups:

- If p^n is the largest power of the prime p to divide the order of a group G then
- G has subgroups of order p^n
 - G has $1 + kp$ such subgroups
 - any two such subgroups are conjugate.

Almost all work on finite groups uses Sylow's theorems. Sylow's original 1872 paper is discussed by Rod Gow in [7] and also by the authors of [6] and the author of [15]. In [11] Winfried Scharlau describes how Sylow was led to his discovery by his study of [Galois's](#) work, in particular of [Galois's](#) criterion for the solvability of equations of prime degree. The paper [11] explains how Sylow used methods from [Galois theory](#) in his proofs.

Sylow became an editor of *Acta Mathematica*, was elected a member of the [Academy of Sciences of Göttingen](#) in 1883, and, in 1894, was awarded an honorary doctorate from the University of Copenhagen.

[Lie](#) had a special chair created for Sylow at Christiania University and Sylow taught at the university from 1898. Note that he was 65 years old before he obtained a university post but, remarkably, he was still able to hold this position for 20 years. [G.A. Miller](#) writes [10]:-

Notwithstanding the advanced age at which Sylow entered the university faculty he is said to have filled the position during twenty years with marked success. The duties of his professorship did not seem to be burdensome to him until the last year of his life when he frequently remarked that he felt tired.

Although at the age of 65 he had at last become a university professor, we should note that he did not receive the salary of a professor. At first, he was paid a headmaster's salary which was approximately half the salary of a university professor. Later he received salary increases.

In 1902 Sylow gave the welcoming address at a conference to mark the centenary of [Niels Abel's](#) birth. He said (see [2]):-

In the early nineteenth century, applied mathematics had already achieved great triumphs, especially in the fields of astronomy and physics. But just at the same time mathematics ... started to turn its gaze back to the pure and abstract theories. [[Gauss](#) and [Cauchy](#)] initiated that great movement, which has run through the whole of the previous [19th] century, and which has reformed mathematics from its foundations at the same time it enriched it with new theories. ... It was in this movement that [Niels Abel](#) took such a significant part that he will forever be counted as one of the greatest mathematicians ever.

In 1902 Sylow, in collaboration with [Elling Holst](#), published [Abel's](#) correspondence. Further [Abel](#) documents had been discovered after the Sylow/[Lie](#) book came out in 1881 and, at the 'Third Scandinavian Congress of Mathematicians' which was held in Kristiania in 1913, Sylow discussed this new material.

A version of the Introduction to Sylow's discussion is at [THIS LINK](#).

In [9] G.A. Miller quotes Georg Frobenius's opinion of Sylow:-

[Sylow] was known to mathematicians of every civilised country on account of a well-known theorem that bears his name. In 1876 Frobenius remarked that "as every educated person knows the Pythagorean theorem so does every mathematician speak of [Abel's theorem](#) and Sylow's theorem".

We must not give the impression that the Sylow theorems and the [Abel](#) material were Sylow's only mathematical contributions. He also published a few papers on elliptic functions, particularly on complex multiplication, as well as papers on group theory. Finally we should say a little about Sylow's life outside mathematics. He never married but was a warm person with a nice sense of humour. He was an avid lover of being out of doors and often spent summer vacations in the mountains, usually in Kongsvoll, where he studied plants. Kongsvoll is a mountain station providing food and shelter on the route between Oslo and Trondheim, erected when the route was used by pilgrims visiting the shrine of St Olav in Trondheim.

[Other Mathematicians born in Norway](#)
[A Poster of Ludwig Sylow](#)

References [\(hide\)](#)

1. H.F. Freudenthal, *Biography in Dictionary of Scientific Biography* (New York 1970-1990). See [THIS LINK](#).
2. A.R. Alexander, *Duel at Dawn: Heroes, Martyrs, and the Rise of Modern Mathematics* (Harvard University Press, 2010).
3. A. Stubbhug, *Niels Henrik Abel and his Times: Called Too Soon by Flames Afar* (Springer, New York, 2000).
4. H. Wussing, *The Genesis of the Abstract Group Concept* (MIT Press, Cambridge).
5. B. Birkeland, Ludwig Sylow's lectures on algebraic equations and substitutions, Christiania (Oslo), 1862: An introduction and a summary, *Historia Mathematica* 23 (2) (1996), 182-199.
6. G. Casadio and G. Zappa, History of the Sylow theorem and its proofs (Italian), *Boll. Storia Sci. Mat.* 10 (1) (1990), 29-75.
7. R. Gow, Sylow's proof of Sylow's theorem, *Irish Math. Soc. Bull.* 33 (1994), 55-63.
8. H.B. Kragemo, Ludwig Sylow (German), *Norsk Matematisk Tidsskrift* 15 (1933), 73-99.
9. J. Lützen, The mathematical correspondence between Julius Petersen and Ludwig Sylow, in *S.S. Demidov, M. Folkerts, D.E. Rowe and C.J. Scriba (eds), Auphora: Festschrift for Hans Wussing on the occasion of his 65th birthday* (Birkhäuser, Basel-Boston-Berlin, 1992), 439-467.
10. G.A. Miller, Professor Ludwig Sylow, *Science, New Series* 49 (1256) (1919), 85.
11. W. Scharlau, Die Entdeckung der Sylow-Sätze, *Historia Math.* 15 (1) (1988), 40-52.
12. T. Skolem, Ludwig Sylow und seine wissenschaftlichen Arbeiten, *Norsk matematisk forenings skrifter* (2) 2 (1933), 14-24.
13. E. Stensholt, Ludwig Sylow and his theorems (Norwegian), *Normat* 31 (1) (1983), 17-29.
14. C. Stormer, Gedächtnisrede auf Professor Dr P.L.M. Sylow, *Norsk matematisk forenings skrifter* (2) 1 (1933), 7-13.
15. W.C. Waterhouse, The early proofs of Sylow's theorem, *Arch. Hist. Exact Sci.* 21 (3) (1979/80), 279-290.

Additional Resources [\(hide\)](#)

Other pages about Ludwig Sylow:

1. [Ludwig Sylow's 1913 Address](#)

Other websites about Ludwig Sylow:

1. [Dictionary of Scientific Biography](#)
2. [Mathematical Genealogy Project](#)
3. [MathSciNet Author profile](#)
4. [zbMATH entry](#)

Cross-references [\(hide\)](#)

1. [History Topics: History of Computational Group Theory to 1993](#)
2. [Societies: Norwegian Academy of Sciences](#)
3. [Other: Earliest Known Uses of Some of the Words of Mathematics \(S\)](#)

Written by J.J.O'Connor and E.F. Robertson
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If you have comments, or spot errors, we are always pleased to [hear from you](#).

Ischia was twice an Artist Colony

Ischia

Ischia developed into a well-known artist colony at the beginning of the 20th century. Writers and painters from all over the world were attracted. Eduard Bargheer, Hans Purrmann and Arrigo Wittler lived on the island. Rudolf Levy, Werner Gilles, Max Peiffer Watenphul with Kurt Craemer and Vincent Weber stayed in the fishing village of **Sant'Angelo** on the southern tip of the island shortly before the outbreak of the Second World War. In 1936 Ischia had a population of 30,418. Spa tourism did not start again until the early 1950s. At that time, a quite remarkable artist colony of writers, composers and visual artists lived in **Forio**, including Ingeborg Bachmann, Elizabeth Taylor and Luchino Visconti stayed here for filming.



Location in the Tyrrhenian Sea



Map of the island



View of Ischia from Procida

See <https://en.wikipedia.org/wiki/Ischia>

This page was last edited on 3 February 2025, at 20:26 (UTC).

Ischia (Insel)

Ischia entwickelte sich Anfang des 20. Jahrhunderts zu einer bekannten Künstlerkolonie. Schriftsteller und Maler aus der ganzen Welt wurden angezogen. Eduard Bargheer, Hans Purrmann und Arrigo Wittler lebten auf der Insel. Rudolf Levy, Werner Gilles, Max Peiffer Watenphul mit Kurt Craemer und Vincent Weber hielten sich im Fischerdorf **Sant'Angelo** am Südpfeil der Insel kurz vor Ausbruch des Zweiten Weltkriegs auf. 1936 hatte Ischia 30.418 Einwohner. Erst Anfang der 1950er Jahre begann der Kurtourismus erneut. Damals lebte in **Forio** eine recht beachtliche Künstlerkolonie aus Schriftstellern, Komponisten und bildenden Künstlern, darunter Ingeborg Bachmann. Elizabeth Taylor und Luchino Visconti hielten sich hier zu Dreharbeiten auf. Auch Charlie Chaplin, Maria Callas, Gina Lollobrigida, Audrey Hepburn, das Ehepaar Hitchcock, Peter Ustinov, die Familie Karajan, aber auch das Ehepaar Angela Merkel & Joachim Sauer, Hans Christian Andersen, Francis Ford Coppola, Jean-Claude van Damme, Alain Delon und Romy Schneider besuchten die Insel.



Silhouette von Ischia



Ischia, Fontana,
400m, Tuffsteinfelsen



Santa Maria del Soccorso in Forio

Siehe [https://de.wikipedia.org/wiki/Ischia_\(Insel\)](https://de.wikipedia.org/wiki/Ischia_(Insel))

[Diese Seite wurde zuletzt am 5. Oktober 2024 um 18:31 Uhr bearbeitet.]

und https://rp-online.de/leben/reisen/news/promi-eiland-ischia-merkels-insel_aid-12200923

Île d'Ischia

Ischia est devenue une colonie d'artistes bien connue au début du XX^e siècle. Des écrivains et des peintres du monde entier ont été attirés. Eduard Bargheer, Hans Purrmann et Arrigo Wittler vivaient sur l'île. Rudolf Levy, Werner Gilles, Max Peiffer Watenphul avec Kurt Craemer et Vincent Weber ont séjourné dans le village de pêcheurs de **Sant'Angelo** à la pointe sud de l'île peu avant le déclenchement de la Seconde Guerre mondiale. Le tourisme thermal n'a recommencé qu'au début des années 1950. À cette époque, une colonie d'artistes assez remarquable d'écrivains, de compositeurs et d'artistes visuels vivait à **Forio**, dont Ingeborg Bachmann, Elizabeth Taylor et Luchino Visconti y ont séjourné pour le tournage.



Paysage de l'île



Ischia Porto: phare et panorama



Vue de Forio



Ischia en 1969

Voir https://fr.wikipedia.org/wiki/%C3%8Ele_d%27Ischia

La dernière modification de cette page a été faite le 2 janvier 2025 à 17:19.

Isola d'Ischia

– 😞



Isolotto di Sant'Angelo



Tramonto da Barano d'Ischia, ben visibile la sagoma dell'isolotto di Sant'Angelo, il punto più meridionale dell'isola



Baia di Sorgeto, Ischia, fonte di acqua calda



Spiaggia di San Pietro a Ischia Porto



Castello Aragonese



Borgo di Sant'Angelo, piccolo borgo abitato un tempo da pescatori diventato tra i principali luoghi di attrazione dell'isola



Sullo sfondo Ischia vista da Posillipo

Vedi https://it.wikipedia.org/wiki/Isola_d%27Ischia

Questa pagina è stata modificata per l'ultima volta l'11 febbraio 2025 alle 15:31.



ON THE p -LENGTH OF p -SOLUBLE GROUPS AND REDUCTION THEOREMS FOR BURNSIDE'S PROBLEM

By P. HALL and GRAHAM HIGMAN

[Received 13 August 1954 — Read 25 November 1954]

Although this Research Article is about simple groups we cannot help to close with a brief attention to p -soluble groups since it is the joint study of the (locally) simple and the (locally) p -soluble groups which directs very reliably the Sylow theory in (locally) finite groups. We start with the famous and groundbreaking paper which introduced the finite p -soluble groups beautifully. It is about Sylow theory!

1. Notations, summary of results, elementary theorems

1.1. THE primary object of this paper is to obtain bounds for the p -length of a finite p -soluble group G in terms of the structure of its Sylow p -subgroup. As applications, we shall obtain bounds for the derived length of a soluble group in terms of the structure of its Sylow subgroups, and prove reduction theorems for Burnside's problem.

Let a group be called a p' -group, where p is a prime, if its order is prime to p , and, as usual, a p -group if its order is a power of p . Then a finite group G is called p -soluble if each of its composition factors is either a p -group or a p' -group (cf. Tchounikhin (16)). Thus G is soluble in the ordinary sense if and only if it is p -soluble for all primes p . Clearly G is p -soluble if and only if it has a series of normal subgroups

$$1 = V_0 < V_1 < \dots < V_n = G \quad (1)$$

for which each factor group V_{i+1}/V_i is either a p -group or a p' -group. For such a group we can therefore define the *upper p -series*

$$1 = P_0 \leq N_0 < P_1 < N_1 < P_2 < \dots < P_l \leq N_l = G \quad (2)$$

inductively by the rule that N_k/P_k is the greatest normal p' -subgroup of G/P_k , and P_{k+1}/N_k the greatest normal p -subgroup of G/N_k . The number l which is the least integer such that $N_l = G$ we call the p -length of G , and we denote it by l_p , or, if necessary, $l_p(G)$. (We can also define, dually, the *lower p -series* of G , but this will not occur in our arguments.)

The p -length of a p -soluble group may also be defined as the smallest number of p -factors occurring in any series such as (1), the minimum being attained for the upper p -series (2). The subgroups P_k and N_k are obviously characteristic subgroups of G , and N_k contains all normal subgroups of G of p -length at most k . We note also that

$$l_p(N_k) = l_p(G/P_{l-k}) = k$$

for $0 \leq k \leq l = l_p(G)$.

Proc. London Math. Soc. (3) 6 (1956)

5388.3.6

B

Subgroups and factor groups of a p -soluble group G are also p -soluble, with p -length not exceeding $l_p(G)$. If G and H are both p -soluble, so is their direct product $G \times H$, and $l_p(G \times H) = \max\{l_p(G), l_p(H)\}$.

Let G be a p -soluble group and S a Sylow p -subgroup of G . It is reasonable to conjecture that the greater the p -length l_p of G is, the greater must be the complexity of the Sylow subgroup S . We shall make this conjecture precise, and prove it, in a number of different ways, by taking different measures of the complexity of S . The most obvious of these measures, the Sylow p -invariants of G , are as follows:

- (i) b_p , where p^{b_p} is the order of S .
- (ii) c_p , the class of S ; that is, the length of the (upper or) lower central series of S .
- (iii) d_p , the length of the derived series of S .
- (iv) e_p , where p^{e_p} is the exponent of S ; that is, the greatest order of any element of S . The exponent of G itself, that is, the l.c.m. of the orders of its elements, is therefore $\prod_p p^{e_p}$.

Clearly the vanishing of any one of the invariants b_p, c_p, d_p, e_p , or l_p is equivalent to G being a p' -group.

In our main theorems we have to restrict ourselves to odd primes p , and even then the results are slightly different according to whether p is a Fermat prime, of the form $2^n + 1$, or not. For instance, we shall prove the following theorem.

THEOREM A. *If G is a p -soluble group, where p is an odd prime, then*

- (i) $d_p \geq l_p$;
- (ii) $e_p \geq l_p$ if p is not a Fermat prime, and $e_p \geq [\frac{1}{2}(l_p + 1)]$ if p is a Fermat prime.

Furthermore these inequalities are best-possible.

We shall also obtain inequalities between b_p and l_p , and between c_p and l_p , but here our results are best-possible only for non-Fermat primes. All these results are trivial for $l_p \leq 1$, and we prove them by induction on l_p . Suppose that $l_p > 1$, and, as always, that G has the upper p -series (2). Let F/N_0 be the Frattini subgroup of the p -group P_1/N_0 . Every element of G induces, by transformation, an automorphism of P_1 , and hence of P_1/F . But, of course, P_1/F is an elementary abelian p -group; it can therefore be identified with the additive group of a vector space over the prime field of characteristic p , and its automorphisms with the linear transformations of this space. The automorphisms of P_1/F induced by elements of G therefore form a linear group over a field of characteristic p . This group is evidently a homomorphic image of G/P_1 , and we shall show that it is actually isomorphic to G/P_1 , and is therefore a p -soluble group with no

ON THE p -LENGTH OF p -SOLUBLE GROUPS 3

normal p -subgroup greater than 1. This situation we study in § 2. Our main result is:

THEOREM B. *Let H be a p -soluble linear group over a field of characteristic p , with no normal p -subgroup greater than 1. If g is an element of order p^m in H , then the minimal equation of g is $(x-1)^r = 0$, where $r = p^m$, unless there is an integer m_0 , not greater than m , such that $p^{m_0} - 1$ is a power of a prime q for which $c_q(H) > 1$, in which case, if m_0 is the least such integer,*

$$p^{m-m_0}(p^{m_0}-1) \leq r \leq p^m.$$

42 ON THE p -LENGTH OF p -SOLUBLE GROUPS

This paper and follow-up papers by A.H.M. Hoare, T.R. Berger, F. Gross, E.G. Bryukhanova and A. Turell prove often best-possible inequalities between the p -length λ_p of a finite p -soluble group and its Sylow p -invariants, namely the order p^{b_p} of a Sylow p -subgroup, its nilpotency class c_p , its solubility length d_p , its exponent p^{e_p} , or the rank r_p of a maximal elementary abelian subgroup, thereby creating what is known as the classical Hall-Higman theory.

REFERENCES

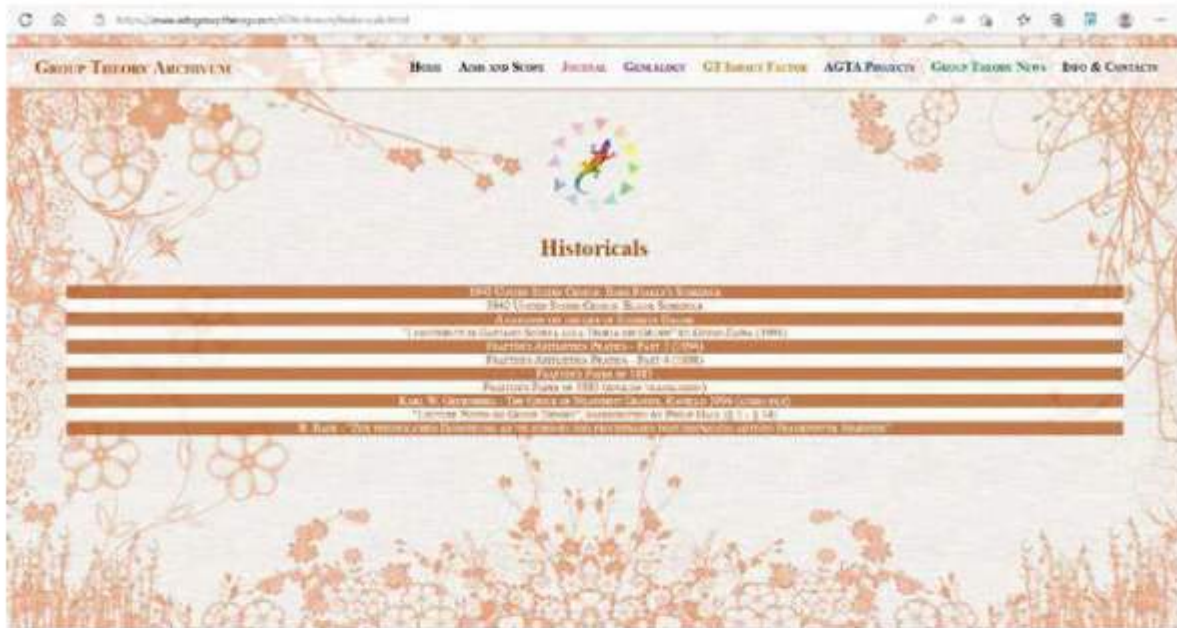
1. E. ARTIN, C. J. NESBITT, and R. M. THRALL, *Rings with minimum condition*, Michigan (1944).
2. W. BURNSIDE, 'On an unsettled question in the theory of discontinuous groups', *Quart. J. Math.* 33 (1902), 230–8.
3. ——— *Theory of groups of finite order* (2nd ed., Cambridge, 1911).
4. J. DIEUDONNÉ, *Sur les groupes classiques* (Paris, 1948).
5. H. FITTING, 'Beiträge zur Theorie der Gruppen endlicher Ordnung', *Jber. Deutsch. Math. Verein.* 48 (1938), 77–141.
6. P. HALL, 'A note on soluble groups', *J. London Math. Soc.* 3 (1928), 98–105.
7. ——— 'A contribution to the theory of groups of prime power order', *Proc. London Math. Soc.* (2) 36 (1933), 29–95.
8. L. KALOUJNINE, 'Sur les p -groupes de Sylow du groupe symétrique du degré p^n (Suite centrale ascendante et descendante)', *C. R. Acad. Sci. Paris*, 223 (1946), 703–5.
9. F. LEVI and B. L. VAN DER WAERDEN, 'Über eine besondere Klasse von Gruppen', *Abh. Math. Sem. Univ. Hamburg*, 9 (1932), 154–8.
10. H. MEIER-WUNDERLI, 'Metabelsche Gruppen', *Comment. Math. Helvet.* 25 (1951), 1–10.
11. G. POLYA, 'Kombinatorische Anzahlbestimmungen für Gruppen, Graphen und chemische Verbindungen', *Acta Math.* 69 (1937), 145–254.
12. R. REMAK, 'Über minimale invariante Untergruppen in der Theorie der endlichen Gruppen', *J. reine angew. Math.* 162 (1930), 1–16.
13. I. N. SANOV, 'Solution of Burnside's problem for exponent 4', *Leningrad State Univ. Annals (Uchenye Zapiski)*, Mat. Ser. 10 (1940), 166–70.
14. O. SCHREIER, 'Die Untergruppen der freien Gruppen', *Abh. Math. Sem. Univ. Hamburg*, 5 (1926), 161–88.
15. ——— and B. L. VAN DER WAERDEN, 'Die Automorphismen der projektiven Gruppen', *ibid.* 6 (1928), 303–22.
16. S. A. TCHOUNIKHIN, 'Sur les p -propriétés des groupes', *C. R. (Doklady) Acad. Sci. U.R.S.S. (N.S.)*, 55 (1947), 477–80.
17. H. ZASSENHAUS, *Lehrbuch der Gruppentheorie* (Leipzig and Berlin, 1937).

King's College,
Cambridge

The University,
Manchester, 13

Our aim is to extend the Hall-Higman theory to the beautiful p -uniqueness p^{a_p} of a Sylow p -subgroup. **Part 2** "The Strong Sylow Theorem for the Prime p in Locally Finite and p -Soluble Groups" of **The Second Trilogy** complements this Research Article. It **first** summarises the work by B. Hartley and A. Rae regarding λ_p and p^{a_p} and the foregoing work on the classical Hall-Higman theory and **then** proves the best-possible **Conjecture 3** "Let p be a prime. Let G be a p -soluble finite group, $\lambda_p(G)$ be its p -length, and $a_p(G)$ be its p -uniqueness. Then $\lambda_p(G) \leq a_p(G) + 1$." not only in **English** but partly in **Portuguese** for historical reasons.

Some Historicals on Group Theory (many more are available ...)




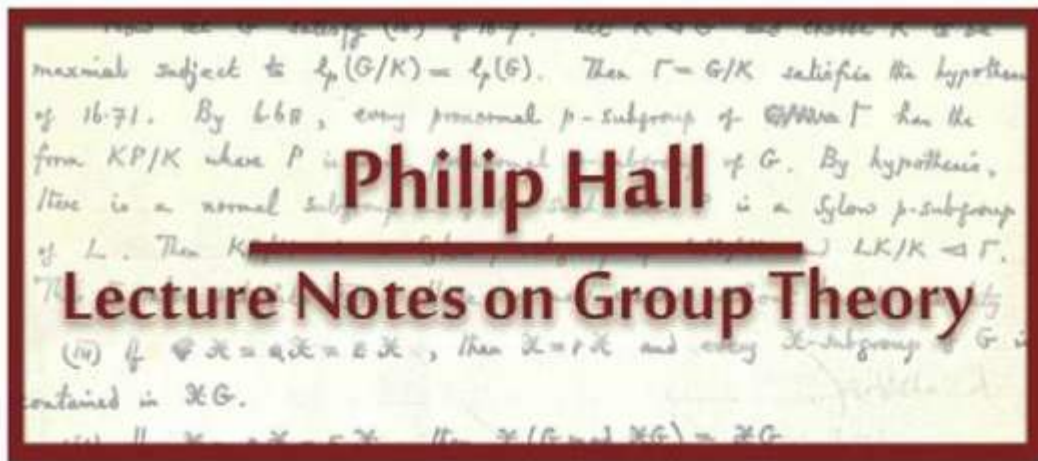
Historicals at the GROUP THEORY ARCHIVUM of Advances in Group Theory and Applications (AGTA) (see <https://www.advgrouptheory.com/GTArchivum/historicals.html>).

For some further items of the Historicals on Group Theory see https://en.wikipedia.org/wiki/History_of_group_theory [This page was last edited on 30 December 2024, at 23:39 (UTC).], https://en.wikipedia.org/wiki/List_of_important_publications_in_mathematics#Group_theory [This page was last edited on 19 March 2025, at 23:14 (UTC).], <https://quod.lib.umich.edu/cgi/t/text/text-idx?c=umhistmath;idno=AAN9280> [Accessed March 30, 2025], <https://www.tandfonline.com/doi/abs/10.1080/0025570X.1986.11977247> [Published online: 13 Feb 2018] and <https://www.amazon.de/Genesis-Abstract-Group-Concept-Contribution/dp/0486458687/> [11 May 2007].

A particularly beautiful Historical are Philip Hall's hand-written Lecture Notes. Though they seem to be slightly incomplete, as the original page numbering suggests, they are nevertheless unrivalled great and are a genuine wonderful introduction to Group Theory.

LECTURE NOTES ON GROUP THEORY, 130 + vii pages (resp. 141 + vii pages), hand-written by Prof. Philip Hall (see [26] and <https://www.advgrouptheory.com/GTArchivum/Hall/HallLectureNotes.pdf> and most notably http://omeka.wustl.edu/omeka/exhibits/show/philip_hall/ but see Page 251 and Page 252 regarding this URL).

This material (see Page 103 to Page 250) is a set of unpublished, hand-written lecture notes composed by Prof. Philip Hall (11 April 1904 [see Page 2, Page 3, Page 56 and Page 76] until 30 December 1982) (see https://en.wikipedia.org/wiki/Philip_Hall [This page was last edited on 22 September 2024, at 07:45 (UTC).] and https://en.wikipedia.org/wiki/Biographical_Memoirs_of_Fellows_of_the_Royal_Society [This page was last edited on 21 July 2024, at 23:56 (UTC).] and <https://londmathsoc.onlinelibrary.wiley.com/doi/abs/10.1112/blms/16.6.603> [First published: November 1984]). They were written from 1951 to 1967 at the University of Cambridge, UK, where Prof. Philip Hall was a professor at King's College, and they were retained by Dr. Edward Lawrence Spitznagel (4 September 1941 until 20 March 2025) (see <https://math.wustl.edu/people/edward-spitznagel> and <https://www.luptonchapel.com/obituary/dr-edward-spitznagel>) at the Department of Mathematics of Washington University in St. Louis (WashU) , Missouri (see under <https://math.wustl.edu/> and https://en.wikipedia.org/wiki/Washington_University_in_St._Louis [This page was last edited on 28 April 2025, at 01:40 (UTC).]).



Philip Hall Lecture Notes on Group Theory

Philip Hall Lecture notes

This material is a set of unpublished, hand-written lecture notes composed by Philip Hall (1904-1982). They were written in the 1960s at the University of Cambridge, where Hall was a professor at King's College.

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I. Elements of Group Theory.
 §1. The Laws of Group Theory. The symmetric group.

(A). The theory of groups originated in the study of the permutations of a finite set of objects: for example, the roots of an algebraic equation; or the faces, edges and vertices of a regular solid. From this beginning the concept of a group was derived by abstraction, that is to say by the elimination of incidentals, in the following way.

A permutation α of a set X (which need not be finite) is by definition any mapping

$$x \rightarrow x\alpha \quad (x \in X)$$

of X into itself with these two properties: (i) $x\alpha = y\alpha$ implies $x = y$; and (ii) every y in X has the form $x\alpha$ for some $x \in X$. (i) states that the mapping α is one-to-one and implies that the solution x of the equation $y = x\alpha$ in \mathbb{E} is for given y unique. Therefore with every permutation α of X there is associated another permutation α^{-1} , the inverse of α , defined by the rule that $y\alpha^{-1} = x$ whenever $x\alpha = y$.

The number of elements in a set X is denoted by $|X|$. If this number is finite, the two properties (i) and (ii) are equivalent: each implies the other.

The set of all possible permutations of X is denoted by $\Sigma(X)$ and is called the symmetric group on X . If $|X| = n$ is finite, then $|\Sigma(X)| = n!$.

Let α and β be in $\Sigma(X)$. Their product $\alpha\beta$ is the mapping of X defined by

$$x(\alpha\beta) = (x\alpha)\beta \quad (x \in X).$$

It is the result of applying first α , then β ; and it is also a permutation of X . Thus the set $\Sigma(X)$ is closed with respect to two operations: (i) inversion, which is a singular operation; and

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BY

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(c) multiplication, a binary operation.

These two operations satisfy the following laws:

$$I. (\alpha\beta)\gamma = \alpha(\beta\gamma), \text{ the associative law of multiplication;}$$

$$II. (\alpha^{-1})^{-1} = \alpha;$$

$$III. (\alpha\alpha^{-1})\beta = \beta = \beta(\alpha\alpha^{-1}), \text{ the law of cancellation.}$$

Note that the product $\alpha\alpha^{-1}$ is the identity mapping of X which maps each element of X into itself. In these laws, α, β and γ are arbitrary permutations of X .

(B). Now let G be any non-empty set in which "inverses" and "products" have been defined. Provided the laws I, II and III hold for all α, β and γ in G , then G is called a group.

The notations $\alpha^{-1}, \alpha\beta$ are the conventional ones used in the general theory of groups. They are not necessarily the most appropriate in every particular instance. For example, in the set \mathbb{Q} of all rational integers, "negatives" and "sums" are defined, and the laws

$$I'. (\alpha+\beta)+\gamma = \alpha+(\beta+\gamma);$$

$$II'. -(-\alpha) = \alpha;$$

$$III'. (\alpha+(-\alpha))+\beta = \beta = \beta+(\alpha+(-\alpha))$$

hold for all α, β and γ in \mathbb{Q} . We express this by calling \mathbb{Q} an additive group. The distinction between additive groups like \mathbb{Q} and multiplicative groups like $\mathbb{Z}(X)$ is not one of principle, but merely of notation.

(C). Some easy consequences of the group-laws I, II and III are to be noted.

Lemma 1.1. In any group G , the element $\beta\beta^{-1}$ is independent of the choice of β in G . It is called the unit element of G and usually denoted by 1 .

The unit element of $\mathbb{Z}(X)$ is the identity mapping of X . In an additive group like \mathbb{Q} , one speaks of the zero element rather than the unit element, and the notation is 0 , not 1 . Thus

$$\beta+(\beta^{-1}) = 1 \text{ for all } \beta \in G.$$

Lemma 1.2. Given elements $\beta_1, \beta_2, \dots, \beta_n$ in a group G , not necessarily distinct, their product in the given order is a uniquely determined element $\beta_1\beta_2 \dots \beta_n$ of G and does not depend on the precise way in which the multiplication is carried out.

For example, when $n=4$, there are five ways of calculating $\beta_1\beta_2\beta_3\beta_4$, viz. $((\beta_1\beta_2)\beta_3)\beta_4, (\beta_1\beta_2)(\beta_3\beta_4), (\beta_1(\beta_2\beta_3))\beta_4, \beta_1((\beta_2\beta_3)\beta_4)$ and $\beta_1(\beta_2(\beta_3\beta_4))$. The associative law I ensures that all five give the same answer. Thus, in writing products of three or more elements of a group, brackets may be dispensed with.

However, the ordering of the factors is usually important. In general $\beta\gamma \neq \gamma\beta$. If it should happen that $\beta\gamma = \gamma\beta$, then the elements β and γ are said to commute. Groups in which every pair of elements commute form a very special class called Abelian groups after N.H. Abel 1802-29. Groups, like \mathbb{Q} above, which are written in additive notation are nearly always Abelian.

The powers of an element α of a group are defined inductively by the equations

$$\alpha^0 = 1, \alpha^1 = \alpha, \alpha^{r+1} = \alpha^r \cdot \alpha, \alpha^{-r-1} = \alpha^{-r} \cdot \alpha^{-1}$$

for $r=1, 2, 3, \dots$

Lemma 1.3. For all m, n in \mathbb{Z} , we have

$$\alpha^m \alpha^n = \alpha^{m+n} = \alpha^n \alpha^m; (\alpha^m)^n = \alpha^{mn}.$$

If α and β commute, we also have $(\alpha\beta)^m = \alpha^m \beta^m$.

(D). The most significant derivation from the group laws is

Lemma 1.4. Let α be an element of the group G . Then the mapping $r(\alpha)$ of G defined by

$$r(\alpha) : \xi \rightarrow \xi\alpha \quad (\xi \in G)$$

is a permutation of G , i.e. $r(\alpha) \in \mathcal{Z}(G)$. Also, $r(\alpha\beta) = r(\alpha)r(\beta)$ and $r(\alpha^{-1}) = r(\alpha)^{-1}$. Finally, $r(\alpha) = r(\beta)$ only if $\alpha = \beta$.

This theorem brings us back to permutations from which we started. The statement that $r(\alpha) \in \mathcal{Z}(G)$ means that, ~~the operation~~ for given α and $\beta \in G$, the equation $\xi\alpha = \beta$ has always a unique solution ξ in G . This unique ξ is $\beta\alpha^{-1}$. For $\xi\alpha = \beta$ implies that $\beta\alpha^{-1} = (\xi\alpha)\alpha^{-1} = \xi(\alpha\alpha^{-1})$ by \mathbb{I} , $= \xi$ by \mathbb{II} ; while $(\beta\alpha^{-1})\alpha = \beta(\alpha^{-1}\alpha)$ by \mathbb{I} , $= \beta$ by \mathbb{II} , since $\alpha = (\alpha^{-1})^{-1}$ by \mathbb{II} .

$r(\alpha)$ is the operation of multiplying the elements of G on the right by α . The operation $l(\alpha)$ of multiplying the elements of G on the left by α is also a permutation of G , because for given α and $\beta \in G$, the equation $\alpha\xi = \beta$ always has the unique solution $\xi = \alpha^{-1}\beta \in G$. However $l(\alpha\beta) = l(\beta)l(\alpha)$ which is different from $l(\alpha)l(\beta)$ unless α and β commute. If G is Abelian, $l(\alpha) = r(\alpha)$ for all $\alpha \in G$.

The permutations $r(\alpha)$ and $l(\alpha)$ with $\alpha \in G$ are called the right and left translations of G . Obviously $r(1) = l(1) =$ the identity mapping of G . The word "translation" is a reminder that, if $\alpha \neq 1$, $r(\alpha)$ and $l(\alpha)$ leave no element of G invariant. For example, $\xi\alpha = \xi$ implies $\alpha = \xi^{-1}\xi = 1$.

The law of inversion for products of group elements

Lemma 1.5 $(\xi_1, \xi_2, \dots, \xi_n)^{-1} = \xi_n^{-1} \dots \xi_2^{-1} \xi_1^{-1}$.

(E) Inverses and products can be defined in a natural way for arbitrary subsets A, B, X_1, X_2, \dots of a group G .

A^{-1} is the set of all inverses α^{-1} of the elements $\alpha \in A$; while AB is the set of all elements ξ of G which are expressible (in at least one way) in the form $\xi = \alpha\beta$ with $\alpha \in A, \beta \in B$.

The laws \mathbb{I}, \mathbb{II} and \mathbb{I}' extend at once to these operations on subsets:

$$(AB)C = A(BC); \quad (A^{-1})^{-1} = A; \quad (X_1 X_2 \dots X_n)^{-1} = X_n^{-1} \dots X_2^{-1} X_1^{-1}.$$

But law \mathbb{III} applies to subsets only in exceptional cases. The inversion $\alpha \rightarrow \alpha^{-1}$ ($\alpha \in G$) is a permutation of G whose square is the identity. Since it is a permutation, we have $|A^{-1}| = |A|$.

By 1.4 we also have

Lemma 1.6 For all subsets A, B and all elements ξ, η of a group we have $|A\xi| = |A|, |\eta B| = |B|$.

Obviously $|AB| \leq |A| \cdot |B|$.

§2. Subgroups, Cosets and Indices, Quotient Groups.

(A). If a non-empty subset H of a group G is closed with respect to the inversion and multiplication in G , it forms a group in its own right. For the group laws I, II and III hold in G and therefore certainly hold also in H . Such a subset is called a subgroup of G . The study of particular groups is largely bound up with the discovery and study of the subgroups they contain.

Let H be a subgroup of G and let $a \in H$. Then $a^{-1} \in H$ and so $aa^{-1} = 1 \in H$. Every subgroup H of G contains the unit element of G and this is at the same time the unit element of H . Since $1^{-1} = 1 = 1$, the unit element taken by itself is a subgroup, the unit subgroup of G . We do not need to make a subgroup distinction between the unit element 1 of G and the unit subgroup which has 1 as its sole element.

Lemma 2.1 Let $(H_\lambda)_{\lambda \in \Lambda}$ be any family of subgroups of a group G . Then their intersection

$$H = \bigcap_{\lambda \in \Lambda} H_\lambda$$

is also a subgroup of G .

For H contains 1 and so it is not empty. And it inherits the requisite closure properties from the subgroups H_λ .

Note that the intersection of two sets X and Y is usually written $X \cap Y$.

(B). Let X be any subset of G . The intersection of all the subgroups of G which contain X is called the subgroup generated by X and written $\langle X \rangle$. It is the smallest subgroup of G which contains X . If X is not empty, $\langle X \rangle$ consists of all elements of G which are expressible in at least one way as a product of elements of the set $X \cup X^{-1}$.

Here $X \cup X^{-1}$ is the union of the sets X and X^{-1} and consists of all elements which belong to ~~both~~ at least one of these two sets. Obviously $\langle X \rangle$ must contain all products of elements of $X \cup X^{-1}$. But the set of all such products is closed with respect to multiplication, and also by I-5 with respect to inversion. Hence it is a subgroup containing X , and contained in $\langle X \rangle$. Therefore it coincides with $\langle X \rangle$ by definition.

If H, K, L, \dots are subgroups of G , then $\langle H \cup K \cup L \cup \dots \rangle$ is usually written $\langle H, K, L, \dots \rangle$. It is the join of H, K, L, \dots ; the smallest subgroup which contains them all.

If $P(\xi, \eta, \dots)$ is a proposition involving certain elements ξ, η, \dots of a group, then $\{ \xi, \eta, \dots \}$ denotes the subgroup generated by all ξ, η, \dots for which $P(\xi, \eta, \dots)$ is true.

A subset X of a group G such that $\langle X \rangle = G$ is called a set of generators of G . Such a set can usually be chosen in many different ways. $X = G$ is always a possible choice.

A group which can be generated by a single element is called cyclic. Every element ξ of a group G generates a cyclic subgroup $\langle \xi \rangle$ and 1.3 shows that this consists of all the powers ξ^m of ξ as $m = 0, \pm 1, \pm 2, \dots$; it shows also that all cyclic groups are Abelian.

(C). If G is a group, $|G|$ is called the order of G . (Unless the contrary is stated), all groups considered will be finite. If $S \subseteq G$, the order of S is also called the order of S .

Let H be a subgroup of G . The sets Hx with $x \in G$ are called the cosets of H in G . If X is any subset of G , the set HX is the union of a certain number of cosets of G and this number is divided by $|HX:H|$. This relation is justified by

Lemma 2-3 Distinct cosets of H in G have no common element, $|HX| = |HX:H| \cdot |H|$ for any subset X of G .

Proof: Let $y \in Hx$. Then $y = yz$ with $z \in H$. Hence $z = y^{-1}y \in H$ since $y^{-1} \in H$. But H contains 1 and is closed with respect to multiplication. Hence $Hx = H$ and so $Hx = Hx$. Therefore if two cosets Hx and Hx' have an element y in common, they both coincide with Hx . By 1-6, $|Hx| = |H|$ for all $x \in G$ and so $|HX| = |HX:H| \cdot |H|$ by definition of $|HX|$.

Obviously $G = HG$. The number $|G:H|$ is the total number of cosets of H in G . It is called the index of H in G . An immediate corollary of 2-3 is

Theorem 2-4 Let H be a subgroup of G . Then $|G| = |G:H| \cdot |H|$. The order of H and also its index in G divides the order of G . In particular, the order of every element of G divides $|G|$.

(D). This theorem is usually attributed to J.L. Lagrange, 1726-1813. For cyclic groups, a much more precise result holds good. This is

Theorem 2-5 Let $G = \langle \xi \rangle$ be a cyclic group of order n . Then the elements of G are $1, \xi, \xi^2, \dots, \xi^{n-1}$, and $\xi^n = 1$. For each divisor d of n , G has one and only one subgroup of order d , viz. $\langle \xi^{n/d} \rangle$ with elements $1, \xi^{n/d}, \xi^{2n/d}, \dots, \xi^{(d-1)n/d}$. All subgroups of a cyclic group are cyclic.

Note that ξ^n is the first positive power of ξ which is equal to 1 . $\xi^d = 1$ if and only if d is a multiple of n . $\xi^d = \xi^n$ if and only if

$d = n$ and n . For this reason, the order n of a cyclic group $\langle \xi \rangle$ is often called the period of ξ .

Suppose $\xi \in \mathbb{Z}(X)$, and let $x \in X$. If r is the least positive integer such that $x\xi^r = x$, then the elements $x, x\xi, x\xi^2, \dots, x\xi^{r-1}$ are all distinct. They form a cycle of ξ of order r . If $y = x\xi^k$, the cycle $y, y\xi, \dots, y\xi^{r-1}$ differs from this only superficially; they contain the same r elements of X in the same cyclic order. As cycles, they are to be considered the same. On this understanding, distinct cycles of ξ contain no common term. The n elements of X fall into a certain number of cycles of ξ which are mutually disjoint. If the orders of these cycles are r_1, r_2, \dots, r_k then $n = \sum_{i=1}^k r_i$. The numbers r_1, \dots, r_k are the parts of a partition of n and this partition is called the cycle-type of the permutation ξ . Obviously the order or period of ξ is the least common multiple of the orders of its cycles. Note that this l.c.m. divides $n!$: the order of $\mathbb{Z}(X)$.

If a partition contain m_1 parts equal to 1 , m_2 parts equal to 2 and so on, it is usually denoted by the symbol $(1^{m_1} 2^{m_2} \dots)$. It is a partition of the number $m_1 + 2m_2 + 3m_3 + \dots$

(E). Transversals. Let K be a subgroup of the group G . A subset T of G which contains exactly one element from each coset of K in G is called a transversal to K in G . Now ξ and η lie in the same coset of K if and only if $\xi\eta^{-1} \in K$. For T to be transversal to K in G it is therefore necessary and sufficient that

$$G = KT \quad \text{and} \quad K \cap T = \{1\}.$$

Now let

$$H = H_0 \cong H_1 \cong \dots \cong H_p = G$$

be a chain of subgroups of G each contained in the next. (Here $X < Y$ means that the set X is a proper part of the set Y .) Then we have the product law of indices:

$$\text{Lemma 2.6} \quad |G:H| = \prod_{i=1}^{p-1} |H_i:H_{i-1}|.$$

Proof: By induction we may assume $p=2$. Suppose then that K is a subgroup of G containing H . Let T be a transversal to K in G and let S be a transversal to H in K . Then $G = KT$ and $K = HS$ so that $G = HST$. Hence $|G:H| \cong |ST| \cong |S| \cdot |T|$.

Suppose that σ_1, σ_2 and σ_3 lie in the same coset of H , where $\sigma_i \in S$, $i=1,2$. Then $\sigma_1\sigma_2^{-1}\sigma_3^{-1} \in H$ and so $\sigma_1\sigma_2^{-1} \in \sigma_3^{-1}H\sigma_3$ which is contained in K since $S \subseteq K$, $H \subseteq K$. Hence $\sigma_1\sigma_2^{-1} \in K$ and so $\sigma_1 = \sigma_2$ since T is transversal to K . It follows that $\sigma_i\sigma_j^{-1} \in H$ and so $\sigma_j = \sigma_i$ since S is transversal to H . Thus we obtain $|G:H| = |S| \cdot |T|$. Since $|S| = |K:H|$ and $|T| = |G|/|K|$, the result follows. Note that ST is a transversal to H in G .

The word transversal with sometimes be used in a more general sense. If \sim is an equivalence relation defined on a set X , then X splits up into the union of a number of disjoint non-empty subsets, the q-classes, each of which consists of all $x \in X$ equivalent to x under \sim to some fixed element of X . A transversal to the q-classes is any subset of X which contains just one member from each q-class.

(F) The product HK of two subgroups H and K is contained in but usually distinct from their join $\langle H, K \rangle$. This is one of the more awkward facts of group theory. Since H and K both contain 1, HK contains both H and K . Hence $HK = \langle H, K \rangle$ if and only if HK is a subgroup. For this there is a simple criterion:

Lemma 2.7 HK is a subgroup if and only if $HK = KH$.

Proof: Suppose HK is a subgroup. Then $HK = (HK)^{-1} = K^{-1}H^{-1} = KH$. Conversely, let $HK = KH$. Then $(HK)^{-1} = K^{-1}H^{-1} = KH = HK$ and $(HK)(HK) = H(KH)K = (HM)(KK) = HK$. So HK is closed with respect to inversion and multiplication; it is a subgroup.

Two subgroups H and K for which $HK = KH$ are called permutabile. This does not imply that their elements commute. We also call two subsets X and Y of a group permutabile if $XY = YX$.

The most important subgroups of a group are usually those which are permutabile with every subset. These are called normal subgroups. In order that a subgroup H of the group G shall be a normal subgroup of G it is necessary and sufficient that

$$H\xi = \xi H$$

for all $\xi \in G$. Now $(H\xi)^{-1} = \xi^{-1}H^{-1} = \xi^{-1}H$. Sets of the form ξH with $\xi \in G$ may therefore be called inverse cosets of H in G . Two distinct inverse cosets of H have no common element. The condition for H to be normal in G is the cosets of H in G shall be the same as the inverse cosets. The normality relation is denoted by

$$H \triangleleft G.$$

A fundamental consequence of normality is

Theorem 2.8. If $H \triangleleft G$ then the cosets of H in G form a group G/H called the quotient group of G by H .

Since $H \triangleleft G$, we have $(H\xi)^{-1} = \xi^{-1}H = H\xi^{-1}$ and $(H\xi)(H\eta) = H(\xi\eta)$ for all $\xi, \eta \in G$. The set G/H whose elements are the $\{G/H\}$ distinct cosets of H in G is therefore closed with respect

It should be read "H is a normal subgroup of G" or "H is normal in G".
A fundamental fact is stated in

Theorem 2.8. Let $H \triangleleft G$. Then

$$(Hx)^{-1} = Hx^{-1} \text{ and } (Hx)(Hy) = HxHy$$

for all x and y in G . The set G/H whose elements are the cosets of H in G is a group. It is called the quotient group of G by H .

For $(Hx)^{-1} = Hx^{-1}$ and $HxHy = HxHy$. These equations show that G/H is closed with respect to inversion and multiplication. Lemma I and II hold for arbitrary subsets of G . As for law III, we need only note that $(Hx)^{-1}(Hy) = Hx^{-1}Hy = Hx^{-1}Hy = Hx^{-1}Hy$ for all $x \in G$ and $y \in H$. So G/H is a group whose unit element is the subgroup H itself. The unit subgroup of G/H is more appropriately denoted by H/H .

The unit subgroup 1 of G is normal in G and $G/1$ need not be distinguished from G itself. We also have $G \triangleleft G$ and G/G is a unit group with only one element.

If G has no normal subgroups other than 1 and G it is called simple. For example, if $|G| = p$ is a prime, then $G = \{1\}$ for every $3 \neq 1$ in G by 2.4. So G is cyclic and has no subgroups at all other than 1 and G . The discovery and study of simple groups of composite order is one of the most interesting, but also, most difficult parts of group theory.

By way of contrast, in an Abelian group every subgroup is normal. Quotient groups H/K , where H is a subgroup of G and $K \triangleleft H$, are called sections of G , following Wielandt. Their study is an essential adjunct to the investigation of the subgroups of G .

§3. Isomorphic Groups and Homomorphic Mappings

(A). Let G and Γ be groups and let f be a mapping of G into Γ , or (what is the same thing) a function whose argument ranges through G and whose values lie in Γ . If

$$f(xy) = f(x)f(y) \tag{1}$$

for all x, y in G , then f is called homomorphic; it is also called a homomorphism of G into Γ .

Taking $\eta = 1$, we find that $f(1)$, if f is homomorphic, then $f(1)$ is the unit element of Γ . Taking $\eta = x^{-1}$, we observe that $f(x^{-1}) = f(x)^{-1}$. \tag{2}

The image of G under f is the set $f(G)$ of all $f(x)$ with $x \in G$.

$f(G)$ is a subgroup of Γ . If $f(G) = \Gamma$, then f is said to map G onto Γ ; it is an epimorphism. On the other hand, if $f(x) = f(y)$ implies $x = y$, then f is a one-to-one mapping; it is a monomorphism. If f is both an epimorphism and a monomorphism, then it is called an isomorphism of G onto Γ . In this case, the inverse mapping f^{-1} exists and this will be an isomorphism of Γ onto G .

Two groups G and Γ are called isomorphic if there exists an isomorphism f mapping one onto the other. This relation is written $G \cong \Gamma$.

The isomorphism f allows us to translate any group-theoretical statement about G into a corresponding statement about Γ . Conversely, using f^{-1} . Hence isomorphic groups are simply copies of each other. They have the same order and their subgroups can be put in one-to-one correspondence, with normal subgroups corresponding to normal subgroups, joins and intersections being preserved and so on. We may describe them as instances of the same type of group. For example, cyclic groups of the same order are isomorphic. If $|X| = |Y|$, then $\mathbb{Z}(X) \cong \mathbb{Z}(Y)$.

Evidently, a homomorphism f of G into Γ is at the same time an isomorphism of G onto $f(G)$, which is a subgroup of Γ .

From 1.4 we have

Theorem 3.1. For any group G , the mapping

$$\xi \rightarrow \pi(\xi) \quad (\xi \in G)$$

is a homomorphism of G into $\Sigma(G)$, called the regular representation of G . Thus $G \cong \pi(G)$; every group is isomorphic with a permutation group.

This remark is due to A. Cayley 1821-95. Note that the mapping

$$\xi \rightarrow \pi(\xi^{-1}) \quad (\xi \in G)$$

is also a homomorphism of G into $\Sigma(G)$.

Another example follows from 2.8. This is

Theorem 3.2. Let $H \triangleleft G$. Then the mapping

$$\xi \rightarrow H\xi \quad (\xi \in G)$$

is an epimorphism of G onto G/H . This is called the natural epimorphism

(B). Every homomorphism can be analyzed into a natural epimorphism followed by a monomorphism. If G and Γ are groups, let

$$\text{Hom}(G, \Gamma)$$

denote the set of all possible homomorphisms of G into Γ . Suppose that $f \in \text{Hom}(G, \Gamma)$, and let $f(\xi) = f(\eta)$. Then $f(\xi\eta^{-1}) = 1$, the unit element of Γ , by (1) and (2). Conversely, $f(\xi\eta^{-1}) = 1$ implies

$f(\xi) = f(\eta)$. The set K of all $\xi \in G$ such that $f(\xi) = 1$ is called the kernel of f . For any $\eta \in G$ and $\xi \in K$, we have

$$f(\eta^{-1}\xi\eta) = f(\eta^{-1})f(\xi)f(\eta) = 1 \text{ since } f(\xi) = 1. \text{ Hence } \eta^{-1}K\eta = K$$

or $K\eta = \eta K$ for all $\eta \in G$. Since K is clearly a subgroup of G , it follows that $K \triangleleft G$. And $f(\xi) = f(\eta)$ if and only if ξ and η belong to the same coset of K . Hence we have a one-to-one

mapping

$$K\xi \rightarrow f(\xi)$$

of the cosets of K onto the elements of $f(G)$. Since K is normal in G

$K\xi K\eta = K\xi f(\eta) = f(\xi\eta)$. So this mapping is an isomorphism of G/K onto $f(G)$. This gives

Theorem 3.3. If f is any homomorphism of G with kernel K , then $G/K \cong G$ and $G/K \cong f(G)$.

Thus the mapping $\xi \rightarrow f(\xi)$, $\xi \in G$, is the product of the natural epimorphism $\xi \rightarrow K\xi$ of G onto G/K , followed by the isomorphism $K\xi \rightarrow f(\xi)$ of G/K onto $f(G)$.

Theorem 3.4. Let H be any subgroup of G and let $K \triangleleft G$. Then KH is a subgroup of G and $K \cap H \triangleleft H$ and $H/K \cap H \cong KH/K$.

Proof: since $K \triangleleft G$, it is permutable with H . So KH is a subgroup of G by 2.7. Obviously $K \triangleleft KH$. Let f be the restriction to H of the natural epimorphism of G onto G/K . Then $f(H) = KH/K$. f is a homomorphism of H with kernel $K \cap H$, the result follows. It is usually called the first isomorphism theorem.

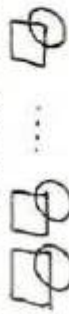
(C). If $K \triangleleft G$ and L is any subgroup of G containing K , then $K \triangleleft L$ and L/K is a subgroup of G/K . Every subgroup of G/K is of this form. Moreover $L \triangleleft G$ if and only if $L/K \triangleleft G/K$. Now let f be the product f of the natural epimorphisms of G onto G/K and of G/K onto Γ ($G/K)/(L/K)$ maps any element ξ of G onto that element $f(\xi)$ of Γ which contains $K\xi$ as one of its members. Thus $f(\xi)$ is simply $L\xi$ considered not as a set of elements of G but as a union of certain cosets of K . Comparing f with the natural epimorphism of G onto G/L we obtain

Theorem 3.5. Let K and L be normal subgroups of G such that L contains K . Then

$$G/L \cong (G/K)/(L/K).$$

This rather obvious fact is sometimes known as the second isomorphism theorem.

(D). Let H/K and L/M be two sections of the group G . There is one simple case where we can be sure that $H/K \cong L/M$. This is when each element Kg ($g \in H$) of H/K "meets" (has non-empty intersection with) exactly one element $M\lambda$ ($\lambda \in L$) of L/M and when conversely each element of L/M meets exactly one element of H/K . The situation is like this:



where for example the squares represent elements of H/K and the circles represent elements of L/M . Two sections related in this way are called incident. If we make each Kg ($g \in H$) correspond to the unique $M\lambda$ ($\lambda \in L$) which it meets, we obtain an isomorphism of H/K onto L/M . Thus incident sections are isomorphic.

Now let H/K and L/M be any two sections of G . By 3.4, $L^* = K(L \cap H)$ is a subgroup of H because $K \triangleleft H$. Similarly $M^* = K(M \cap H)$ is a subgroup and $K \triangleleft M^* \leq L^* \leq H$. By 3.4 again, $M \cap H \triangleleft L \cap H$ because $M \triangleleft L$. Hence if $g \in L \cap H$ we have $gK = Kg$ and $g(M \cap H) = (M \cap H)g$. Consequently $gM^* = M^*g$. But $K \triangleleft M^*$ and so every element of K is also permutable with M^* . It follows that every element of $L^* = K(L \cap H)$ is permutable with M^* . Consequently $M^* \triangleleft L^*$.

We call the quotient group L^*/M^* the projection of L/M onto H/K . Similarly we can form the projection H^*/K^* of H/K into L/M . Here $H^* = M(H \cap L)$ and $K^* = M(K \cap L)$. Then we have the Zassenhaus Lemma:

Theorem 3.6 If H/K and L/M are any two sections of a group, then their projections L^*/M^* and H^*/K^* in each other are incident. Hence $K^* \triangleleft M^*$ we have $K(L \cap H)/K(M \cap H) \cong M(H \cap L)/M(K \cap L)$.

Proof: Since $K^* \triangleleft M^* \leq L^*$, we have $L^* = M^*(L \cap H)$ and every element of L^*/M^* has the form $M^*\lambda$ with $\lambda \in L \cap H$. Similarly every element of H^*/K^* has the form $K^*\lambda$ with $\lambda \in L \cap H$. Hence each element of L^*/M^* "meets" some element of H^*/K^* and vice versa.

Suppose $K^*\lambda$ meets both $M^*\lambda_1$ and $M^*\lambda_2$ where λ, λ_1 and λ_2 are in $L \cap H$. Since $M \triangleleft L$, we have $K^* = M(L \cap K) = (L \cap K)M$ and so $K_1, \lambda_1 \in M^*\lambda$, and $K_2, \lambda_2 \in M^*\lambda_2$ for some K_1, K_2 in $L \cap K$ and some μ_1, μ_2 in M . Hence $(K_1, \mu_1, \lambda_1)(K_2, \mu_2, \lambda_2)^{-1} = K_1, \mu_1, \mu_2^{-1}, K_2^{-1} \in L^* \leq H$ and so $\mu_1, \mu_2^{-1} \in M \cap H$. Since $K \triangleleft H$, it follows that $K_1, \mu_1, \mu_2^{-1}, K_2^{-1} \in K(M \cap H) = M^*$. Hence $M^*\lambda_1 = M^*\lambda_2$ and $K^*\lambda$ meets only one element of L^*/M^* . Similarly each element of L^*/M^* meets only one element of H^*/K^* . Thus the two projections are incident, as stated.

The first isomorphism theorem is a special case of the Zassenhaus Lemma, viz. the case $H = G, M = 1$.

(E). Let $L = H_0 \triangleleft H_1 \triangleleft \dots \triangleleft H_n = G$. Such a chain of subgroups is called a series; each term is normal in the next. Let

$$L = L_0 \triangleleft L_1 \triangleleft \dots \triangleleft L_3 = G \quad (1)$$

be any other series, also going from L to G . Let $L_{ij} = H_{j-1}(L_i \cap H_j)$ and $H_{ji} = L_{i-1}(H_j \cap L_i)$.

Then we obtain two new series:

$$L = L_{01} \triangleleft L_{11} \triangleleft \dots \triangleleft L_{21} = H_1 = L_{21} \triangleleft \dots \triangleleft L_{31} = H_2 = L_{31} \triangleleft \dots$$

$$\dots \triangleleft L_{3r-1} = H_{r-1} = L_{3r-1} \triangleleft L_{3r} = H_r = L_{3r} \triangleleft \dots \triangleleft L_{3n} = H_n = G \quad (2)$$

and $L = H_{01} \triangleleft H_{11} \triangleleft \dots \triangleleft H_{21} = L_1 = H_{21} \triangleleft \dots \triangleleft H_{23} = L_3 = G$. Since each term of (2) occurs also in (1), we call (2) a refinement of (1). It is obtained from (1) by the insertion of more terms.

Similarly (1) is a refinement of (2). The number r is called the length of the series (1) and the

s groups H_i/H_{i-1} ($i=1, 2, \dots, s$) are called factor groups of G ; they are the factors of the series (2). The series (2) is of length s and its factors are the s groups L_i/L_{i-1} ($i=1, 2, \dots, s$). The two series (3) and (4) have the same length s . Now the factor L_i/L_{i-1} of (3) is the projection of H_i/H_{i-1} in H_i/H_{i-1} and the factor H_i/H_{i-1} is the projection of L_i/L_{i-1} in L_i/L_{i-1} . Hence $L_i/L_{i-1} \cong H_i/H_{i-1}$ and indeed, by 3.6, these two factor groups of G are incident for all $i=1, 2, \dots, s$ and all $j=1, 2, \dots, s$. Then we have

Theorem 3.7 Any two series of G possess refinements which are of equal length and with factors which are incident in pairs.

This is known as the refinement theorem: O. Schreier 1901-26.

(F.) If $H_{i-1} = H_i$ in (1), the factor H_i/H_{i-1} is called trivial. By omitting repeated terms, we can replace any given series by one which has the same non-trivial factors but has no trivial factors. A series without trivial factors, i.e. without repeated terms, is called proper. A composition series of G is a proper series which has no proper refinements distinct from itself. By 3.5, a composition series may also be defined as one in which all the factors are simple groups. Obviously every proper series can be refined to a composition series. If we apply 3.7 to two given composition series of G and then detrivialize the two refinements, we obtain

Theorem 3.8 The factors of any two composition series of G are incident in pairs.

This is the Jordan-Hölder Theorem: C. Jordan 1858-1922 and O. Hölder 1859-1937.

A soluble group is one which has a series with all the factors Abelian. Since subgroups and quotient groups of Abelian groups are always Abelian, the composition factors of a soluble group, i.e. the factors of any composition series, are also Abelian. But all subgroups of an Abelian group are normal, and so by 2.5 the only simple Abelian groups are the groups of order a prime. Hence a group G is soluble if and only if all its composition factors are of prime order. Insoluble groups are distinguished by the fact that one at least of their composition factors is a simple group of composite order.

§4. Automorphisms, Representations, Conjugates

(A) An automorphism of a group G is an isomorphic mapping of G onto itself: it is a permutation of G which leaves the group-structure of G invariant.

More generally, let X be any set, X^n be the set of all ordered n -tuples

$$x = (x_1, x_2, \dots, x_n)$$

with n coordinates $x_i \in X$. The word "multiplet" reminds us that these coordinates need not be distinct. Thus $|X^n| = |X|^n$. Only when x_1, \dots, x_n are all different shall we call x an ordered set. An n -ary relation connecting the elements of X determines the set R of all $x \in X^n$ for which the given relation holds. Mostly, we can identify the relation with this subset R of X^n . A structured set X can be specified by a certain number of relations R_1, R_2, \dots , not necessarily all with the same n . For example we can think of the structure of a group G as specified by a single ternary relation, consisting of all triplets $(\xi, \eta, \zeta) \in G^3$ for which $\xi\eta = \zeta$.

Any $\alpha \in \Sigma(X)$ induces a mapping $x \rightarrow x\alpha$ of X^n into itself, defined by the equations $(x\alpha)_i = x_i\alpha_i$, $i=1, 2, \dots, n$. For any relation R , let $R\alpha$ be the set of all $x\alpha$ with $x \in R$. The set of all $\alpha \in \Sigma(X)$ such that $R\alpha = R$, $R_1\alpha = R_1, \dots$ is clearly a subgroup of $\Sigma(X)$: it is called the group of automorphisms of X with respect to the structure defined by the relations R, R_1, \dots . An automorphism is a structure preserving permutation.

For example, let X consist of n real numbers and let R consist of all $x \in X^n$ such that

$$\prod_{i=1}^n (x_i - x_j) > 0.$$

As group of automorphisms of X with respect to R we obtain the

alternating group $\Sigma^+(X)$ of order $\frac{1}{2}n!$ consisting of all even permutations of X . A permutation of cycle-type $(1^{m_1} 2^{m_2} \dots)$ is even or odd according as the number $m_2 + m_4 + \dots$ of cycles of even order is even or odd.

The group of automorphisms of a group G will be denoted by $\text{Aut } G$.

(B). Let α, β, ξ, η be elements of a group G . We write $\alpha^{-1}\xi\alpha = \xi^\alpha$.

The element ξ^α is called the transform of ξ by α . Clearly $(\xi^\eta)^\alpha = \xi^{\eta\alpha}$. Hence the mapping $t(\alpha) = \xi(\alpha^{-1})\alpha \in \Sigma(G)$ defined by

$$t(\alpha): \xi \rightarrow \xi^\alpha \quad (\xi \in G)$$

is an automorphism of G . These automorphisms of G which have the form $t(\alpha)$ for some $\alpha \in G$ are called inner automorphisms of G . We also have $\xi^{\alpha\beta} = (\xi^\alpha)^\beta$ and so $t(\alpha\beta) = t(\alpha)t(\beta)$. Hence the mapping

$$\alpha \rightarrow t(\alpha) \quad (\alpha \in G)$$

is homomorphic: it is an epimorphism of G onto the group $\ell(G)$ of all inner automorphisms of G . Now $\xi = \xi^\alpha$ if and only if $\alpha\xi = \xi\alpha$. Hence the kernel of this epimorphism consists of all $\alpha \in G$ which commute with every element of G . This kernel is called the centre of G and denoted by $Z(G)$. By 3-3, $Z(G) \trianglelefteq G$ and $\ell(G) \cong G/Z(G)$. G is Abelian if and only if $G = Z(G)$. The only inner automorphism of an Abelian group is the identity.

Now let τ be any automorphism of G , and let $\xi \in G$. It is usual to denote the image of ξ under τ by ξ^τ . Calculating the transform $\tau^{-1}t(\alpha)\tau$ of the inner automorphism $t(\alpha)$ by τ , we have the scheme

$$\xi^\tau \xrightarrow{\tau^{-1}t(\alpha)\tau} \xi \xrightarrow{t(\alpha)} \xi^\alpha \xrightarrow{\tau} \xi^{\tau\alpha}$$

since $(\xi^\tau)^\tau = (\xi^{\tau\tau}) = \xi$. But $\tau \in \Sigma(G)$ and so ξ^τ is an arbitrary element of G if ξ is chosen suitably. Hence we obtain the formula

$$t(\alpha)^\tau = t(\alpha^\tau).$$

Summing up, we have

Theorem 4-1 For any group G , with centre Z , we have
 $G/Z \cong \text{Inn}(G) \triangleleft \text{Aut}(G)$.

The quotient group $\text{Aut}(G)/\text{Inn}(G)$ is sometimes called the group of outer automorphisms of G . But it is more natural to call any automorphism outer if it is not inner.

(C). If X is any subset of the group G and $\alpha \in G$, then we write $\alpha X \alpha^{-1} = X^\alpha$. Then X is invariant under $\ell(\alpha)$ if and only if $X^\alpha = \alpha X$. Hence a normal subgroup of G is precisely one which is invariant under every inner automorphism of G .

$K \triangleleft G \iff K^\alpha = K$ for all $\alpha \in G$.

If a subgroup of G is invariant under every automorphism of G both outer and inner, it is called a characteristic subgroup of G . By 2.5, in a cyclic group every subgroup is characteristic. We express that K is a characteristic subgroup of G by writing $K \text{ char } G$.

If $K \triangleleft G$ and $\alpha \in G$, then the restriction $\ell_K(\alpha)$ of $\ell(\alpha)$ to K is an automorphism of K , but in general $\ell_K(\alpha)$ is not an inner automorphism of K . However we have

Lemma 4-2 If $H \text{ char } K$ and $K \triangleleft G$, then $H \triangleleft G$.

If $H \text{ char } K$ and $K \text{ char } G$, then $H \text{ char } G$.

But $H \triangleleft K$ and $K \triangleleft G$ do not in general imply that $H \triangleleft G$.

(D). A homomorphism f of a group G into a permutation group $\Sigma(X)$ is called a representation of G by permutations of X .

The number $|X|$ is called the degree of the representation f . We have already noted two such representations: the regular representation τ of G and the representation ℓ of G by inner automorphisms. If the kernel of a representation f is 1, it is called faithful, for then

$f(\alpha) \neq 1$. τ is always faithful by 1-4; but ℓ is faithful only if $G = 1$, by 4-1.

If we are studying one particular representation f of G , it is often convenient to eliminate the special symbol f by writing αx instead of $\alpha f(x)$, where $\alpha \in G$, $x \in X$ and $f \in \text{Hom}(G, \Sigma(X))$. We are then left with the set X , the group G and a multiplication $XG \rightarrow X$ such that

$$\begin{aligned} I^* \quad \alpha(\alpha\beta) &= (\alpha\alpha)\beta; \\ \alpha 1 &= \alpha, \end{aligned}$$

for all $\alpha \in X$ and all $\alpha, \beta \in G$. Here 1 is the unit element of G . The laws I^* and II^* ensure that the mapping

$$f(\alpha): x \rightarrow \alpha x \quad (x \in X)$$

is a permutation of X and that $f(\alpha\beta) = f(\alpha)f(\beta)$ for all $\alpha, \beta \in G$. In this way the representation f is recovered.

Two elements x, y in X are related under G if $\alpha x = y$ for some $\alpha \in G$. If $\alpha x = y$ and $\beta y = z$ then $\alpha(\beta y) = z$; also $y\alpha^{-1} = x$. So this is an equivalence relation. If every pair of elements of X are related under G , f is called a transitive representation. In this case, we say that G permutes X transitively.

In general, however, X will split up into a number of transitivity classes X_i :

$$X = X_1 \cup X_2 \cup \dots \cup X_r$$

each of which is permuted transitively by G . Each X_i consists of all $x \in X$ which are related under G to any given element of X_i .

And, of course, $X_i \cap X_j$ is empty if $i \neq j$. If f_i is the permutation of X_i induced by $f(\alpha)$, then f_i is a transitive representation of G . f_1, f_2, \dots, f_r are called the transitive components of f . Since they determine f uniquely, we express this decomposition by writing

$$f = f_1 \oplus f_2 \oplus \dots \oplus f_r.$$

Summing up, we have

Theorem 4-1 For any group G , with centre Z , we have
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(D). A homomorphism f of a group G into a permutation group $\Sigma(X)$ is called a representation of G by permutations of X .

The number $|X|$ is called the degree of the representation f . We have already noted two such representations: the regular representation τ of G and the representation ℓ of G by inner automorphisms. If the kernel of a representation f is 1, it is called faithful, for then

odd order r , then $\tau = (\tau_1 \tau_2) (\tau_3 \tau_4) \dots (\tau_{r-1} \tau_r)$ commutes with ξ and is odd. If ξ has an even cycle of even order, we can take τ to be this cycle. Again τ is odd and commutes with ξ . If $\xi = \tau_1 \tau_2 \dots \tau_r$ where the τ_i are the cycles of ξ and if these cycles are all of different orders, then the only elements of $\Sigma(X)$ which commute with ξ are those of the form $\tau_1^{m_1} \tau_2^{m_2} \dots \tau_r^{m_r}$. If the τ_i are all of odd order, these products are all even permutations.

(F) Let $f \in \text{Hom}(G, \Sigma(X))$ and let $x \in X$. The set of all $\alpha \in G$ such that $\alpha x = x$ is a subgroup of G called the stabilizer of x in G we denote it by $S_G(x)$. Let $\xi \in G$ and suppose that $x\xi = y$.

Then every element of the coset $S_G(x)\xi$ maps x into y . Conversely, if $x\eta = y$ then $\xi\eta^{-1} \in S_G(x)$ and $\eta \in S_G(x)\xi$. The cosets of $S_G(x)$ in G correspond one-to-one with the elements $y \in X$ which are related to x under G . In particular, we have

Lemma 4-3 If G permutes X transitively, then for all $x \in X$ we have $|G : S_G(x)| = |X|$. The degree of a transitive representation of G divides the order of G .

Let $\xi \in G$. Then the set of all $\alpha \in G$ which commute with ξ is a subgroup $S_G(\xi)$ called the centralizer of ξ in G . Applying 4-3 to the ξ -representation of G , we have the

Corollary 1 If $\xi \in G$, then $|G : C_G(\xi)|$ is equal to the number of conjugates of ξ in G . This number always divides $|G|$.

Each $\xi(x)$, $\alpha \in G$, is an automorphism of G and maps subgroups into subgroups. Two subgroups H and K of G are called conjugate in G if and only if there $H^a = K$ for some $a \in G$. If $t^a(x)$ is the permutation induced by $t(x)$ on the subgroups of G , then t^a is a representation of G ; the transitivity classes of t^a are the classes of conjugate subgroups of G . Each of these classes consists of all the subgroups of G which are conjugate in G to any particular subgroup

if K_i is the kernel of f_i , then the kernel of f is $K = \bigcap_{i=1}^r K_i$.
 (E). Two elements ξ and η of G are called conjugate in G if and only if $\xi^a = \eta$ for some $a \in G$. Two conjugate elements are simply elements related under G in the representation t . Therefore G splits into a certain number of mutually disjoint classes of conjugates G_i :
 $G = G_1 \cup G_2 \cup \dots \cup G_r$.

The unit element of G is conjugate only to itself, so we may take $G_1 = 1$. The number k of G_i is an important numerical invariant of G . We call it the class number of G . If G is Abelian, each G_i consists of a single element of G and so in this case $k = |G|$.

Lemma 4-3 In $\Sigma(X)$, two elements ξ and η are conjugate if and only if they have the same cycle-type. If $|X| = n$, the class number of $\Sigma(X)$ is equal to the number of partitions of n .

Proof. Suppose that $\eta = \xi^a$ with $a \in \Sigma(X)$ and let (x_1, x_2, \dots, x_r) be any cycle of ξ . Writing $x_{i+1} = x_i a$, we have $x_i \xi \rightarrow x_{i+1} \rightarrow x_{i+1} a \rightarrow x_{i+2}$

so that $(x_1 a, x_2 a, \dots, x_r a)$ is a cycle of η of the same order r . Thus ξ and η have the same cycle-type. Conversely, if ξ and η have the same cycle-type, we can find a permutation τ of X which maps each cycle of ξ into a corresponding cycle of η . If the cycle-type of ξ and η is $(1^{m_1} 2^{m_2} \dots)$, this τ can be chosen in precisely $\prod_{h=1}^{\infty} \frac{h!^{m_h} m_h!}{h^{m_h}}$ distinct ways.

Lemma 4-4 Taking $\xi = \eta$ gives the first part of with a given element ξ of type $(1^{m_1} 2^{m_2} \dots)$ is $\prod_{h=1}^{\infty} \frac{h!^{m_h} m_h!}{h^{m_h}}$. There is always an odd τ which commutes with ξ excepting only when the different cycles of ξ have different odd orders i.e. when $m_{2i} = 0$ and $m_{2i-1} \leq 1$ for all $i = 1, 2, \dots$

For if ξ has two cycles (x_1, x_2, \dots, x_r) and (y_1, y_2, \dots, y_s) of

H of the class in question. The set of all $\alpha \in G$ such that $H^\alpha = H$ is subgroup $N_G(H)$ of G which contains H . It is called the normalizer of H in G . Since $H^\alpha = H$ is equivalent to $\alpha H = H\alpha$, the normalizer of H is the largest subgroup N of G such that $H \triangleleft N$. Applying 4.5 to the representation χ^H of G , we obtain

Corollary 2. If H is any subgroup of G , then $|G : N_G(H)|$ is equal to the number of subgroups conjugate to H in G .

Note that a subgroup is always conjugate to itself. $H \triangleleft G$ if and only if $N_G(H) = G$. A normal subgroup is conjugate only to itself.

(G) Let G permute X transitively. Then, for given x and y in X , we have $x\alpha = y$. Then $x\beta = x$ if and only if $y\alpha^{-1}\beta\alpha = y$. Hence $S_G(y) = \alpha^{-1}S_G(x)\alpha$. In a transitive representation, the stabilizer forms a class of conjugate subgroups. Suppose we have a second representation in which G permutes X' transitively. These two representations are called equivalent if and only if there is a one-to-one mapping θ of X onto X' such that $x\theta\alpha = x'\alpha'$ for all $\alpha \in G$. If this is the case, then $|X| = |X'|$ and $S_G(x\theta) = S_G(x)$ for all $x \in X$. So equivalent representations have the same stabilizers, and of course the same degree.

Conversely, suppose that for some $x \in X$ and $x' \in X'$ we have $S_G(x) = \theta S_G(x') = H$. Let T be a transversal to H in G . Then each element of X is expressible uniquely in the form $x\tau$ with $\tau \in T$. Similarly each element of X' is expressible uniquely in the form $x'\tau'$ with $\tau' \in T$. Hence we have a one-to-one mapping θ of X onto X' defined by $(x\tau)\theta = x'\tau'$. Given $\alpha \in G$, $\tau \in T$ there is a unique $\tau' \in T$ such that $H\tau\alpha = H\tau'$. We then have $(x\tau)\theta\alpha = x'\tau'\alpha = x'\tau'\alpha = (x'\tau')\theta = (x\tau\alpha)\theta$ and so the two representations are equivalent.

Given any subgroup H of G , the regular representation τ of G induces a representation τ_H of G by permutations of the cosets of H . For $\alpha \in G$, $\tau_H(\alpha)$ is by definition the mapping

$$\tau_H(\alpha): H\xi \rightarrow H\xi\alpha \quad (\xi \in G).$$

For any ξ, η in G , $\tau_H(\xi\eta)$ maps $H\xi$ into $H\xi\eta$. Thus τ_H is a transitive representation of G . The stabilizer of the coset H in this representation is precisely H itself. Summing up, we have

Theorem 4.6. In any transitive representation of a group G , the stabilizers form a class of conjugate subgroups of G . Two transitive representations are equivalent if and only if they have the same stabilizers. If H is one of the stabilizers for τ , then τ is equivalent to τ_H .

In any representation of G , the kernel is the intersection of the stabilizers. In the representation τ_H , the stabilizers are the conjugates of H in G . Hence the kernel of τ_H is the group

$$K = \bigcap_{\beta \in G} H^\beta = K_G(H).$$

If $L \triangleleft G$ and $L \leq H$, then $L = L^\beta \leq H^\beta$ for all $\beta \in G$. Hence $L \leq K$. Thus $K_G(H)$ is the largest normal subgroup of G contained in H .

Note that $\tau_2 \circ \tau$ is precisely the regular representation itself.

(H) If H is any subgroup of G and $N = N_G(H)$, we have a representation t_H of N by automorphisms of H . Here $t_H(x)$ is for any $x \in N$ the restriction of $t(x)$ to H . The kernel of this representation t_H is called the centralizer of H in G and denoted by $C_G(H)$. It consists of all elements of G which commute with every element of H . Hence $t_H(N) \cong N_G(H)/C_G(H)$. This group $t_H(N)$ will be called the automorphism of H in G and denoted for preference by $A_G(H)$. It consists of all automorphisms of H which can be induced by transforming H by some element of G . Hence

$$A_G(H) \cong N_G(H)/C_G(H)$$

Note that $C_G(G) = Z(G)$ is the centre of G and $A_G(G) = \mathcal{L}(G)$ is the group of inner automorphisms of G .

If we are given a representation f of a group Γ by automorphisms of another group G , we shall say that G admits Γ as group of operators. If G is a multiplicative group, it is usually convenient to use the notation

$$\xi \rightarrow \xi^\alpha \quad (\xi \in G)$$

for the automorphism $f(\alpha)$ of G . Here $\alpha \in \Gamma$. If G is an additive group, as often happens, we write $\xi\alpha$ instead of ξ^α . The laws which apply are

$$(\xi\eta)^\alpha = \xi^\alpha\eta^\alpha; \quad \xi^\alpha = \xi; \quad \xi^{\alpha\beta} = (\xi^\alpha)^\beta$$

or $(\xi + \eta)\alpha = \xi\alpha + \eta\alpha; \quad \xi\alpha = \xi; \quad \xi(\alpha\beta) = (\xi\alpha)\beta$

according to the case. Here $\xi, \eta \in G$ and $\alpha, \beta \in \Gamma$, while ξ is the unit element of Γ .

A subgroup H of G is F-admissible if $H^\alpha = H$ for all $\alpha \in \Gamma$. This and intersections of admissible subgroups are admissible. A section H/K of G is admissible if both H and K are admissible. If this is the case, then the mapping

$$K\xi \rightarrow K\xi^\alpha \quad (\xi \in H)$$

is well-defined for $\alpha \in \Gamma$. If we denote it by $f_{H/K}(\alpha)$, it is well-known an automorphism of H/K .

then $f_{H/K}$ is a representation of Γ , and H/K admits Γ as group of operators. If G is any other group admitting Γ as group of operators, then we say that G and G' are operator-isomorphic, or more precisely F-isomorphic if there is an isomorphism $\xi \rightarrow \xi'$ of G onto G' , such that $(\xi')^\alpha = (\xi)^\alpha$ for all $\xi \in G$ and all $\alpha \in \Gamma$. This notion of F-isomorphism is analogous to that of equivalent representations. It is clear that if two admissible sections of G are incident, they are necessarily F-isomorphic. Hence we may state as a corollary of 3.7 and 3.8

Theorem 4.7. Let the group G admit the Γ as group of operators. Then any two F-admissible series of G have refinements whose factors are F-isomorphic in pairs. In other words, any two F-composition series of G , the factors are F-isomorphic in pairs.

Here, a F-composition series is a proper series whose terms are all F-admissible but which cannot be refined any further without losing this property. In other words, each factor group H/K of a F-composition series is F-simple, in the sense that no subgroup L exists such that $K < L < H$. This implies that H/K has no characteristic subgroup L/K other than the two obvious ones with $L=K$ and $L=H$. A group G which has exactly two characteristic subgroups is sometimes called characteristically-simple; but this does not mean that G is simple, for G may have normal subgroups M with $1 < M < G$ which are not characteristic.

The most important special case of 4.7, apart from the one already covered in 3.8, is the case $\Gamma = G$. A G-composition series of G is called a chief series of G and its factors are called chief factors of G . H/K is a chief factor of G if and only if $K \triangleleft G$ and H/K is a minimal normal subgroup of G/K . (M is a minimal normal subgroup of G if $M \neq 1$, $M \triangleleft G$ and M contains no normal subgroup L of G such that $1 < L < M$.)

11.

§5. p -groups. Sylow subgroups. p -groups

(A) Let p be a prime. A group whose order is a power of p is called a p -group.

p -groups have many special properties which do not belong to finite groups generally. A good many of these follow from Lemma 5.1. Let the p -group G be represented by permutations of a set X and let Y be the set of all elements of X which are left invariant by G . Then $|X| \equiv |Y| \pmod{p}$.

For by 4.5, if X_i is one of the transitivity classes in this representation of G , then $|X_i|$ divides the order of G . Since $|G|$ is a power of p , either $|X_i| = 1$ or else $|X_i|$ is divisible by p . Hence $|X| = \sum |X_i| \equiv |Y| \pmod{p}$, since $|Y|$ is the number of classes X_i with a single element.

Theorem 5.2 Let G be a p -group.

- (i) Let $K \triangleleft G$ and $K \neq 1$. Then $K \cap ZG \neq 1$.
- (ii) If $G \neq 1$, then $ZG \neq 1$.
- (iii) Every minimal normal subgroup of G has order p and lies in ZG .
- (iv) Let H be a proper subgroup of G . Then $H \triangleleft N_G(H)$.
- (v) Every subgroup of index p is normal in G .
- (vi) Every maximal subgroup of G is of index p , hence normal.
- (vii) Every chief factor of G has order p .

Proof: (i) Apply 5.1 to the representation ρ_K of G by automorphisms of K . This representation is faithful since $K \neq 1$. But $|K| = |K| = p^r$ with some $r > 0$ by 3.4, since $K \neq 1$. Hence $|Y|$ is at least p , by 5.1.

(ii) is the special case $K = G$ of (i).

(iii) If K is a minimal normal subgroup of G , then $K \neq 1$. Hence $K \cap ZG \neq 1$. But every subgroup of ZG is normal in G . Hence $K \cap ZG = K$ has order p by the minimality of K .

(iv) By 3.4, H is a p -group and $|G:H| = p^s$, with $s > 0$ since $H \triangleleft G$. The representation ρ_H of G is of degree p^s . Apply 5.1 to the restriction f of ρ_H to H . Then X is the set of all cosets of H in G and among these is H itself, which is invariant under H since $HH = H$. Since $|X| = p^s$, it follows that $|Y|$ is at least p , for it is not zero. Hence $H \cap H = H \neq 1$ for some element ξ of G which is not in H . This implies that $\xi H \xi^{-1} \subseteq H$ and so $\xi^{-1} \in N_G(H)$. Since $\xi \notin H$, we then find $H \triangleleft N_G(H)$.

(v) and (vi) are immediate consequences of (iv).

(vii) follows from (iv) since a chief factor H/K of G is a minimal normal subgroup of the p -group G/K .

A section L/M of any group G is called a chief factor of G if $M \triangleleft G$ and $L/M \cong Z(G/M)$. This implies that $L \triangleleft G$ and 5.2 (vii) shows that in a p -group G , every chief factor is a chief factor.

A group G is called nilpotent if it has a series whose factors are all chief factors of G . Hence all p -groups are nilpotent.

(B) Approaches the proof of 5.2 (iv), we note here Lemma 5.3. If H is any subgroup of a group G and $N = N_G(H)$, then the centralizer of $\rho_H(G)$ in $Z(G)$ consists of all the H -invariant mappings

$$\rho_H(\alpha): H \rightarrow \alpha H \alpha^{-1} = H \cap \alpha H \alpha^{-1} \quad (\alpha \in G)$$

with $\alpha \in N$. It is isomorphic with N/H and its order $|N:H|$ is the number of H -invariants of H in the representation ρ_H .

We have already noted that $H \cap \alpha H \alpha^{-1} = H \cap \alpha H \alpha^{-1}$ if and only if $\alpha \in N$ so that $|N:H|$ is the number of H -invariants of H in the representation ρ_H . If $\alpha \in N$, then $\alpha H \alpha^{-1} = H$ and $\rho_H(\alpha)$ is therefore a permutation of the set X of all cosets of H in G . If $\alpha \in H$, $\rho_H(\alpha)$ is the identity on X , but if $\alpha \notin H$ then $\rho_H(\alpha)$ maps H into $H \alpha \neq H$. Further, $\rho_H(\alpha\beta) = \rho_H(\beta)\rho_H(\alpha)$ so that $\alpha \rightarrow \rho_H(\alpha^{-1})$

(ii) $(\alpha \in N)$ is a representation of N with kernel H . Thus $\ker(\alpha) = N/H$. It is clear that $\ker(\alpha)$ commutes with $\tau_H(g)$ for all $\alpha \in N, g \in G$. If τ belongs to the centralizer of $\tau_H(G)$ in $Z(G)$ and if τ maps H into the exact H^0 , then τ must map H^0 into H^0 for all $g \in G$. Hence $H^0 \tau = H^0 \tau g$ for all $g \in H$, so $H^0 = H^0 \tau H$ and $H^0 = H^0 \tau$ with $\alpha \in N$. Thus $\tau = \ker(\alpha)$ and 5.3 is proved.

(c) Let p be a prime and let G be a group of order $p^n m$ where $(m, p) = 1$. Any subgroup of G of order p^k is called a Sylow p -subgroup of G . Thus S is a Sylow p -subgroup of G if and only if S is a p -group and $|G:S|$ is prime to p . If p does not divide $|G|$, then $n=0$ and the only Sylow p -subgroup of G is the unit subgroup.

The following theorem, due to L. Sylow 1832-1918, is fundamental.

- Theorem 5.10 (i) Every group G has at least one Sylow p -subgroup.
 (ii) The Sylow p -subgroups of G are all conjugate in G .
 (iii) Every p -subgroup of G is contained in at least one Sylow p -subgroup.
 (iv) The number of Sylow p -subgroups of G is $\equiv 1 \pmod{p}$.

Proof: (i) See 5.10. Let X be a subset of G with $|X| = p^n$ and let $g \in G$. Then $|Xg| = p^n$ by 1.6. Hence the regular representation of G induces a representation f of G :

$$f(g) : X \rightarrow Xg \quad (|X| = p^n)$$

on the $(\mathbb{F}_p)^m$ sets X . Since $(m, p) = 1$, the numbers $p^m - \tau$ and $p^m - \tau$ are divisible by the same highest power of p for each value of $\tau = 0, 1, 2, \dots, p^m - 1$. So the degree of f is prime to p and f has at least one transitive component f_i with degree m_i prime to p . Let X_i be any member of the corresponding transitivity class and let S be the stabilizer of X_i in G . Then $|X_i| = p^{m_i}$ and $|G:S| = m_i$ and $X_i S = X_i$. So X_i is the union of certain increase orbits of S and hence $|S| \equiv |X_i| \pmod{p}$. But $(m_i, p) = 1$ and $|G:S| = m_i$. Hence p^n divides $|S|$. Thus $|S| = p^n$ and S is a Sylow p -subgroup of G .

(ii) and (iii). Let H be any subgroup of G . In the proof of 5.2 (ii) we used the restriction to H of the representation τ_H of G . We now use the restriction f of τ_H to a second subgroup K of G . The transitivity class of f which contains a given exact H^0 consists of all cosets of H which are contained in $H^0 K$. The set $H^0 K$ is called a double coset of H, K ; so then double cosets of H, K in G have no common elements. Let T be a transversal to the double cosets of H, K in G , i.e. let T contain exactly one element from each of these double cosets. Then $G = \bigcup_{\tau \in T} H \tau K$ and the terms $H \tau K$ are disjoint in pairs. Hence

$$|G:H| = \sum_{\tau \in T} |H \tau K:H|.$$

The numbers $|H \tau K:H|$ are the degrees of the transitive components of the representation f of K . Now $\tau \notin H$ and τg lie in the same coset of H if and only if $\tau \notin g^{-1} \tau' \in H$ i.e. $\tau g^{-1} \tau' \in H$. Taking $\tau, \tau' \in H$ we obtain

$$\begin{aligned} \text{Lemma 5.5.} \quad & \text{Let } H \text{ and } K \text{ be subgroups of } G, \text{ let } \tau \in G \\ & \text{and let } T \text{ be a transversal to the double cosets of } H, K \text{ in } G. \text{ Then} \\ & |H \tau K:H| = |K:H \cap K| \\ & \text{and } |G:H| = \sum_{\tau \in T} |K:H \cap K|. \end{aligned}$$

Now let $H = S$ be a Sylow p -subgroup of G and let K be any p -subgroup of G . Then $|G:S|$ is prime to p and so, by 5.5, $|K:S \cap K|$ is prime to p for some $\tau \in T$. Since K is a p -group, it follows that $|K:S \cap K| = 1$ and so $K \subseteq S^\tau$, which is a Sylow p -subgroup of G . This proves (ii). If K is actually a Sylow p -subgroup of G , then $|K| = |S|$ and so $K = S^\tau$ is conjugate to S in G . This proves (iii).

(iv) Here we take $H = N_G(S)$ and $K = S$ in 5.5. By 4.5 Cor. 2, the number l of Sylow p -subgroups of G is equal to $|G:H|$, since they are all conjugate to S by (ii). If S is of H^τ , then $|S:H^\tau \cap S|$ is a positive power of p . If $S \subseteq H^\tau$, then

$S_1 = \tau S \tau^{-1} \trianglelefteq H$ and S_1, S are Sylow p -subgroups of H . But $S \triangleleft H$ and so $S = S_1$ by (iii). Hence $\tau \in H$ and $H \tau S = HS$. Thus there is exactly one double coset of H, S via the product HS , for which $(S : H \tau S)$ is not divisible by p . For this exceptional one, $\tau \in H$ and $(S : H \tau S) = 1$. Thus $\ell \equiv 1 \pmod p$ and Sylow's Theorem is completely proved.

(D) A few immediate consequences of Sylow's Theorem are worth recording as corollaries.

Corollary 5.1.1 If p^r divides $|G|$, then G has subgroups of order p^i for $i = 0, 1, \dots, r$. For by 5.2 (iii), a Sylow p -subgroup S of G has subgroups of every order $1, p, p^2, \dots, p^r = |S|$.

On the other hand, the tetrahedral group Σ_4 has subgroups of order 12 has no subgroups of order 6. (The symmetric and alternating groups $\Sigma(X)$ and $\mathbb{Z}^+(X)$ with $|X| = n$ will often be denoted by Σ_n, \mathbb{Z}_n^+ when it is not necessary to indicate the set X . This may be taken as the set of integers $1, 2, \dots, n$.)

Note that a subgroup H of index 2 in G is always normal:

$$|G:H| = 2 \implies H \triangleleft G.$$

For $\exists H = Hg$ for all $g \in G$, while if $\exists \in G - H$ (in G but not in H), then again $\exists H = Hg$ since each of these sets contains $G - H$.

Corollary 5.1.2 Every normal p -subgroup \mathbb{Z} of G is contained in $K_G(S) = \bigcap_{g \in G} S^g$, where S is a Sylow p -subgroup of G . $K_G(S)$ is the unique maximal normal p -subgroup of G .

~~For by 5.1.1 (iii), $M \trianglelefteq S^g$ for some $g \in G$. Since $M \trianglelefteq G$, we then have $M = M^g \trianglelefteq S^g$ for all $g \in G$. Hence $M \trianglelefteq K_G(S)$.~~

For by 5.4 (iii), $M \trianglelefteq S^g$ for some $g \in G$. Since $M \trianglelefteq G$, we then have $M = M^g \trianglelefteq S^g$ for all $g \in G$. Hence $M \trianglelefteq K_G(S)$.

Corollary 5.1.3 If H and K are subgroups of G such that $N \trianglelefteq H \trianglelefteq K$, where N is the normalizer in G of the Sylow p -subgroup S of G , then $|K:H| \equiv 1 \pmod p$.

For S is a Sylow p -subgroup of both H and K ; and $N = N_H(S) = N_K(S)$. Hence $|H:N|$ and $|K:N|$ are the numbers of Sylow p -subgroup of H and K , respectively, by 5.4 (ii). So $|H:N| \equiv |K:N| \equiv 1 \pmod p$ by 5.4 (iv). Consequently $|K:H| \equiv 1 \pmod p$.

More important is

Lemma 5.5.6 Let $K \triangleleft G$ and let S be a Sylow p -subgroup of G . Then KS/K is a Sylow p -subgroup of G/K and $S \cap K$ is a Sylow p -subgroup of K .

Proof: By 5.5.5, $|KS:K| = |S:K \cap S|$ which is a power of p ; while the index of KS/K in G/K is $|G:KS|$ by 3.5, and this divides $|G:S|$ by 2.6. Hence $|G:KS|$ is prime to p .

Again, $|S \cap K|$ is a power of p ; while $|K:S \cap K| = \frac{|K|}{|S \cap K|}$ is a power of p and $|KS:K|$ is prime to p .

Corollary: If $|G:K|$ is prime to p , then $S \trianglelefteq K$. If $|G:K|$ is a power of p , then $KS = G$.

Let ω be any set of primes. A ω -number is a number whose prime factors all lie in ω . Let ω' be the complementary set of primes. Every positive integer $n = \omega \omega'$, where ω is a ω -number and ω' is a ω' -number uniquely determined by n . A S_{ω} -subgroup of a group G is any subgroup S of order $|S|_{\omega}$ i.e. such that $|S|$ is a ω -number and $|G:S|$ is a ω' -number. When $\omega = p$ is a single prime, we come back to the notion of a Sylow p -subgroup which is the same as an S_p -subgroup. G is called a ω -group whenever $|G|_{\omega} = |G|$ or what is the same $|G|_{\omega'} = 1$.

Remark: The symbol ω should not be confused with ω , the last letter of the Greek alphabet. In fact ω is a special form of π .

often used by mathematicians: it is not omega but p_1 in the sky.
 Clearly the proof of 5.6 applies equally well with p replaced by an arbitrary set of primes. So we may state
Lemma 5.7 Let $K \triangleleft G$ and let S be any Set-subgroup of G . Then KS/K is an S -subgroup of G/K and $S \cap K$ is an S -subgroup of K .

We also need
Lemma 5.8 If G has a normal S -subgroup K , then every ω -subgroup and every ω -subgroup of G is contained in K .

For let H be a ω -subgroup of G . Since $K \triangleleft G$, KN is a subgroup of G . By 5.5, $|KN| = |K| \cdot |N|$. Hence $H \cap KN$ is a ω -number and $|K| = |G|_{\omega}$. Hence $H \cap KN = K$ and $H \subseteq K$.

Consequences of Sylow's Theorem.

§6. Sylow's Theorem on Normal Subgroups. Nilpotent Groups

(A) Let H be a subgroup of G .

(1) We call H subnormal in G if and only if it belongs to some series of G i.e. if and only if there exist subgroups H_i such that $H = H_0 \triangleleft H_1 \triangleleft H_2 \triangleleft \dots \triangleleft H_n = G$. (2)

(2) We call H normal in G if and only if H is conjugate to H^z in $\{H, H^z\}$ for all $z \in G$.

(3) We call H characteristic in G if and only if $N_G(H) = H$.

(4) We call H self-normalizing in G if and only if it is both normal and characteristic in G .

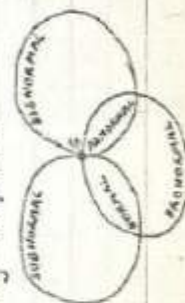
These relations will be handled as follows:

$H \text{ s.n. } G, H \text{ p.n. } G, H \text{ char. } G, H \text{ s.n. } G$ respectively.

Lemma 6.1 $H \triangleleft G$ if and only if H is both subnormal and normal in G .

Obviously $H \triangleleft G$ implies $H \text{ s.n. } G$ and also $H \text{ p.n. } G$, for $H = H^z$ for all $z \in G$. Suppose $H \text{ s.n. } G$ but that H is not normal in G . We may assume that (1) holds and that H is not normal in H_2 . Then $H \neq H^z$ for some $z \in H_2$. Then $H^z \triangleleft H_1$ since $H_1 \triangleleft H_2$ and so $J = \langle H, H^z \rangle \triangleleft H_1$. But $H \triangleleft H_1$ so $H \triangleleft J$ and H cannot be conjugate to H^z in J . Hence H is not normal in G . Thus $H \text{ s.n. } G$ and $H \text{ p.n. } G$ imply $H \triangleleft G$.

The only subgroup of G which is both subnormal and characteristic in G is itself. So we may picture these five classes of subgroups diagrammatically as follows:



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Theorem 6.4 Let H be a p -subgroup of G . Then H is normal in G if and only if each Sylow p -subgroup of G contains exactly one of the conjugates of H in G .

Choosing P to contain H , this situation is expressed by saying that H is weakly closed in P with respect to G . This is the Wicklund terminology; but here we shall simply say H is normal in G .

Corollary 6.4.1 If $H \leq P$ where P is a Sylow p -subgroup of G and if H is normal in G , then $N_G(H) \leq N_G(P)$ for every subgroup Q and that $H \leq Q \leq P$.

Corollary 6.4.2 If H_1 and H_2 are normal p -subgroups of G contained in the same Sylow p -subgroup P of G , then $\langle H_1, H_2 \rangle$ is normal in G .

Note that H_1 and H_2 are normal in P by 6.4.1. If $(H_1, H_2)^{g^{-1}} \leq H_1$, then $H_1^g \leq P$ and so $H_1^g = H_1$ ($g \in P$) since H_1 is normal in P . So $(H_1, H_2)^{g^{-1}} \leq H_1$. So H_1, H_2 are normal in P by 6.4.1. If $(H_1, H_2)^{g^{-1}} \leq H_2$, then $H_2^g \leq P$ and so $H_2^g = H_2$ ($g \in P$) since H_2 is normal in P . So $(H_1, H_2)^{g^{-1}} \leq H_2$. Thus $\langle H_1, H_2 \rangle$ is normal in P .

It follows from 6.4.2 that if Q is any p -subgroup of G , then Q is a uniquely determined maximal normal p -subgroup of G , contained in P . If this is H , then $H \leq K \leq Q$ implies that K is not normal in G . By 6.4.1, $H \leq K \leq Q$. The normal subgroups of G contained in P are Sylow p -subgroups of G for a lattice $\mathcal{L} = \text{Con}_G(P)$. But this is not a sublattice of the lattice of all subgroups of G in general. For join H_1, H_2 in \mathcal{L} , although $H_1, H_2 \in \mathcal{L}$ by 6.4.2, in general $H_1 \vee H_2$ does not belong to \mathcal{L} , i.e. $H_1 \vee H_2$ is usually smaller than H_1, H_2 and even smaller than $[H_1, H_2]$. For example, if G is the dihedral group of order $2p$, then \mathcal{L} is an \mathcal{L} -group. The subgroups of order 2 in G are all normal, but none of the subgroups of order 2 is normal. So \mathcal{L} consists here of \mathcal{L} itself, 1 and the three subgroups of order 2 in \mathcal{L} .

Corollary 6.4.3 If $K \leq G$, then any Sylow subgroup of K is normal in G . For a Sylow subgroup P of G contains only one Sylow subgroup of K , viz. $P \cap K$.

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This picture represents a general case.

Lemma 6.3 Let K be a subgroup of G containing H . If H is a Sylow p -subgroup of K , then H is a Sylow p -subgroup of G if and only if H is normal in G .

This is clear except for \Rightarrow , in which case we may state the stronger result:

Lemma 6.3.1 Let H be a Sylow p -subgroup of K and let L be any subgroup of G . Then $L \cap H$ is a Sylow p -subgroup of $L \cap K$.

(i) If H is normal in G , then $H \leq L \cap K$ and $L \cap H$ is a Sylow p -subgroup of $L \cap K$.

(ii) If H is not normal in G , then $H \leq L \cap K$ and $L \cap H$ is a Sylow p -subgroup of $L \cap K$.

Proof: (i) We may assume $H = H_1 \leq H_2 \leq \dots \leq H_r = K$. Then $L \cap H_i \leq L \cap H_{i+1}$ for each $i=1, 2, \dots, r$ by 3.1.4 and so $L \cap H$ is a Sylow p -subgroup of $L \cap K$. (ii) It is sufficient to show that $H \leq L \cap K$ and $L \cap H$ is a Sylow p -subgroup of $L \cap K$. By (i), $H \leq L \cap K$ and so $L \cap H$ is a Sylow p -subgroup of $L \cap K$.

(3) Suppose that $J = \langle H, H^g \rangle$ is a p -group, $J \leq G$ and H is a Sylow p -subgroup of J . By 6.2, H is normal in J . By 5.2 (1), $H \leq J$. Hence $H \leq J$ by 6.1.3 and so $H = H^g$ since H is normal in J .

Conversely, let H be a Sylow p -subgroup of G . By 5.4 (ii), H is contained in some Sylow p -subgroup S of G . Suppose that no other conjugate of H in G is contained in S . Let $T = \langle H, H^g \rangle$, $J \leq G$. By 5.4 (ii), $H \leq T \leq S$, where T is a Sylow p -subgroup of J and S is a Sylow p -subgroup of G . Also by 5.4 (ii) and (i), we have $H^g \leq T$ for some $g \in G$. Since S and J are conjugate in G by 5.4 (ii), they contain the same number of conjugates of H in G . By hypothesis, this number is one. Hence $H^g = H$ and H is normal in G .

Thus we have proved.

These normal subgroups of G which have the form $P \cap K$ with $K \in \mathcal{G}$ form a special subalgebra \mathcal{P}_0 of \mathcal{P} . Unlike \mathcal{P} , \mathcal{P}_0 is also a sublattice of the lattice of all subgroups of G . For if K_1 and K_2 are normal in G , so is $K_1 \cap K_2$ and $P \cap (K_1 \cap K_2) = (P \cap K_1) \cap (P \cap K_2)$. Further, $P \cap K_1 K_2 = (P \cap K_1)(P \cap K_2)$.

(C) The normal subgroups of a group G can also be characterized by their relation to the transitive permutation representations of G . Suppose that G is represented transitively by permutations of a set X . Let H be a subgroup of G and let $N = N_G(H)$. Let Y be the set of all $y \in X$ which are invariant under H . If $y \in Y$ and $z \in N$, we have $yzH = y$ and so $yH = yzH = yz$. Hence $y \in Y$. Thus N leaves Y invariant. $YN = Y$.

Let S_y be the stabilizer of $y \in G$. If y and z are in Y , we have $yz = z$ for some $g \in G$ since G permutes X transitively. Since $H \leq S_y \cap S_z$, we have $H^g \leq S_y^g = S_z$. If H perm G , it is conjugate to $H^g \leq \{H, N\}^g$ and hence also in S_z . So $H^g \leq H$ for some $g \in S_y$. So $g \in N$ and $yH = z$. Thus N permutes Y transitively whenever H is normal in G .

Conversely, suppose that in any transitive representation of G , N permutes transitively the symbols left invariant by H . Let $z \in G$ and let $J = \{H, H^z\}$. Then $JH = J$ and $JH^z = J$ or $JH^z = JH^z$. Hence, in the transitive representation of G , H leaves fixed the sets J and J^z . By hypothesis, $J = J^z$ for some $g \in N$. Then $g^z \in J$ and transforms H^z into $H^g = H$. This is true for all $z \in G$. Hence H perm G .

Thus we have proved the first part of

Theorem 6.1. Let H be a subgroup of G and let $N = N_G(H)$. Then (1) H perm G if and only if, in every transitive representation of G , N permutes transitively the symbols left invariant by H .
 (2) H also perm G if and only if, in every transitive representation of G , H leaves at most one of the symbols invariant.

To prove (3), let H also perm G and let G permute X transitively. Then by (1) N permutes transitively the set Y of elements of X which are left invariant by H . But $H = N$. Hence $|Y| = 1$.

Conversely, if $|Y| = 1$ for every transitive representation of G , then H perm G by (1). If we take the representation of G to be \mathcal{P}_0 , then $|Y| = |N \cap H|$ by 5.3 and so $N = H$ is normal in G . Hence H also perm G .

By 4.6, every transitive representation of G is equivalent to \mathcal{P}_0 for some subgroup K of G . In \mathcal{P}_0 , H leaves the set $K \setminus \{1\}$ invariant if and only if $H \leq K^g$. Thus $|Y| = 0$ unless H is contained in some conjugate of K . We may therefore restate the criterion for abnormal subgroups in the following form:

Corollary 6.1. Let H be a subgroup of G . Then the following conditions are equivalent.

(1) H also perm G .
 (2) For any subgroup K of G and any $g \in G$, $H \leq K \cap K^g$ implies $g \in K$.
 (3) Every subgroup of G containing H is normal in G , and H is not contained in the intersection of any two distinct conjugate subgroups of G .

We note also the following

Lemma 6.7. (1) If H perm G and $K \leq G$, then $H \cap K$ perm G .
 (2) If $K \leq G$ and $K \leq H$, then H perm G if and only if H/K perm G/K .
 (3) If $K \leq G$, H perm HK and $H \cap K$ perm G , then H perm G .
 (4) If $G = \langle H, K \rangle$ and $H \cap K = 1$, then $G \cong H \rtimes K$ and H perm G if and only if K perm G .

Since $H \cap K$ perm G , there is an element $g \in G$ such that $g \in H$ and $g \in K$. Such an element is trivial.

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If K pm G , then K pm N by 6.2 and so K/H pm N/H . Conversely, if K/H pm N/H , then K pm N . For any $g \in G$, there is an element ξ in $\langle H, H^g \rangle \leq J = \langle K, K^g \rangle$ such that $H^{\xi} = H$, since N pm G . Then $H \leq K^{\xi}$ and so $K^{\xi} \leq N$. Since K pm N , we have $K^{\xi} = K$ for any $\xi \in J = \langle K, K^g \rangle$. But $\xi \in J$ and so $J \leq J$ and hence $J \leq J$. Thus K and K^g are conjugate in J . This holds for all $g \in G$ and so K pm G .

Lemma 6.6f Let $K \trianglelefteq G$ and let H/K be a normal p -subgroup of G/K . Then any Sylow p -subgroup I of M is normal in G .

Proof: Since H/K is a p -group, we have $H = K \cdot I$. Let $J \leq G$ and let $J = \langle P, P^g \rangle$. Then $\langle H, H^g \rangle = JK$ and since H pm G , we have $H^{\xi} = H$ for some $\xi \in J, \xi \in K$. Then P and P^{ξ} are Sylow p -subgroups of $H = H^{\xi}$ and hence they are conjugate in their join $J = \langle P, P^{\xi} \rangle$. But $J \leq J$ since $g \in J$. Hence P and P^g are conjugate in J . This holds for all $g \in G$ and so I pm G .

This lemma shows that every normal p -subgroup of G/K is the image in the natural epimorphism of G into G/K of some normal p -subgroup of G .

Lemma 6.6g Let $G = LM$ where M is a normal p -subgroup of G and $L \cap M = 1$. Let I be a Sylow p -subgroup of L , $I_1 = [M, I]$ and $H = PP$. Then H is normal in G .

Proof: $Q = PM$ is a Sylow p -subgroup of G . Let $N = N_G(I)$. Since G is G/M have the same number of Sylow p -subgroups, we have $N_G(I) = NM$. Let $H^g = Q$. Then $Q^g = P^g M$ since $M \trianglelefteq G$ and $P^g \leq Q, P^g M \trianglelefteq NM$. Hence $Q^g = Q$ and $g \in N, g \in M$. But $P^g \leq P$ and so $P^g = P$ and hence $H^g = H$. Then $H^g = H^g$. But $P \trianglelefteq M$ by 7.1(5) and $P^g \leq P [P, M] = H$. Hence $H^g = H$, and so Q contains only one conjugate of H in G . So H pm G as stated.

(D) We now need some easy consequences of 6.6.

Lemma 6.6i The normalizers of normal subgroups are abnormal.

Lemma 6.6j Any subgroup containing an abnormal subgroup is abnormal.

Lemma 6.6k Let H pm G and let $N = N_G(H)$. If two elements of the centre of H are conjugate in G , they are conjugate in N . If two normal subgroups of H are conjugate in G , they are conjugate in N .

Lemma 6.6l Let H pm G and let $N = N_G(H)$ and let $K = \langle H^g \rangle$ be the normal closure of H in G . Then $KN = G$. Further, if $H \leq L$ with $L \trianglelefteq G$, then $K \leq L$, and so $LN = G$.

Proof of 6.6k. Let $\xi \in G$. Then $J = \langle H, H^{\xi} \rangle \leq K$ and so $H^{\xi} \leq H$ for some $g \in K$. Hence $\xi = (g^{\xi})g^{-1} \in NK = KN$, and so $KN = G$.

Lemma 6.6l. Let $L \trianglelefteq G$. Since $H \leq L$, $L \trianglelefteq G$, we have $K \leq L$. Since H pm G , we have $K = \langle H^g \rangle$ and hence $H \leq L$ and so $L \trianglelefteq G$.

Lemma 6.6m. Let $L \trianglelefteq G$. Then $J = \langle H, H^g \rangle \leq K$ and so $H^g \leq H$ for some $g \in K$. Hence $\xi = (g^{\xi})g^{-1} \in NK = KN$, and so $KN = G$.

Lemma 6.6n. Let $L \trianglelefteq G$. Since $H \leq L$, $L \trianglelefteq G$, we have $K \leq L$. Since H pm G , we have $K = \langle H^g \rangle$ and hence $H \leq L$ and so $L \trianglelefteq G$.

Lemma 6.6o A normal subgroup cannot be permutable with any of its conjugates (other than itself).

Lemma 6.6p. Let H pm G and $H^g \trianglelefteq H, g \in G$. Then we can choose $\xi \in J = \langle H, H^g \rangle$. But $H^{\xi} = H^g H$ would imply $J = H H^g$ by 2.7, contrary to 7.6(5), sorry!

Lemma 6.6q. Let H pm G and let $H \leq K \leq N = N_G(N)$. Then $N_G(K) \leq N$ and K pm G if and only if K/H pm N/H .

Lemma 6.6r. Suppose $K^g = K, g \in G$. Since $K \leq N$, we have $H^g \leq K$. But $H^g \leq K$ and $H \leq J = \langle H, H^g \rangle$ which is contained in K . Hence $H^g = H$ and $J \trianglelefteq N$.

(E). Theorem 6.5 The following conditions on a group G are equivalent:
 (i) G is nilpotent.
 (ii) No proper subgroup of G is diagonal in G .
 (iii) Every subgroup of G is subnormal in G .
 (iv) Every pronormal subgroup of G is normal in G .
 (v) Every maximal subgroup of G is normal in G and therefore contains G' .
 (vi) Every Sylow subgroup of G is normal in G .

Proof: (i) \Rightarrow (ii). For let H be a proper subgroup of G and let G_i be the i -th term in the series $1 = G_0 < G_1 < \dots < G_r = G$ be a central series of G . Let G_i be the first term in this series which does not contain H , let $y \in H$ and $y \notin G_i$. Since G_i/G_{i-1} is in the centre of G/G_{i-1} , we have $G_{i-1}y = yG_{i-1}$. Hence $G_{i-1}y^3y' = G_{i-1}y^3y' \in H$, since $G_{i-1} \leq H$ by definition of i . Therefore $y^3y' \in H$. This holds for all $y \in H$ and $y \in G_i$. Hence $G_i \leq N_G(H)$. Since $G_i \not\leq H$, we conclude that H is not diagonal in G .

(ii) \Rightarrow (iii) is clear.
 (iii) \Rightarrow (iv) follows from 6.1.
 (iv) \Rightarrow (v). For by 6.7, a maximal subgroup is either normal or abnormal, and in the second case it is pronormal. Alternatively, (v) \Rightarrow (vi) is clear.

(v) \Rightarrow (vi). For by 6.61 and 6.62 a maximal subgroup which contains the normaliser of a non-normal Sylow subgroup is abnormal and therefore not normal. Alternatively, (vi) \Rightarrow (v) since Sylow subgroups are pronormal.

(vi) \Rightarrow (i). By induction on $|G|$, we need the following easy Lemma 6.9. Let $H \triangleleft G$, $K \triangleleft G$ and $H \cap K = 1$. Then every element of H commutes with every element of K .
 For let $y \in H$, $y \in K$ and $y = y^3y' \in K$. Then $y^3y' \in K$ and so $y \in K$, while $y^3y' \in H$ and so $y \in H$. Since $H \cap K = 1$, we have $y = 1$ and $y^3y' = 1$.

Suppose now that $G \neq 1$ and let S be a Sylow p -subgroup of G , $S \neq 1$ and $Z = Z(S)$. Then $Z \neq 1$ by 5.2(ii). Let T be a Sylow q -subgroup of G with $q \neq p$. Then $Z \cap T = 1$. By hypothesis $T \triangleleft G$. Also $S \triangleleft G$ and Z char S , so $Z \triangleleft G$ by 4.2. Hence T centralises Z by 6.7. Also S centralises Z by 3.5. Thus $C = C_G(Z)$ contains a Sylow q -subgroup of G for all primes q , including $q = p$. So $C = G$ and $Z \triangleleft G$. By 5.6, ZT/Z is a Sylow q -subgroup of G/Z . Since $Z \triangleleft G$ and $T \triangleleft G$, we have $ZT \triangleleft G$ and so $ZT/Z \triangleleft G/Z$. Hence the Sylow subgroups of G/Z are all normal, by 5.4(i). By the induction hypothesis, G/Z has a central series with terms G_i/Z where $Z = G_0 < G_1 < \dots < G_r = G$. Since $Z \triangleleft G$, $Z = G_0 < Z = G_1 < \dots < G_r = G$ is a central series of G . Thus G is nilpotent.

Proof: (i) $\xi^2 = \xi[\xi, \eta]$. Hence $(\xi\eta)^2 = \xi[\xi, \eta]\xi\eta$ but also $\xi^2\eta^2 = \xi[\xi, \eta]\eta\xi$ and comparing gives (i).
 (ii) follows from (i) by inversion, using (1).
 (iii) In (i), let $\xi, \eta \in H$ and $\xi \in K$. Then $\xi[\xi, \eta]^2 = \xi[\xi, \eta]\xi[\xi, \eta]$ and $\xi^2\eta^2 = \xi[\xi, \eta]\eta\xi$. This is true for all such ξ, η, ξ . Hence $\xi[\xi, \eta]^2 = \xi[\xi, \eta]\xi[\xi, \eta]$. Similarly, in (ii), let $\xi \in H$ and $\eta, \xi \in K$. We obtain $K \subseteq N_G([H, K])$. Hence $[H, K]$ normalizes $[H, K]$; and also contains it. So (iii) follows.
 (iv) $\xi[\xi, \eta]^2 = [\xi^2, \eta^2]$ for all $\xi, \eta \in G$ and $\eta \in \Gamma$. Hence $[H, K]^2 = [H^2, K^2]$. If H and K are normal in G , the fact that $[H, K] \subseteq H \cap K$ follows from (v).
 (v) Let $\xi \in H, \eta \in K$. If H normalizes K , then $\xi[\xi, \eta] = \xi^2\eta$. If K normalizes H , then $\xi[\xi, \eta] = \xi\eta^2$. Hence $[H, K] \subseteq K$. Conversely, the latter relation implies that $\xi^2\eta^2 \in K$ for all $\xi \in H, \eta \in K$. So H normalizes K .
 (vi) Let $\xi \in L$ and $\eta \in G$. If L/M is a central factor of G , then $M\langle \xi \rangle$ commutes with $M\eta$, so $[M\xi, M\eta] = 1$. But $M \triangleleft G$ and so $[M\xi, M\eta] = 1$ in G . Then $[L, G] \subseteq M$. Conversely, let $[L, G] \subseteq M$. Since $M \leq L$, it follows that $[M, G] \subseteq M$ and so $M \triangleleft G$ by (v). Further, $[G, \eta] \in M$ and so $[M\xi, M\eta] = 1$. Hence $L/M \subseteq Z(G/M)$ and L/M is a central factor of G .
 (vii) If $K \triangleleft G$ and G/K is Abelian, then G/K is a central factor of G and so $G' \leq K$ by (vi). Conversely, if $G' \leq K$, then $[K, G] \leq K$ and $K \triangleleft G$ by (v); and so G/K is a central factor of G by (vii), hence G/K is Abelian.
 (viii) Take $\xi, \eta, \xi \in U$ and (i) to be arbitrary elements of H, K, L respectively and use (i) which ensures the normality of $[H, L], [K, L], \dots$

Upper - Lower Central Series

§7 Let ξ and η be elements of a group G . We define $[\xi, \eta] = \xi^{-1}\eta^{-1}\xi\eta = \xi^{-1}\xi\eta^{-1}\eta$ to be the commutator of ξ with η . Note that $[\eta, \xi] = [\xi, \eta]^{-1}$ and that $[\xi, \eta] = 1$ if and only if ξ and η commute. Let H and K be subgroups of G . We define $[H, K] = \langle [\xi, \eta] : \xi \in H, \eta \in K \rangle$. Hence $[H, K] = [K, H]$ and $[H, K] = 1$ if and only if every element of H commutes with every element of K .

The group $G' = [G, G]$ is called the (first) derived group of G . The second, third, ... derived groups of G are $G'' = [G', G']$, $G''' = [G'', G'']$ and so on. Besides 6.9, we have the following easy results concerning commutators

- Lemma 7.1
- (i) $[\xi\eta, \gamma] = [\xi, \gamma][\eta, \gamma]$.
 - (ii) $[\xi, \eta\xi] = [\xi, \eta][\xi, \eta]^{\xi}$.
 - (iii) $[H, K] \triangleleft \langle H, K \rangle$.
 - (iv) If G admits a group of operators Γ , then $[H, K]^{\xi} = [H^{\xi}, K^{\xi}]$ for all $\xi \in \Gamma$. In particular, if $H \triangleleft G$ and $K \triangleleft G$, then $[H, K] \triangleleft G$ and $[H, K] \subseteq H \cap K$.
 - (v) H normalizes K if and only if $[H, K] \leq K$.
 - (vi) If A is a section L/M of G is a central factor of G , if and only if $[L, G] \leq M$.
 - (vii) $K \triangleleft G$ and G/K is Abelian if and only if $G' \leq K$.
 - (viii) Let H, K and L be normal subgroups of G . Then $[HK, L] = [H, L][K, L]$ and $[H, KL] = [H, K][H, L]$.

The upper central series of G consists of the groups $Z^i G$ defined inductively as follows:

$$Z^0 G = Z(G) = \{g \in G \mid g \text{ is central}\}$$

$$Z^1 G = Z(G/Z^0 G) = \{g \in G \mid gZ^0 G \text{ is central in } G/Z^0 G\}$$

$$Z^2 G = Z(G/Z^1 G) = \{g \in G \mid gZ^1 G \text{ is central in } G/Z^1 G\}$$

$$\dots$$

Also we write $Z^i G = 1$. Hence

$$Z^0 G \leq Z^1 G \leq Z^2 G \leq \dots$$

and $Z^i G / Z^{i-1} G$ is the centre of $G/Z^{i-1} G$. We have

$$Z^i G \text{ char } G$$

for all i . $Z^0 G, Z^1 G, \dots$ are called the second, third, ... centres of G . We define

$$Z^{\infty} G = \bigcup_{i=0}^{\infty} Z^i G$$

to be the hypercentre of G . If r is the least non-negative integer such that $Z^r G = Z^{\infty} G$, then

$$Z = Z^0 G \leq Z^1 G \leq \dots \leq Z^r G = Z^{\infty} G$$

Suppose that $Z = Z_0 \leq Z_1 \leq Z_2 \leq \dots$ and that each Z_i / Z_{i-1} is a central factor of G . Then we have an ascending central series of G . By induction on n , we obtain

$$G_n \leq Z^i G \quad (n=0, 1, 2, \dots)$$

The upper central series is the uppermost ascending central series of G . Hence G is nilpotent if and only if $G = Z^r G = Z^{\infty} G$ for some $r \geq 0$. The least such r is called the class of G , following W.B. Rieu.

Abelian groups are the same as nilpotent groups of class 1.

~~Lemma 7.2 (1) If $H \trianglelefteq G$ and $H \cap Z^i G > H \cap Z^{i-1} G$ for some $i \geq 1$, then $H \cap Z^i G > H \cap Z^{i-1} G$ for all $i=1, 2, \dots, r$.~~

~~(2) If $|G| = p^n$ and $|H| = p^r$ and $H \trianglelefteq G$, then $H \leq Z^i G$.~~

~~(3) If $G' \neq 1$, then $G/Z^i G$ is not cyclic.~~

~~(4) If $|G| = p^n$, then, for $n > 1$, the class of G is at most $n-1$.~~

~~In particular, groups of order p^n are Abelian.~~

~~Proof: (1) Let $Z \in H \cap Z^i G$, $Z \notin H \cap Z^{i-1} G$ and let $r > 1$, $Z = z^{p^{r-1}}$. Then there is an element y of G such that $[Z, y] \neq 1$. Since $H \trianglelefteq G$, $[y, Z] \in H$ by 7.1(1). Also $[y, Z] \in Z^{i-1} G$ since $Z \in Z^i G$. But $Z y \neq Z$ and so $y \notin Z^{i-1} G = Z$. Hence $H \cap Z^{i-1} G > H \cap Z^{i-2} G$.~~

~~(2) follows from (1).~~

~~(3) If $G/Z^i G$ is cyclic, then $G = \langle Z^i G, x \rangle$ for some x and $[x, Z^i] = 1$ for all $Z \in Z^i G$. Hence G is Abelian.~~

~~(4) Let $G = Z^i G > Z^{i-1} G > \dots > Z^0 G = 1$ so that e is in the class of G . If $e = 1$, the result follows from $n > 1$. If $e > 1$, then $G/Z^i G$ is of order at least p^2 and so $|G| \geq p^{2i}$.~~

~~Lemma 7.2 (1) If $H \trianglelefteq G$ and $H \cap Z^i G > H \cap Z^{i-1} G$ for some $i \geq 1$, then $H \cap Z^i G > H \cap Z^{i-1} G$ for all $i=1, 2, \dots, r$.~~

~~(2) If $|G| = p^n$ and $|H| = p^r$ and $H \trianglelefteq G$, then $H \leq Z^i G$.~~

~~(3) If $G' \neq 1$, then $G/Z^i G$ is not cyclic.~~

~~(4) If $|G| = p^n$, then, for $n > 1$, the class of G is at most $n-1$.~~

~~In particular, groups of order p^n are Abelian.~~

~~Proof: (1) Let $Z \in H \cap Z^i G$, $Z \notin H \cap Z^{i-1} G$ and let $r > 1$, $Z = z^{p^{r-1}}$. Then there is an element y of G such that $[Z, y] \neq 1$. Since $H \trianglelefteq G$, $[y, Z] \in H$ by 7.1(1). Also $[y, Z] \in Z^{i-1} G$ since $Z \in Z^i G$. But $Z y \neq Z$ and so $y \notin Z^{i-1} G = Z$. Hence $H \cap Z^{i-1} G > H \cap Z^{i-2} G$.~~

~~(2) follows from (1).~~

~~(3) If $G/Z^i G$ is cyclic, then $G = \langle Z^i G, x \rangle$ for some x and $[x, Z^i] = 1$ for all $Z \in Z^i G$. Hence G is Abelian.~~

~~(4) Let $G = Z^i G > Z^{i-1} G > \dots > Z^0 G = 1$ so that e is in the class of G . If $e = 1$, the result follows from $n > 1$. If $e > 1$, then $G/Z^i G$ is of order at least p^2 and so $|G| \geq p^{2i}$.~~

~~(5) If X is any subset of the group G , we denote by X^G the set of all x^g with $g \in G$. Then $\{X^G\}$ is the least normal subgroup of G containing X . It is called the normal closure of X in G . Note that any subgroup of G which is generated by the union of a certain number of classes of conjugates in G is normal in G .~~

~~Theorem 7.3 Let H be a subgroup of G and let H_1, H_2, \dots, H_r be r conjugates of H in G , so that $r = |G : N_G(H)|$. Then $H^G = \langle H_1, H_2, \dots, H_r \rangle$.~~

~~The theorem is due to Itô.~~

~~Proof. Let $K = \langle H_1, H_2, \dots, H_r \rangle$ so that $K \trianglelefteq (H^G)$. Every element $g \in K$ can be expressed in the form~~

$$g = z_1 z_2 \dots z_r \quad (2)$$

~~where $z_i \in H_{i_i}$, $i=1, 2, \dots, r$. We say that the expression (2) the~~

The groups \mathcal{G}_n & defined inductively by $\mathcal{G} = \mathcal{G}_1$, $[\mathcal{G}_n, \mathcal{G}] = \mathcal{G}_{n+1}$, $n=1,2,\dots$ are characteristic subgroups of \mathcal{G} , by 7.1(6) and the series $\mathcal{G} = \mathcal{G}_1 \supseteq \mathcal{G}' = \mathcal{G}_2 \supseteq \mathcal{G}'' = \mathcal{G}_3 \supseteq \dots$ is called the lower central series of \mathcal{G} .
 Suppose that $\mathcal{G} = \mathcal{G}_1 \supseteq \mathcal{G}_2 \supseteq \mathcal{G}_3 \supseteq \dots$ and that each $\mathcal{G}_i/\mathcal{G}_{i+1}$ is a central factor of \mathcal{G} . Then we have a descending central series of \mathcal{G} .
 By induction on n , we obtain $\mathcal{G}_n \supseteq \mathcal{G}_n^c$ ($n=1,2,3,\dots$).
 The lower central series of \mathcal{G} is the lowest descending central series of \mathcal{G} .
 \mathcal{G} is nilpotent if and only if $\mathcal{G}_n = 1$ for some $r \geq 0$ and the least such r is the class of \mathcal{G} . If c is the class of \mathcal{G} , then $\mathcal{G}^c \supseteq \mathcal{G}_{c-1}^c$ ($n=0,1,\dots,c$).

Lemma 7.5 (i) If $G = \langle X \rangle$, then \mathcal{G}_n is generated by all the commutators $[x_1, x_2, \dots, x_n]$ with $x_i \in X$ together with their conjugates in G .
 (ii) If $H \triangleleft G$ and $\mathcal{G}_r \subseteq H$ for some r , then $\mathcal{G}_k \subseteq H$ for all $k \geq r$.
 (iii) If G is nilpotent and H is any subgroup such that $H\mathcal{G} = G$, then $H = \mathcal{G}$.
 (iv) If H is a subgroup of G and $\mathcal{G}_n \subseteq H$ for some n , then H is a normal subgroup of G .
 (v) By hypothesis, every element of \mathcal{G}_2 has the form $[x, y]$ with $x, y \in \mathcal{G}_1$. Hence \mathcal{G}_2 is generated by elements of the form $[x, y] = [x, z]$ with $x \in \mathcal{G}$. Hence $[x, z] \in H$ since $H \triangleleft G$ and $[x, z] \in K = \mathcal{G}_2$. Since $K \triangleleft G$, we then have $\mathcal{G}_2 \subseteq H$.
 (vi) This is a consequence of (v). If $H \triangleleft G$, let M be a maximal subgroup of G containing H . Then by 6.8(v), $M \triangleleft G$ and so G/M is cyclic of prime order, and $M \supseteq \mathcal{G}'$ by 7.1(6). Hence $H\mathcal{G}' \subseteq M\mathcal{G}' = M$ contrary to hypothesis.
 (vii) An obvious generalization on n .

belongs to the ordered multiplet $i = (i_1, i_2, \dots, i_p)$. Choose (i) so that s is as small as possible; and among the possible expressions for β with minimum s , choose (i) so that i comes as early as possible in the lexicographic ordering of ordered multiplets. Suppose if possible that for some $\lambda < p$, we have $i_\lambda \geq i'_\lambda$. Then we have $\beta = \beta' / \beta'' \dots \beta''$ (ii) where $\beta_r = \beta'_r$ for $r < \lambda$ and also for $r > p$; $\beta'_\lambda = \beta''_\lambda$; and for $\lambda < r < p$ $\beta'_r = \beta''_r \beta_{r-\lambda}$. Then (ii) belongs to the ordered multiplet $j = (j_1, \dots, j_p)$ of length s ; and j precedes i in lexicographic order since $j_p < i_p$. This contradicts the hypothesis of (i). Hence $i_r \leq i'_r \leq \dots \leq i_p$. But if $i_\lambda = i'_\lambda$, then the two neighbouring factors $\beta'_\lambda, \beta''_\lambda$ in (ii) can be combined to a single factor which likewise belongs to $H_\lambda = H_{\lambda+1}$, again contrary to the minimality of s . Hence $i_1 < i'_1 < \dots < i'_p$ and so $\beta \in H_1 H_2 \dots H_p$. Since β was any element of K , we then obtain $K = H_1 H_2 \dots H_p$ as stated.

(P). Lemma 7.6 Let $G = \langle X \rangle$ and let $H = \langle Y \rangle$. Then $[H, \mathcal{G}] = \langle Z \rangle$ where Z is the set of all commutators $[y, x]$ with $y \in Y, x \in X$.
 Proof: By 7.1(6), $K = [H, \mathcal{G}] \triangleleft G$; also $Z \subseteq K$. Hence $K_1 = \langle Z \rangle$ is contained in $K \subseteq H$. Let $H_1 = \langle \mathcal{G} \text{ mod } K_1 \rangle$. Since $[y, x] \in Z \subseteq K_1$ for all $y \in Y$ and all $x \in X$, we have $[K_1, \mathcal{G}] = K_1$. Hence also G/K_1 is generated by the cosets $K_1 y, y \in X$. Hence $K_1 y \in \mathcal{G}/K_1$ i.e. $y \in H_1$ for all $y \in Y$. But $H_1 \triangleleft G$ and so $H_1 \supseteq H = \langle Y \rangle$. Then H/K_1 is a central factor of G and so $K = [H, \mathcal{G}] \subseteq K_1$ by 7.1(6). Since $K_1 \subseteq K$, we obtain $K = K_1$ as required.

We define inductively $[x_1, x_2, \dots, x_n] = [x_1, \dots, x_{n-1}], x_n$, $[H_1, H_2, \dots, H_n] = [[H_1, \dots, H_{n-1}], H_n]$, for $n=2, 3, \dots$, where the β'_i and H_i 's are respectively elements and subgroups of G .

(E) Let $H_i \triangleleft G, K_j \triangleleft G, \dots, M_r \triangleleft G$ for $i=1, \dots, s; j=1, \dots, k; \dots$
 $r=1, \dots, n$. Then $\prod_{i=1}^s (H_i) \triangleleft G$ gives the general distributive law of
 commutation.

$$\prod_{i=1}^s H_i \triangleleft \prod_{j=1}^k K_j \triangleleft \dots \triangleleft \prod_{r=1}^n M_r \triangleleft G \quad (1)$$

by induction. The order of the factors is irrelevant since all the groups are normal.

In particular, if $H \triangleleft G$ and $K \triangleleft G$, we obtain

$$\mathcal{O}_p(HK) = \prod [L_1, L_2, \dots, L_n] \quad (2)$$

where the product is taken over all 2^n terms having each L_i equal to either H or K . Suppose that $L = [L_1, L_2, \dots, L_n]$ just r terms are equal to H and s terms equal to K . By 7.1(2) and induction, we then obtain $L \leq \mathcal{O}_p(H)^r \mathcal{O}_p(K)^s$. Hence we must interpret $\mathcal{O}_p(H) = \mathcal{O}_p(K) = 1$. Consequently (2) gives

$$\mathcal{O}_p(HK) \leq \prod_{i=1}^n (\mathcal{O}_p(H) \cap \mathcal{O}_p(K)) \quad (3)$$

Hence again the $n+1$ factors are normal in G since e.g. $\mathcal{O}_p(H) \triangleleft G$. Suppose now that H and K are nilpotent, of class a and b respect if we take $n = a+b+1$ in (3), we then have in each factor either $r \geq a+1$ or $s \geq b+1$. But then $H = \mathcal{O}_p(H), K = 1$. Hence $\mathcal{O}_p(HK) = 1$ and we obtain

Theorem 7.6. If H and K are normal subgroups of G and if H and K are nilpotent, of class a and b respectively, then HK is nilpotent of class at most $a+b$. (Filling)

~~that the product of two normal nilpotent subgroups H and K is nilpotent follows also, as noted by Liebert, from 6.8(4). For if $L = HK$ and $M = H \cap K$, then $|L| = |H| \cdot |K| \cdot |M|$ by 5.5 (or equally 3.4) and so $|L|, |M|, |K| \equiv 1 \pmod{p}$. If S and T are the unique Sylow p -subgroups of H and K respectively, then $S \triangleleft H$ and $T \triangleleft K$. Similarly $T \triangleleft G$ and so $ST \triangleleft G$. Now $|S \cap T| \leq 1$ Hence $|ST| = |S| \cdot |T| \equiv |S| \cdot |T| \pmod{p}$. Hence the Sylow subgroups of L are all normal and so L is nilpotent by 6.8(5).~~

Corollary 7.6.1. Every group G has a normal nilpotent subgroup $\mathcal{N}(G)$ which contains every other normal nilpotent subgroup of G . $\mathcal{N}(G)$ is called the Fitting subgroup of G . If $H \triangleleft \text{Inn } G$, then $\mathcal{N}(H) = H \cap \mathcal{N}(G)$. In particular every subnormal nilpotent subgroup of G is contained in the Fitting subgroup of G .

Proof: By 7.6, a normal nilpotent subgroup of maximal order in G contains every normal nilpotent subgroup of G and hence is unique. Clearly $\mathcal{N}(G) \triangleleft G$. Hence, if $H \triangleleft G$, $\mathcal{N}(H) \triangleleft G$ and so $\mathcal{N}(H) \leq H \cap \mathcal{N}(G)$. But $H \cap \mathcal{N}(G) \triangleleft G$ and is nilpotent, so $H \cap \mathcal{N}(G) \leq \mathcal{N}(H)$. Thus $\mathcal{N}(H) = H \cap \mathcal{N}(G)$ if $H \triangleleft G$. This extends at once to the case of subnormal H . If H is itself nilpotent, then $H = \mathcal{N}(H)$.

~~Note that a subgroup L of a nilpotent group M of class c is nilpotent of class at most $c-1$, since $\mathcal{O}_p(L) \leq \mathcal{O}_p(M) \leq \mathcal{O}_p(M)$. This remark should have been inserted before.~~

(G) As a corollary of 7-7 (ii), we have
 Theorem 7-8 (i) Let $H_0 \geq H_1 \geq H_2 \geq \dots$ be a series of normal subgroups of G and let K be a subgroup of G such that $[H_i, K] \leq H_{i+1}$ for all $i = 0, 1, 2, \dots$. Then $[H_i, \mathcal{O}_p(K)] \leq H_{i+1}$ for all $i \geq 0$ and $n \geq 1$.
 (ii) In any group G , $[\mathcal{O}_p(G), \mathcal{O}_p(G)] \leq \mathcal{O}_{p^2}(G)$ and, for $\ell \geq n$, $[\mathcal{O}_p^\ell(G), \mathcal{O}_p(G)] \leq \mathcal{O}_{p^{\ell+1}}(G)$.
 (iii) If $G^{(n)}$ is the n -th derived group of G , then $G^{(n)} \leq \mathcal{O}_{p^2}(G)$.
 Proof: (i) by induction on n . For $n=1$, the theorem is true by hypothesis. Let $n > 1$. In 7-7 (ii), take $H = \mathcal{O}_p(G)$, $L = H_i$ and $M = H_{i+1}$. Then $U = [K, H_i, \mathcal{O}_p(K)] = [H_i, K, \mathcal{O}_p(K)] \leq [H_{i+1}, \mathcal{O}_p(K)]$ by hypothesis and so $U \leq M$ by induction. On the other hand, $V = [H_i, \mathcal{O}_p(K), K] \leq [H_{i+1}, K]$ by induction and so $V \leq M$ by the hypothesis. Since $M \leq [H_{i+1}, K]$ it follows that $W = [\mathcal{O}_p(K), K, H_i] = [H_i, \mathcal{O}_p(K)] \leq M = H_{i+1}$ as required.
 (ii) If each of the sections H_i/H_{i+1} is a central factor of G , we can take $K = G \leq U$. Applying this remark to the lower and upper central series of G gives (ii).
 (iii) By induction on n , we may assume $G^{(n-1)} \leq \mathcal{O}_{p^2}(G)$ for some n , since $G^{(0)} = G = \mathcal{O}_p(G)$. Then $G^{(n)} = [G^{(n-1)}, G^{(n-1)}] \leq [\mathcal{O}_{p^2}(G), \mathcal{O}_{p^2}(G)] \leq \mathcal{O}_{p^3}(G)$ by (ii).
 (iv) If $x \in A = \text{Aut } G$ and $f \in G$, then $\tau(f^x) = \tau(f)^x = x^{-1} \tau(f) x$. Thus A normalizes $R = \tau(G)$. The group $H = AR$ is called the Sylow subgroup of G . Note that H is a transitive subgroup of $\Sigma(G)$ and that A is the stabilizer of 1 in H . By 5-3, the centralizer of R in $\Sigma(G)$ is $L = \mathcal{C}(R)$. Both R and L are invariant with G and $R \cap L = Z$ consists of all the permutations $\tau(Y) = \ell(Y)$ with $f \in \mathcal{O}_p(G)$. The stabilizer of 1 in RL is $T = \ell(G)$ the group of

(F) Let ξ, η, ζ be elements of any group G , let $\alpha = \xi\xi\xi\xi\xi$ and let β and γ be obtained from α by cyclic permutation of ξ, η, ζ . Then $[\xi, \eta^2, \zeta^2] = \eta^2(\eta\xi^2\eta^2)\xi^2(\xi^2\zeta^2\xi^2)\zeta^2 = \alpha^2\beta$. Thus we obtain the first part of
 Theorem 7-7 (i) $[\xi, \eta^2, \zeta^2] = [\eta, \xi^2, \zeta^2] = [\zeta, \xi^2, \eta^2] = \alpha^2\beta$.
 (ii) Let H, K and L be subgroups of G . Then any normal subgroup M of G which contains two of the groups $U = [K, L, H]$, $V = [L, H, K]$ and $W = [H, K, L]$ also contains the third.
 (iii) In particular, if H, K and L are normal in G , then $U \leq VW$, $V \leq WU$ and $W \leq UV$.
 (iv) If $L \leq [H, K]$ and $\eta^2 \notin [H, K, L] = 1$, then $[L, H, K] = [L, K, H] = \langle \eta^2 \rangle$.
 Proof: (i) In (i), let $\xi \in H$, $\eta \in K$ and $\zeta \in L$, and suppose that M contains U and V . The three factors in (i) belong respectively to W^3 , U^3 and V^3 . Since $M \leq G$, $U^3 \leq M^3 \leq M$ and similarly $V^3 \leq M$. Hence $[L, \eta^2, \xi] \in \eta^2 M^3 \leq M$, and so M commutes with M^3 and η^2 for all $\xi \in H$, $\eta \in K$. Therefore the centralizer of M^3 is G/M and contains M^3 . This is true for all $\xi \in L$. Hence $[H, K, L] \leq W \leq M$ as required. (ii) is a corollary of (i). By 7-1 (iv).
 (iv) Let C be the centralizer of $[H, K]$ in $J = \langle H, K \rangle$. By 7-1 (iv), $C \leq J$ and by hypothesis $L \leq C$. Hence $[L, H] \leq C$ by 7-1 (v). Let $\tau \in H$, $\eta \in K$ and $\zeta \in [L, M]$. Then $\tau \in C$ and $[\xi, \eta^2] \in [H, K]$ so that τ commutes with $[\xi, \eta^2]$. But $\eta^2 = [\xi, \eta^2]\eta$ and so 7-1 (vi) gives $[\tau, \eta^2] = [\tau, \eta]$. Hence $[L, H, K^3] = [L, H, K]$. Since $\xi \in H$, we also have $[L, H] = [L, H^3]$ by 7-1 (vi). Hence $[L, H, K]^3 = [[L, H^3], K^3] = [L, H, K^3] = [L, H, K]$. Thus H normalizes $[L, H, K]$. By 7-1 (vi), K also normalizes $[L, H, K]$. So $[L, H, K] \leq J$. But $[K, H, L] = [H, K, L] = 1$ by hypothesis. Interchanging H and K , we therefore find that $[L, K, H] \leq J$ also. Since $[L, K, H] = [K, L, H]$, (ii) now follows from (i).

invert automorphisms of G . Hence $A \cap BL = T$. The general picture is as shown here:



Given a group Γ of automorphisms of G , i.e. any subgroup of A , it is often convenient to think of Γ and G as subgroups of a larger group ΓG with $\Gamma \cap G = 1$, $G \triangleleft \Gamma G$, and $\xi^{\alpha} = \alpha^{-1} \xi \alpha$ for $\xi \in G, \alpha \in \Gamma$. Such a group can be obtained by from $\Gamma \cdot \sigma(G)$ by identifying $\sigma(\xi)$ with ξ . We will use the isomorphic extensions of G by Γ . This device gives a meaning to the commutators $[\xi, \alpha]$ with $\xi \in G, \alpha \in \Gamma$ via $\xi^{\alpha} \xi^{-1}$. Note that the device is purely a notational one.

Theorem 7.1 Let Γ be a group of automorphisms of G and suppose that there exists a chain of subgroups $G = G_0 > G_1 > G_2 > \dots > G_{m-1} = 1$ such that $[G_{i-1}, \Gamma] \leq G_i$ for each $i=1, 2, \dots, m$. Then:

- (i) Every prime divisor p of $|\Gamma|$ also divides $|G|$.
- (ii) Γ is nilpotent of class at most $\frac{1}{2}m(m-1)$.
- (iii) $[G, \Gamma]$ is a normal nilpotent subgroup of G and $\Gamma [G, \Gamma]$ is a normal nilpotent subgroup of ΓG .

Proof: (i) By induction on m , since $\Gamma = 1$ when $m=1$. Let α be any element of Γ of prime order p . If $[G, \alpha] \neq 1$, then p divides $|G|$ by induction. If $[G, \alpha] = 1$, there is an element $\xi \in G - G_1$ such that $\xi = [\xi, \alpha] \neq 1$, since otherwise $\alpha = 1$, which is not the case. Then $\xi^{\alpha} \neq \xi$ and so $\xi^{\alpha} \xi^{-1} \neq 1$ by hypothesis. Also $\xi^{\alpha^2} = \xi^{\alpha \alpha} = \xi^{\alpha}$, hence $\xi^{\alpha^2} = \xi^{\alpha}$ by induction on $r=1, 2, \dots$. Since $\alpha^{p-1} = 1$, it follows that $\xi^{\alpha^{p-1}} = \xi$. Hence p divides $|G|$ in this case also.

(ii) Define $G_i^{\alpha} = G, G_{i+1}^{\alpha} = [G_i^{\alpha}, \Gamma], i=0, 1, 2, \dots$. Then by induction on i , we have $G_i^{\alpha} \leq G_i$ for all i . By 7.1 (i), $G_{i+1}^{\alpha} \leq [G_i^{\alpha}, \Gamma]$.

Hence there is no loss of generality in assuming that $G_{i+1}^{\alpha} \leq [G_i^{\alpha}, \Gamma]$ for $i=0, 1, \dots, m-1$. Since $\alpha = 1$ implies $\Gamma = 1$, we use induction on m and suppose $m > 1$. Hence if $\ell = \frac{1}{2}(m-1)(m-2)$, we may assume that $G_{\ell}^{\alpha}, \Gamma$ centralizes G_1 . Write $\Gamma = \langle \gamma, \Gamma \rangle$, so that $[G, \gamma, \Gamma] = 1$ for all $r \geq \ell_1$ and therefore, by 7.7 (ii), $[\Gamma, G, \Gamma] = [\Gamma, \Gamma, G] = [G, \Gamma, G]$ for all $r \geq \ell$. Hence $[\Gamma_{\ell+1}^{\alpha}, G] = [\Gamma_{\ell+1}^{\alpha}, G, \Gamma] = [\Gamma_{\ell+1}^{\alpha}, G, \Gamma, \Gamma] = \dots$

Then $\Gamma_{\ell+1}^{\alpha}$ centralizes G and so $\Gamma_{\ell+1}^{\alpha} = 1$ and Γ is nilpotent of class at most $\ell + m - 1 = \frac{1}{2}m(m-1)$ as required.

(iii). By 7.1 (ii), $[G, \Gamma] \leq G$. Hence $\bar{\Gamma} = \Gamma [G, \Gamma]$ is the normal closure of Γ in ΓG . Taking $G_1 = [G, \Gamma], G_2 = [G, \Gamma, \Gamma], \dots$ we find that ΓG_{i+1}^{α} is the normal closure of Γ in ΓG_i . Then we have $\bar{\Gamma} = \Gamma G_m^{\alpha} \triangleleft \Gamma G_{m-1}^{\alpha} \triangleleft \dots \triangleleft \Gamma G_1^{\alpha} = \bar{\Gamma} \triangleleft \Gamma G$.

Hence $\bar{\Gamma}$ sh ΓG . By (i), $\bar{\Gamma}$ is nilpotent. Hence by 7.1 (i), $\bar{\Gamma} \leq \mathcal{N}(\Gamma G)$ the Fitting subgroup of ΓG . Hence $\bar{\Gamma} \leq \mathcal{N}(\Gamma G)$ and so $\bar{\Gamma}$ is nilpotent. A fortiori, $[G, \Gamma]$ is nilpotent.

(ii) is a corollary of 7.8 (i).

§ 8 Direct Products, Central Products, Residual Products: Abelian from Semi-simple groups, Mal'tsev's component of irreducible from
 (A) A group G is called the direct product of its subgroups G_1, G_2, \dots, G_n if (i) each element $g \in G$ is uniquely expressible in the form $g = \xi_1 \xi_2 \dots \xi_n$ with $\xi_i \in G_i$ ($i=1, 2, \dots, n$) and (ii) $[G_i, G_j] = 1$ for $i \neq j$.

If $\eta = \eta_1 \eta_2 \dots \eta_n$ with $\eta_i \in G_i$, it follows from (ii) that $\xi \eta = \xi \eta$ has the expression $\xi_1 \xi_2 \dots \xi_n$ with $\xi_i = \xi_i \eta_i$. Also $\xi^{-1} = \xi_1^{-1} \xi_2^{-1} \dots \xi_n^{-1}$. Then G is determined to within isomorphism by the direct factors G_1, \dots, G_n .
 Given n groups G_1, \dots, G_n not necessarily distinct, the set of all ordered multiplets $\xi = (\xi_1, \xi_2, \dots, \xi_n)$ with $\xi_i \in G_i$ becomes a group if we define multiplication by the rule $\xi \eta = (\xi_1 \eta_1, \dots, \xi_n \eta_n)$. This implies that $\xi^{-1} = (\xi_1^{-1}, \dots, \xi_n^{-1})$. This group is called the Cartesian product of G_1, \dots, G_n and is denoted by $G_1 \times G_2 \times \dots \times G_n$. If G is the direct product of subgroups G_1, \dots, G_n we have a natural isomorphism of G onto the Cartesian product $G_1 \times \dots \times G_n$.

In a direct product, the order of the direct factors is irrelevant. In a Cartesian product, any permutation of the factors determines an isomorphism onto another Cartesian product.
 If G is the direct product of the subgroups $(G_i)_{i \in A}$ and if A is expressed as the union of a number of disjoint subsets A_1, \dots, A_r , then G is the direct product of the subgroups G_1, \dots, G_n , where G_i is the product (also direct) of the G_α with $\alpha \in A_i$.

Lemma 8.1 Let G be generated by the subgroups G_1, \dots, G_n . In order that G shall be the direct product of these subgroups it is necessary and sufficient that (i) each $G_i \triangleleft G$ and (ii) $G_i \cap G_1 G_2 \dots G_{i-1} = 1$ for $i=2, 3, \dots, n$.
 Proof: By (i), $G_i^* = G_i G_1 \dots G_{i-1}$ is a normal subgroup of G for each i . By (ii) and 6.9, it follows that $[G_i, G_j^*] = 1$ and so $[G_i, G_j] = 1$ for $i \neq j$. Then (i) further ensures that each $\xi \in G$ is uniquely expressible in

the form $\xi_1 \xi_2 \dots \xi_n$ with $\xi_i \in G_i$.
Lemma 8.2 (i) A nilpotent group is the direct product of its Sylow subgroups.

(ii) If G is the direct product of subgroups G_1, \dots, G_n whose orders m_1, \dots, m_n are coprime in pairs, then and if H is any subgroup of G , then H is the direct product of the subgroups $H_i = H \cap G_i$ ($i=1, 2, \dots, n$) and $\text{Aut } G \cong \text{Aut } G_1 \times \dots \times \text{Aut } G_n$.
 (iii) If in (ii), the subgroups G_i are all cyclic, then G is cyclic.
 (iv) If G is any group and ξ is any element of G of order $l = l_1 l_2 \dots l_n$ where $(l_i, l_j) = 1$ for $i \neq j$, then $\xi = \xi_1 \xi_2 \dots \xi_n$ with $[l_i, l_j] = 1$ for $i \neq j$ and with ξ_i of order l_i for each $i=1, 2, \dots, n$; moreover the elements $\xi_i \in G$ are uniquely determined by these conditions and are powers of ξ .

Proof: (i) Let G_1, \dots, G_n be the distinct Sylow subgroups $\neq 1$ of the nilpotent group G . Then $G_i \triangleleft G$ for each i , by 6.8(vi). Also $|G| = \prod |G_i|$ and so $G = G_1 G_2 \dots G_n$, and we have uniqueness in the expressions $\xi = \xi_1 \dots \xi_n$ of the elements $\xi \in G$ with $\xi_i \in G_i$. Also $[G_i, G_j] = 1$ for $i \neq j$, by 6.9. The result now follows.

(ii) If η_i is the set of primes dividing m_i , then G_i is a normal Sylow-subgroup of G . We have $H_i \triangleleft H$ and $H_i \triangleleft G_i$ so $[H_i, H_j] = 1$ for $i \neq j$, since then $[G_i, G_j] = 1$. $H/H_i \cong G_i/H/G_i$ and so $|H/H_i|$ is a m_i^* -number since $|G_i/H : G_i|$ divides $|G : G_i| = m_1 m_2 \dots m_{i-1} m_{i+1} \dots m_n$. Hence H_i is a normal Sylow-subgroup of H . Since (H) divides $|G| = m_1 m_2 \dots m_n$ it follows that $H = H_1 H_2 \dots H_n$ and uniqueness for the expression of the elements of H in the form $\xi = \xi_1 \dots \xi_n$ with $\xi_i \in H_i$ follows from the corresponding result for G . Hence H is the direct product of H_1, \dots, H_n .

Let $\alpha \in \text{Aut } G$. Since G_i char G by 5.8, α leaves each G_i invariant and induces in G_i an automorphism α_i . α is uniquely determined by $\alpha_1, \alpha_2, \dots, \alpha_n$ since $G = G_1 G_2 \dots G_n$. Given $\alpha = \alpha_1, \dots, \alpha_n \in G$ with $\xi_i \in G_i$ and given $\alpha_i \in \text{Aut } G_i$, the mapping $\xi \rightarrow \xi^\alpha$ of G

θ_i to H has kernel M_i , so that $\overline{H}_i \cong H/M_i$; and similarly $\overline{H}_2 \cong H/M_2$.
 Suppose $\xi = \xi_1 \xi_2$ and $\eta = \eta_1 \eta_2$ lie in H where $\xi_i, \eta_i \in G_i$ and hence in \overline{H}_i .
 Then $\xi \eta^{-1} = (\xi_1 \eta_1^{-1})(\xi_2 \eta_2^{-1}) \in H$. Hence $\xi_1 \eta_1^{-1} \in H_1$, implies $\xi_1 \eta_1^{-1} \in H_2$ and conversely.
 Hence the correspondence $H_1 \eta_1 \rightarrow H_2 \eta_2$ is one-to-one and is an isomorphism mapping \overline{H}_1/H_1 onto \overline{H}_2/H_2 .

Conversely, let \overline{H}_1/H_1 and \overline{H}_2/H_2 be any two isomorphic sections of G_1 and G_2 respectively and let θ be any isomorphism of the first onto the second. Let H be the set of all $\xi = \xi_1 \xi_2 \in G$ such that $\xi_i \in \overline{H}_i$ and $(H_1, \xi_1)^{\theta} = (H_2, \xi_2)$. If $\eta = \eta_1 \eta_2$ also lies in H , then $\xi_1 \eta_1^{-1} \in H_1$ and $(H_1, \xi_1 \eta_1^{-1})^{\theta} = (H_2, \xi_2 \eta_2^{-1}) = (H_2, \xi_2 \eta_2^{-1})$ and so $\xi_2 \eta_2^{-1} \in H_2$. Thus H is a subgroup of G . Thus we can state

Theorem 8-3. Let G be the direct product of the subgroups G_1 and G_2 . Then there is a one-to-one correspondence between the subgroups H of G and the set of all isomorphisms θ of a section \overline{H}_1/H_1 of G_1 onto a section \overline{H}_2/H_2 of G_2 . In this correspondence, $H_1 = H \cap G_1$ and $\overline{H}_1 = G_1 \cap H G_2$, $\overline{H}_2 = G_2 \cap H G_1$. An element $\xi = \xi_1 \xi_2$ of G with $\xi_i \in G_i$ lies in H if and only if $\xi_i \in \overline{H}_i$ ($i=1,2$) and $(H_1, \xi_1)^{\theta} = (H_2, \xi_2)$. (Conversely)

(D). It is important to consider the case in which G divides a group of operators Γ leaving the direct factors G_1 and G_2 of G invariant. We wish to know in this case which subgroups H of G are Γ -invariant. If H is Γ -invariant, then so are the subgroups H_i and \overline{H}_i . Hence any $\alpha \in \Gamma$ induces an automorphism in each of the two \overline{H}_i sections \overline{H}_1/H_1 and \overline{H}_2/H_2 . If $\xi, \xi_2 = \xi \in H$ with $\xi_i \in \overline{H}_i$, we then have also $\xi^{\alpha} = \xi_1^{\alpha} \xi_2^{\alpha} \in H$ and hence $(H_1, \xi_1^{\alpha})^{\theta} = (H_2, \xi_2^{\alpha}) = (H_2, \xi_2)^{\alpha}$. In other words, the isomorphism θ of 8-3 must commute with every $\alpha \in \Gamma$.
 Conversely, if this condition is fulfilled in addition to the Γ -invariance of the H_i, \overline{H}_i , then H is clearly Γ -invariant. This gives, as an essential supplement to 8-3.

defined by $\xi^{\alpha} = \xi_1^{\alpha} \xi_2^{\alpha}$ is an automorphism of G which induces α_i on G_i . If α and $\beta \in A$, then $(\alpha\beta)_i = \alpha_i \beta_i$ for each i . Hence $A \cong A_1 \times \dots \times A_n$, as stated.

(B) If H is cyclic of order $m_1 m_2 \dots m_n$ with $(m_i, m_j) = 1$ for $i \neq j$, then H has a subgroup H_i of order m_i and H_i is cyclic by 2-5. By (A) H is the direct product of H_1, \dots, H_n . Hence $H \cong H_1 \times \dots \times H_n$ and any direct product of cyclic groups of orders m_1, \dots, m_n is isomorphic with H , hence cyclic.

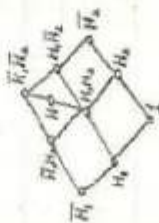
(C). Take $H = \langle \xi \rangle$ with $m_i = \xi_i$, $i=1, 2, \dots, n$. Then $\xi = \xi_1 \xi_2 \dots \xi_n$ with $\xi_i \in H_i$, $|H_i| = \xi_i$. The order of ξ_i is precisely ξ_i since otherwise the order of ξ would be less than $\xi_1 \xi_2 \dots \xi_n$. Each ξ_i is a power of H_i . Hence $[\xi_i, \xi_j] = 1$ for all i, j . Conversely, given elements ξ_1, \dots, ξ_n in G with these properties, we have $\xi^{\xi_i} = \xi_1^{\xi_i} \dots \xi_n^{\xi_i}$ and so $\xi^{\xi_i} = \xi_i^{\xi_i} \xi$, which has order ξ_i since otherwise ξ would be of order less than ξ . Hence ξ_i is a power of $\xi_i^{\xi_i}$ and therefore $\xi_i \in H$ for each i . Thus ξ_1, \dots, ξ_n are uniquely determined by ξ .

(C) 8-2 (E) shows that the properties of direct products of groups whose orders are coprime are easily reducible to a study of the structure of the direct factors. ~~The theory of direct products of groups~~
 particular, the theory of nilpotent groups reduces trivially to the theory of p -groups. When the direct factors are not of coprime order, however, more difficult questions arise. For example, in this case, the direct factors need not be characteristic subgroups.

Consider the subgroups H of the direct product $G = G_1 G_2$ of two groups G_1, G_2 . Let $H_1 = H \cap G_1$ ($i=1,2$) and let $\overline{H}_1 = G_1 \cap H G_2$, $\overline{H}_2 = G_2 \cap H G_1$. Let $\xi = \eta \xi_2 \in \overline{H}_1$ where $\eta \in H_1$, $\xi_2 \in G_2$ and let $\alpha \in H_1$. Then $\alpha^{\eta} \in H_1 = H \cap G_1$, since $\alpha \in G_1 \rightarrow G_2$ and $\alpha^{\xi_2} = \alpha^{\xi_2}$ since $[G_1, G_2] = 1$. Hence $H_1 \rightarrow \overline{H}_1$ and similarly $H_2 \rightarrow \overline{H}_2$. If $\xi = \xi_1 \xi_2 \in G$ with $\xi_i \in G_i$, then the mapping $\xi \rightarrow \xi_i$ is homomorphic. Call this mapping θ_i ($i=1,2$). We have $\theta_i(H) = \overline{H}_i$ and the restriction of

Theorem 8.4 If in 8.3 the group G admits a group of operators Γ leaving G_1 and G_2 invariant, then the subgroup H of G is Γ -invariant if and only if the subgroups \bar{H}_1, \bar{H}_2 ($i=1,2$) are also Γ -invariant and in addition $(H_1 \bar{H}_1)^\theta = (H_1 \bar{H}_1)^{\theta\gamma}$ for all $\alpha \in \Gamma$ and $\bar{H}_1 \in \bar{H}_1$. In particular, $H \rightarrow G$ if and only if \bar{H}_i/H_i is a central factor of G_i for $i=1,2$.

In the case $H \rightarrow G$, we have $\Gamma = G$ and so the H_i and \bar{H}_i must be normal in G or, what is equivalent, normal in G_i ($i=1,2$). In addition, taking $\alpha \in G$, we find that $(H_1 \bar{H}_1)^\theta = (H_1 \bar{H}_1)^{\theta\alpha} = (H_1 \bar{H}_1)^\theta$ since then $[G_1, \alpha] = 1$. Hence $H_1 \bar{H}_1 = H_1 \bar{H}_1$ for all $\alpha \in G$ and $\bar{H}_1 \in \bar{H}_1$ so that \bar{H}_1/H_1 is a central factor of G_1 . Similarly, taking $\beta \in G_2$, we have $(H_2 \bar{H}_2)^\theta = (H_2 \bar{H}_2)^{\theta\beta} = (H_2 \bar{H}_2)^\theta$ so that $H_2 \bar{H}_2 = H_2 \bar{H}_2$ for all $\beta \in G_2$ and $\bar{H}_2 \in \bar{H}_2$. Thus \bar{H}_2/H_2 is a central factor of G_2 . Conversely, these conditions are sufficient to ensure that $(H_1 \bar{H}_1)^\theta = (H_1 \bar{H}_1)^{\theta\gamma}$ for all $\gamma \in G$, and so they imply $H \rightarrow G$.



Note that $H \rightarrow G$ implies $H_1 H_2 \rightarrow G$ and $G/H_1 H_2$ is the direct product of $G_1 H_2 / H_1 H_2$ with $G_2 H_1 / H_1 H_2$. It is therefore isomorphic with the direct Cartesian product $G_1/H_1 \times G_2/H_2$. In this Cartesian product $\bar{H}_1/H_1 \times \bar{H}_2/H_2$ is a subgroup of the centre $Z(G_1/H_1) \times Z(G_2/H_2)$. Thus the step from G to G/H may be thought to take place in two steps. First we form the direct product G_1, G_2 when $\bar{H}_i \cong G_i/H_i$. Then $H \cong G_1 G_2 / K$, where K is a subgroup of the centre of $G_1 G_2$ identified consisting of all elements $\bar{H}_1 \bar{H}_2$ with $\bar{H}_1 \in \bar{H}_1$ and $\bar{H}_2 \in \bar{H}_2$ with $\bar{H}_1 = \bar{H}_1$, where \bar{H}_1 is an isomorphism of K , onto K_2, K , and K_2 being necessarily isomorphic subgroups of $Z(G_1, Z(G_2)$, respectively. The group $G_1 G_2 / K$ is then obtained from the direct product $G_1 G_2$ by 'identifying' corresponding

$\bar{H}_1 \cong (\bar{H}_1)^{\gamma}$ is an isomorphism $\bar{H}_1 \rightarrow (\bar{H}_1)^{\gamma}$ of a subgroup K_1 of \bar{H}_1 onto a subgroup K_2 of \bar{H}_2 . A group constructed in this way is often called a central product of the two groups \bar{G}_1 and \bar{G}_2 .

The last part of 8.4 may now be stated as follows: any homomorphism G/H of the direct G of two groups G_1 and G_2 is isomorphic with some central product of homomorphic images $\bar{G}_1 = G_1/H_1$ and $\bar{G}_2 = G_2/H_2$ of G_1 and G_2 .

(E). Let H be a subgroup of the Cartesian product $G = G_1 \times G_2 \times \dots \times G_n$ if for each $i=1,2,\dots,n$ and each $i_i \in G_i$, there is an element of H whose i -th coordinate is precisely i_i ; then H is called a coordinate product of G_1, G_2, \dots, G_n ; or sometimes though less appropriately a subdirect product of the G_i . If $\bar{H} = (H_1, \dots, H_n) \in G$, the mapping $\theta_i: \bar{H} \rightarrow H_i$ is an isomorphism of G onto G_i . H is a residual product of the G_i if and only if $H^{\theta_i} = G_i$ for each i .

Lemma 8.5 Let $K_i \rightarrow G$ ($i=1,2,\dots,n$) and let $K = \bar{K}_1 \bar{K}_2 \dots \bar{K}_n$. Then G/K is isomorphic with a residual product of the groups G_i/K_i ($i=1,2,\dots,n$). The mapping

$$K \bar{H} \rightarrow (K_1 \bar{H}_1, \dots, K_n \bar{H}_n) \quad (\bar{H} \in G)$$

is the isomorphism in question. For this mapping is homomorphic owing to $K_i \rightarrow G$ for each i . If $K_1 \bar{H}_1$ and $K_2 \bar{H}_2$ have the same image, then $K_1 \bar{H}_1 = K_2 \bar{H}_2$ for all i and so $\bar{H}^{\theta_i} \in K = \bar{K}_1 \bar{K}_2 \dots \bar{K}_n$, whence $K \bar{H} = K \bar{H}$. Thus the mapping is an isomorphism. The image group is obviously a residual product of the G_i/K_i .

$\exists \in \Omega_m G$ if and only if $\langle i, j^m \rangle = 0 \pmod{p^i}$ for each i , which means p^m divides for $\langle i, j \rangle$ if $\lambda_i \geq m$ and p^i divides if $\lambda_i \leq m$. Hence $|\Omega_m G| = p^{\lambda_1 + \lambda_2 + \dots + \lambda_m}$, where λ_i is the number of values of i for which $\lambda_i \geq m$. We have $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_m \geq \lambda_{m+1} = 0$, where $\lambda = \lambda_1 + \dots + \lambda_m$. The partition λ whose parts are $\lambda_1, \lambda_2, \dots, \lambda_m$ is the conjugate-partition to λ and the relation between λ and λ is symmetrical: $\lambda = \lambda', \lambda' = \lambda$. Now G determines λ uniquely because $|\Omega_m G| = p^{\lambda_m}$ for $m=1, 2, \dots$. Hence G determines λ uniquely.

Note that the mapping $\lambda \rightarrow \lambda'$ ($\lambda \in G$) is a homomorphism of G into $\mathcal{U}_m G$ with kernel $\Omega_m G$. Hence $G/\Omega_m G \cong \mathcal{U}_m G$ and $|\mathcal{U}_m G| = p^{\lambda_m} = |\Omega_m G|$.

Corollary 8.7.1. Every finite Abelian group G is the direct product of cyclic subgroups of orders $\lambda_1, \lambda_2, \dots, \lambda_r$ such that $\lambda_1 \geq 1$ and λ_i divides λ_j for each $i, j, 1 \leq i < j \leq r$. The numbers $\lambda_1, \dots, \lambda_r$ are uniquely determined by G subject to these conditions. They are called the elementary divisors of G .

Note that if G is an Abelian p -group of type λ , then the elementary divisors of G are the numbers $p^{\lambda_1}, p^{\lambda_2}, \dots, p^{\lambda_r}$ where $\lambda_1, \dots, \lambda_r$ are the parts of λ .

Proof. By 8.2(1), G is the direct product of its Sylow subgroups G_1, \dots, G_r where $|G_i| = p_i^{\lambda_i}$. Let the invariants of G_i be $\lambda_{i1}, \dots, \lambda_{ir}$ so $\lambda_{i1} \geq \lambda_{i2} \geq \dots \geq \lambda_{ir} > 0$ and let $r = \max \lambda_{ij}$ ($i=1, 2, \dots, r$). Let $\lambda_i = \lambda_{i1} \lambda_{i2} \dots \lambda_{ir}$ ($i=1, 2, \dots, r$). Then $\lambda_i = |\mathcal{U}_i G_i|$ is of order $\lambda_i = p_i^{\lambda_{i1} + \lambda_{i2} + \dots + \lambda_{ir}}$ where $\lambda_{ij} = 0$ if $\lambda_{ij} < i \leq r$. Also λ_i is the direct product of the cyclic groups $\langle \lambda_{i1} \rangle, \langle \lambda_{i2} \rangle, \dots, \langle \lambda_{ir} \rangle$. Hence G is the direct product of $\lambda_1, \dots, \lambda_r$ and by construction the numbers $\lambda_1, \dots, \lambda_r$ satisfy the conditions of 8.7.1. Conversely, these conditions ensure that $\lambda_{ij} \geq \lambda_{jr} \geq \dots \geq \lambda_{ir} \geq 0$ for each $j=1, 2, \dots, r$ and therefore imply that the Sylow p_i -subgroup G_i of G has type λ_i with where

Lemma 8.6. Let G be an Abelian p -group and let λ be any element of G whose order is no larger as possible. Then G is the direct product of $X = \langle \lambda \rangle$ and Y , where Y is any subgroup of G which is maximal with respect to the condition $X \cap Y = 1$.

Proof. We have only to show that $XY = G$ since G is Abelian. Let $\bar{X} = G/Y, \bar{Y} = XY/Y \cong X$. By the maximal property of Y , every subgroup $\bar{H} \neq 1$ of \bar{G} contains elements $\neq 1$ of the cyclic subgroup $\bar{X} = \langle \bar{\lambda} \rangle$. Let $\bar{\eta}$ be any element of \bar{G} and let $\bar{\eta}^m$ be the first positive power of $\bar{\eta}$ to lie in \bar{X} , i.e. $\langle \bar{\eta}^m \rangle = \langle \bar{\lambda} \rangle \cap \bar{X}$. Since $|\bar{X}| = |X|$, the order of $\bar{\eta}$ cannot exceed the order of $\bar{\lambda}$, by our choice of $\bar{\lambda}$. Hence $\langle \bar{\eta}^m \rangle = \langle \bar{\lambda} \rangle$ for some $s \geq r$, and so for a suitable integer m , the element $\bar{\eta}^m$ has order p^r and $\bar{\eta}^m \in \langle \bar{\lambda} \rangle$. Then η has intersection 1 with \bar{X} . Hence $\bar{\eta}^m = 1$ and so $\bar{\eta} \in \bar{X}$. Thus $\bar{G} = \bar{X}$ and $G = XY$ as required.

As an Abelian p -group G , the elements λ such that $\lambda^{p^r} = 1$ form a characteristic subgroup $\Omega_r G$ and the elements η of the form $\eta = \lambda^{p^r}$ for some $\lambda \in G$ form another characteristic subgroup $\mathcal{U}_r G$.

An immediate corollary of 8.6 is Theorem 8.7. (6) Every Abelian p -group G is the direct product of a certain number of cyclic subgroups $X_i = \langle \lambda_i \rangle$ ($i=1, 2, \dots, r$). If $|X_i| = p^{\lambda_i}$, we can arrange that $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_r > 0$. The partition λ whose parts are $\lambda_1, \lambda_2, \dots, \lambda_r$ is called the type of G and the numbers λ_i are the invariants of G . The ordered set $\lambda_1, \lambda_2, \dots, \lambda_r$ is called a basis of G .

(6) Two Abelian p -groups are isomorphic if and only if they have the same type.

Proof. (1) follows from 8.6 by induction on $|G|$, since we can then write $X_1 = X$ and that Y is the direct product of suitable cyclic subgroups X_2, \dots, X_r . (2) Every element of G is uniquely expressible in the form $\lambda = \lambda_1^{\alpha_1} \lambda_2^{\alpha_2} \dots \lambda_r^{\alpha_r}$ with $0 \leq \alpha_i < p^{\lambda_i}$.

$\lambda_i^{(i)}$ is the partition whose parts are those λ_j which are positive. Hence the elementary divisors $\lambda_1, \dots, \lambda_r$ are uniquely determined by \mathcal{G} .

(G). An Abelian p -group \mathcal{G} of type (λ^r) is called elementary.

An Abelian p -group \mathcal{G} is elementary if and only if it has no elements of order p^2 .

If $\lambda_1, \dots, \lambda_r$ is a basis of the Abelian p -group \mathcal{G} and if $\alpha \in \text{Aut } \mathcal{G}$ then $\lambda_1^{\alpha}, \dots, \lambda_r^{\alpha}$ is also a basis of \mathcal{G} . $\lambda_i^{\alpha} = \lambda_i$ for all $i=1, 2, \dots, r$ implies that α is the identity on \mathcal{G} . If η_1, \dots, η_r is an arbitrary basis of \mathcal{G} , then the mapping $\lambda_i \rightarrow \eta_i$ ($i=1, 2, \dots, r$) determines uniquely an automorphism β of \mathcal{G} .

$$\lambda_1^{\beta} = \eta_1, \lambda_2^{\beta} = \eta_2, \dots, \lambda_r^{\beta} = \eta_r \quad (0 \leq \alpha_i < p^{k_i})$$

where λ is the type of \mathcal{G} . Hence the number $|\text{Aut } \mathcal{G}|$ of automorphisms of \mathcal{G} is equal to the number of bases of \mathcal{G} .

When \mathcal{G} is elementary, $\eta_1, \eta_2, \dots, \eta_r$ is a basis of \mathcal{G} if and only if $\eta_i = (\eta_1, \eta_2, \dots, \eta_r)$ is of order p^{α_i} for each $i=1, 2, \dots, r$. Given η_{i-1} , this leaves precisely $p^{\alpha_i - p^{\alpha_{i-1}}}$ choices for η_i . Hence $|\text{Aut } \mathcal{G}|$ is equal to $(p^{\alpha_1 - 1})(p^{\alpha_2 - 1}) \dots (p^{\alpha_r - 1})$.

Suppose next that \mathcal{G} has n invariants all equal to n , so that $|\mathcal{G}| = p^n$ and $\mathcal{G}/\mathcal{U}\mathcal{G}$ is elementary of order p^n . If $\lambda_1, \dots, \lambda_n$ is a basis of \mathcal{G} , then $H\lambda_1, \dots, H\lambda_n$ generates \mathcal{G}/H where H is any subgroup of \mathcal{G} .

Taking $H = \mathcal{U}\mathcal{G}$, this implies that $H\lambda_1, \dots, H\lambda_n$ is then a basis of \mathcal{G}/H . Conversely, if this is so, then each λ_i has order p^n and the group

$X = \langle \lambda_1, \dots, \lambda_n \rangle$ coincides with \mathcal{G} . For if $X \leq \mathcal{G}$, then X is a subgroup Y of index p in \mathcal{G} such that $X \leq Y$ and we have $H\mathcal{U}\mathcal{G} \leq Y$, whence

$$HX \leq \mathcal{G} \text{ and } H\lambda_1, \dots, H\lambda_n \text{ could not be a basis of } \mathcal{G}/H. \text{ Since } |H| = p^{n(n-1)}, \text{ we thus have } |\text{Aut } \mathcal{G}| = p^{n(n-1)}(p^n - p) \dots (p^n - p^{n-1})$$

in this case. Note that the automorphisms α of \mathcal{G} which transform the characteristic quotient group $\mathcal{G}/\mathcal{U}\mathcal{G}$ identically form a normal subgroup A , of

$$A = \text{Aut } \mathcal{G}, \text{ that } |A| = p^{n(n-1)} \text{ and that } A/A, \text{ is isomorphic with}$$

$\text{Aut } \mathcal{G}/\mathcal{U}\mathcal{G}$.

Now let \mathcal{G} be any Abelian p -group, let $\lambda_1, \dots, \lambda_r$ and η_1, \dots, η_r be any two bases of \mathcal{G} and suppose that \mathcal{G} has exactly n invariants equal to n . Let $\lambda_{i_1}, \lambda_{i_2}, \dots, \lambda_{i_m}$ be those λ_j 's which have order $\leq p^n$ and let $X = \langle \lambda_{i_1}, \lambda_{i_2}, \dots, \lambda_{i_m} \rangle$ be the subgroup generated by the remaining λ_j 's. Then no element of order p in X can belong to the set $\mathcal{U}_{i_1} \mathcal{G} \cup \dots \cup \mathcal{U}_{i_m} \mathcal{G}$. On the other hand, every element of order p in X lies in $\mathcal{U}_{i_1} \mathcal{G} \cup \dots \cup \mathcal{U}_{i_m} \mathcal{G}$. Hence $X \cap Y = 1$ and so \mathcal{G} is the direct product of the two subgroups X and Y . It follows that

$$\lambda_1, \lambda_2, \dots, \lambda_n, \eta_{i_1}, \eta_{i_2}, \dots, \eta_{i_m}, \eta_{i_{m+1}}, \dots, \eta_r$$

is a basis of \mathcal{G} . In other words, we can exchange the elements of given order p^n in the λ -basis for the corresponding elements of the η -basis and obtain a new basis. This is called the exchange property of the basis of an Abelian p -group.

Now the number of different subgroups Y, Y_1, Y_2, \dots of \mathcal{G} such that \mathcal{G} is the direct product of X with Y_i , i.e. the number of subgroups Y_i such that $X \cap Y_i = 1$ and $X Y_i = \mathcal{G}$, is equal to the number of distinct homomorphisms of Y into X . This is a particular case of $\mathcal{E} \cdot \mathcal{E}$. Hence the number of different ordered sets $\eta_{i_1}, \dots, \eta_{i_m}$ of elements of order p^n which can occur in some basis or other of \mathcal{G} is equal to

$$a_n = |\text{Hom}(Y, X)| = |\text{Aut } Y|. \quad (4)$$

And by the exchange property, $|\text{Aut } \mathcal{G}| = \prod_{i=1}^n a_i$, where $a_i = 1$ if \mathcal{G} has no invariants equal to i .

Lemma 8.17. Let X and Y be Abelian groups, let α and β be homomorphisms of Y into X and define the sum $\alpha + \beta$ to be the mapping $\eta \rightarrow \eta\alpha + \eta\beta$ ($\eta \in Y$). Then $\alpha + \beta \in \text{Hom}(Y, X)$ and with this addition, H becomes an additive (Abelian) group.

If X and Y are Abelian p -groups of types λ and μ respectively, then H is an Abelian p -group of type $\lambda \oplus \mu$, where the parts of $\lambda \oplus \mu$ are the numbers $\min(\lambda_i, \mu_j) + (\lambda_i - \mu_j)$, $(i=1, 2, \dots, r; j=1, 2, \dots, s)$, and where

$|\text{Aut } G| = p^{\lambda} \prod_{i=1}^r f_{\tau_i}(\frac{1}{p})$, where $\lambda = (p^{\lambda_1} \tau_1, \dots)$ and so the numbers τ_i are the multiplicities of the different parts of λ .

(H) A group is called semisimple if it is the direct product of one or more simple subgroups each of composite order, or if it is 1.

Lemma 8-91 (i) Let H be a semisimple group. Then every subnormal subgroup of H is normal and is a direct factor of H and is itself semisimple.

(ii) Let H and K be normal semisimple subgroups of G , let $M = HK$ and let $L = C_{HK}(M)$. Then HK is semisimple and is the direct product of L and M .

Proof: (i) Let the direct factors of H be H_1, H_2, \dots, H_n , each H_i being simple of composite order, and let $\xi = \xi_1 \xi_2 \dots \xi_n$ with $\xi_i \in H_i$. Then if $\eta_i \in H_i$, we have $[\xi_i, \eta_i] = 1$. If $\xi_i \neq 1$, it follows that $X = \langle \xi^H \rangle$ contains H_i , since H_i is simple but not Abelian. Hence every normal subgroup of H is the product of certain of the simple direct factors H_i . Thus M is also semisimple. It then follows that every subnormal subgroup of H is a direct factor of H . Clearly H has exactly 2 distinct direct factors.

(ii) Since $K \trianglelefteq G$, we have $M \trianglelefteq H$ and so $H = ML_1$ is the direct product of M with a subgroup L_1 , and both M and L_1 are semisimple. Similarly K is the direct product of M with a semisimple subgroup L_2 . We have $L_1 \trianglelefteq L$ and $L_2 \trianglelefteq L$ and so $[L_1, L_2] \leq [H, K] \cap L$. But $[M, K] \leq M$, by the normality of H and K ; while $M \cap L = 1$ since $\xi \in M = 1$. Hence $[L_1, L_2] = 1$. But $L_1 \cap L_2 \leq L \cap M = 1$ and so the product $L_1 L_2$ is direct. Since $|HK| = |M| \cdot |L_1| \cdot |L_2|$, we have $HK = M L_1 L_2$; and $L \cap M = 1$ ensures that $L = L_1 L_2$ is the direct product of the semisimple groups L_1 and L_2 , hence itself semisimple; while HK is the direct product of L and M , since L and M are normal in G .

λ and μ have r and s parts respectively.

Note that if $\nu = \lambda \oplus \mu$, then the parts of the conjugate ν' of ν are the numbers $\lambda'_i p^i + \mu'_i p^i, \dots, \lambda'_r p^r + \mu'_r p^r$, where $t = \min(\lambda_i, \mu_i)$ and λ'_i, μ'_i are the partitions conjugate to λ, μ respectively.

The verification that H is an additive Abelian group is immediate. The zero element of H maps ν into the unit subgroup of X . In the p -group case, if ξ_1, \dots, ξ_r and η_1, \dots, η_s are bases of X and Y respectively, then the elements α_{ij} ($i=1, \dots, r; j=1, \dots, s$) of H defined by

$$\alpha_{ij} : \eta_j \rightarrow \xi_i^{p^{\min(\lambda_i, \mu_j)}} \text{ and } \eta_j \rightarrow 1 \text{ (if } i > j \text{)}$$

form a basis of H . The order of α_{ij} is precisely $p^{\min(\lambda_i, \mu_j)}$. The number of pairs (i, j) such that $\min(\lambda_i, \mu_j) \geq m$ is equal to $\lambda'_m \mu'_m$.

Note that $\text{Hom}(Y, X)$ and $\text{Hom}(X, Y)$ have the same type. An Abelian group isomorphic with $H = \text{Hom}(Y, X)$ is often called the tensor product of X and Y .

We now use 8-8 and equation (1) to calculate $|\text{Aut } G|$ for an Abelian p -group G of arbitrary type $\lambda = (p^{\lambda_1} \tau_1, \dots)$. Here γ is of type (m^{τ_m}) with τ_m invariant equal to m . Define the polynomial f_γ by the equation

$$f_\gamma(x) = (1-x)(1-x^2) \dots (1-x^m) \quad (m=1, 2, \dots)$$

and understand $f_\gamma(0) = 1$. Then we have seen that $|\text{Aut } \gamma| = p^{m \tau_m} f_\gamma(\frac{1}{p})$. The conjugate of the partition (m^{τ_m}) is the partition $(\tau_m, \tau_m, \dots, \tau_m)$ with m parts all equal to τ_m . The conjugate of the type of X has parts $\lambda'_1 - \tau_1, \lambda'_2 - \tau_2, \dots, \lambda'_m - \tau_m, \lambda'_{m+1}, \dots$. Hence $H = \text{Hom}(Y, X)$ has type ν where ν' has the parts $\tau_m(\lambda'_1 - \tau_1), \tau_m(\lambda'_2 - \tau_2), \dots, \tau_m(\lambda'_m - \tau_m)$. Now $\tau_m = \lambda'_m - \lambda'_m$ and $|H| = p^{-m \tau_m^2 + \sum_{i=1}^m \tau_m \lambda'_i}$. Hence $|\text{Aut } G| = p^{\lambda} \prod_{i=1}^r f_{\tau_i}(\frac{1}{p})$ where $\lambda = \sum_{i=1}^r \tau_i \lambda'_i = \sum_{i=1}^r \tau_i^2$. Thus we obtain

Corollary 8-81 Let G be an Abelian p -group of type λ , let g be the sum of the squares of the parts of the partition conjugate to λ and let $f_\lambda(x) = (1-x)(1-x^2) \dots (1-x^g)$, with $f_\lambda(0) = 1$. Then

simple and $H \cong B_i \neq 1$, it follows that $H = G$. Each $G_i \cong G_j$ and every proper normal subgroup of G_i is normal in G . Hence G_i is simple. If $|G_i| = p$ is prime, then G is elementary. Otherwise G is isotropic and semi-simple.

(J) Let V be an elementary p -group which admits G as a group of operators. We use the additive notation for V . V is called G -irreducible if $V \neq 0$ and if V and 0 are the only G -invariant subgroups of V . In any case if $V \neq 0$, a minimal G -invariant subgroup $X \neq 0$ in V is necessarily G -irreducible.

Lemma 8.9. Let V be G -irreducible and let $H \triangleleft G$. Then $V = V_1 \oplus V_2 \oplus \dots \oplus V_r$ is the direct sum of a certain number $r \geq 1$ of H -invariant subgroups V_i with the following properties:

(1) Each V_i is the direct sum of a certain number $s_i \geq 1$ of H -irreducible subgroups $X_{i1}, X_{i2}, \dots, X_{is_i}$, where $X_{ij} \cong X_{kl}$ if and only if $i=j$.

(2) If Y is any H -invariant subgroup of V , then $Y = Y_1 \oplus \dots \oplus Y_r$ where $Y_i = Y \cap V_i$. In particular, if Y is H -irreducible, then Y is H -isomorphic with X_{ij} for some i and in that case $Y \leq V_i$.

(3) If $\tau \in G$, then the mapping $V_i \rightarrow V_i \tau$ ($i=1, 2, \dots, r$) is a permutation of V_1, \dots, V_r and G is represented transitively by these permutations.

Proof. Let $W = W_1 \oplus \dots \oplus W_s$ be a direct sum of H -irreducible subgroups W_i of V , and choose W as large as possible. Let $Z \leq G$. Every subgroup of $W_i Z$ has the form $Z \tau$ where Z is a subgroup of W_i . If $q \in H$, $u \in Z$ then $u \tau q = u (\tau q \tau^{-1}) \tau$ and, since $H \triangleleft G$, $Z \tau$ is H -invariant only if Z is H -invariant. Hence $W_i Z$ is H -irreducible if $W_i Z \neq W$, then $W_i Z \neq W$ for some i and hence $W \cap W_i Z = 0$ and the sum $W + W_i Z$ is direct, contrary to the choice of W . It follows that $W \tau = W$ for all $\tau \in G$. Since V is G -irreducible, we must

Let \mathcal{K} be any class of groups, such that (i) all groups of order 2 belong to \mathcal{K} , and (ii) if $G \in \mathcal{K}$ and $G \cong G_1$, then $G_1 \in \mathcal{K}$. We shall always understand the expression "class of groups" to imply (i) and (ii).

Lemma 8.9.2. Suppose that \mathcal{K} is a class of groups G the product of two normal \mathcal{K} -subgroups always belongs to \mathcal{K} , then every group G has a uniquely determined maximal normal \mathcal{K} -subgroup $\mathcal{K}G$. If H is a G , then $\mathcal{K}H = H \cap \mathcal{K}G$. In particular, $\mathcal{K}G$ contains every maximal \mathcal{K} -subgroup of G .

The argument is the same as for 7.61, which is the special case $\mathcal{K} = \mathcal{N}$ the class of all nilpotent groups. 8.91 shows that $\mathcal{K}G = \mathcal{K}$ the class of all semi-simple groups is another admissible choice. A third choice would be $\mathcal{K} = \mathcal{O}_p$ the class of all p -groups. This is admissible by 5.5 or 3.4, which shows that products of normal \mathcal{K} -subgroups are \mathcal{K} -groups.

The subgroup $\mathcal{K}G$ is characteristic in G and may be called the \mathcal{K} -radical of G . Here "radical" is equivalent to "uniquely determined maximal normal subgroup" of the appropriate class.

(F). A semi-simple group will be called isotropic if its simple direct factors are all isomorphic.

Lemma 8.9.3. A characteristically-simple group G is either (1) an elementary Abelian p -group for some prime p or else (ii) an isotropic semi-simple group.

Proof. Let G_i be a minimal normal subgroup of G and let H be the direct product of G_1, G_2, \dots, G_n where each $G_i = G_i^{G_i}$ for some $\alpha_i \in A = \text{Aut } G_i$. Choose H as large as possible and let $\beta \in A$. If $H \beta \neq H$, then $G_i \beta \neq H$ for some i . But $H \triangleleft G$ and $G_i \beta$ is a minimal normal subgroup of G . Hence $H \cap G_i \beta = 1$ and the product $H G_i \beta$ is direct by 6.9, contrary to the definition of H . It follows that $H \beta = H$ for all $\beta \in A$ and so H char G . Since G is characteristically

therefore have $V = V_i$.

We now relabel the subgroups W_j as X_{iS} so that (ii) is satisfied and defines V_i as the direct sum of the X_{iS} ($S=1, 2, \dots$), so that $V = V_1 \oplus V_2 \oplus \dots \oplus V_r$ if there are r classes of H -isomorphic H -invariant subgroups among the W_j . To prove (iii), suppose $W_j \neq 0$. Then $\forall i, W_j = 0$ since W_j is H -invariant and Y is H -invariant, and so the sum $Y + W_j = 0$ is direct. It follows that $V = Y \oplus W_j \oplus \dots \oplus W_r$ for some $t \geq 0$ and suitable W_1, \dots, W_t . Hence Y is H -isomorphic with $V / \sum_{i=1}^t W_i$ and so sufficient to H_1, \dots, H_t . Thus Y is a direct sum of H -invariant subgroups and we need therefore only consider the case in which Y itself is H -invariant. Let $\overline{W}_j = W_j \oplus \dots \oplus W_r$ and let Z be the first subgroup W_i such that $Y \leq \overline{W}_j$. Then $\overline{W}_j \cap Y = 0$ by the irreducibility of Y and $\overline{W}_j = \overline{W}_j \oplus Y$ by the irreducibility of W_j . So Y is H -isomorphic with $W_j = X_{iS}$ say. By the same argument applied with the W_j 's rearranged so that all those H -isomorphic with Y come first (i.e. X_{i1}, X_{i2}, \dots), we obtain $Y \leq V_i$, as required.

If $Z \in G$, then as we have seen X_{iZ} is H -irreducible. If X_{iZ} is H -isomorphic with X_{i1} , then $X_{iZ} \leq V_i$ by (ii). If $z \rightarrow z'$ is an H -isomorphism of X_{iZ} onto X_{i1} , $z \in X_{iZ}$, then $zS \rightarrow z'S$ is an H -isomorphism of X_{iZ} onto X_{i1} , since $z'S = u(z'S')$ for each $S=1, 2, \dots$ and so $V_i \leq V_j$. Hence $X_{iZ} \leq V_i$ for each $S=1, 2, \dots$ and so $V_i \leq V_j$. Here $f = f(S, Z)$ depends only on i and on the automorphism $f_{iZ}(S)$ induced in H by transposing with Z . It now follows that $V_i \leq V_j$ and the mapping $V_i \rightarrow V_j$ ($i=1, \dots, r$) is a permutation of V_1, \dots, V_r . The representation of G by these permutations must be transitive, since otherwise V would not be G -irreducible. Hence each V_i is the direct sum of the same number ρ of H -irreducible subspaces and (ii) is completely proved.

We call V_1, \dots, V_r the H -isomorphism components of V with respect to H .

§9. Frattini subgroups. Split extensions. p -soluble groups. S_p -subgroup

(A). We consider next how to construct larger groups from smaller ones. Let $K \triangleleft G$ and suppose that $I < K < G$. If there exists in G a proper subgroup H such that $KH = G$, we say that K is partially complemented in G . If in addition $K \cap H = I$, so that H is transverse to K in G , we say that K is complemented in G .

The intersection of the maximal subgroups of G is called the Frattini subgroup of G and denoted by $\phi(G)$. Clearly $\phi(G)$ char G . Theorem 9.1. Let $F = \mathbb{F}_p$ and $\Omega = \phi(G)$ be respectively the Frattini and the Frattini subgroups of G . Then

- (i) $G = \langle \Omega, \dots, \Omega_n \rangle$ if and only if $G = \langle \Omega, \Omega_1, \dots, \Omega_n \rangle$.
- (ii) Let $K \triangleleft G$. Then K is partially complemented in G if and only if $K \not\leq \Omega$.
- (iii) $F' \leq \Omega \leq F$, so that Ω is nilpotent and F/Ω is Abelian. Moreover, $F/\Omega = \mathbb{F}_p(\Omega/\Omega)$ the Frattini subgroup of G/Ω .

Proof: (i) Let $H = \langle \Omega_1, \dots, \Omega_n \rangle < G$. Then there exists a maximal subgroup M of G such that $H \not\leq M$. Since $\Omega \leq M$, it follows that $\langle \Omega, \Omega_1, \dots, \Omega_n \rangle \leq M < G$.

(ii) Let H be a proper subgroup of G such that $KH = G$. If $K \leq \Omega$, we should have $\Omega H = G$ and so $H = G$ by (i), contrary to $H < G$. Hence $K \not\leq \Omega$. Conversely, suppose $K \not\leq \Omega$. Then there is a maximal subgroup M of G which does not contain K . Since $K \triangleleft G$, it follows that KM is a group with M as a proper subgroup, hence $KM = G$ and K is partially complemented in G .

(iii) Let $K = \mathbb{F}_p(\Omega \text{ mod } \Omega)$ so that K/Ω is the Frattini subgroup of G/Ω . Let S be a Sylow p -subgroup of K . Since K/Ω is nilpotent, $\Omega S/\Omega$ is a characteristic subgroup of K/Ω . Also $K \triangleleft G$, $\Omega \leq \Omega$ hence $\Omega S \triangleleft G$. As a Sylow subgroup of a normal subgroup of G , S is permutable in G and so, if $N = N_G(S)$, we have $\Omega N = \Omega S N = G$ by 6.64.

Have $N = G$ by (6) and so $S \trianglelefteq G$. It follows that K is subnormal, $K \in \mathcal{F} \cap \mathcal{N} \cap \mathcal{O}$.
 A fortiori \mathcal{O} is subnormal. Since F/\mathcal{O} is subnormal and normal in G/\mathcal{O} , we have $F \leq N_G(\mathcal{O} \text{ mod } \mathcal{O}) = K$. Combining gives $F = K$.

We show that F/\mathcal{O} is Abelian by proving a more precise result.
Theorem 9.2. Let M be a maximal subgroup of G , let $L = K_G(M)$ and let $N = N_G(\mathcal{O} \text{ mod } L)$. Then either $N = L$ or else N/L is a chief factor of G . In the latter case, N/L is an elementary Abelian p -group for some p and is the only minimal normal subgroup of G/L . Further, $N = C_G(N/L)$ and $M \cap N = L$, so that N/L is represented faithfully by automorphisms of N/L . Also $M \cap N = G$ so that $(G, M) = (N, L) = 1$ for some n .

Corollary. In a soluble group, every maximal subgroup is of index a power of a prime.

Proof: Suppose that $N > L$ and let P/L be any chief factor of G such that $P \leq N$. P/L cannot be trivial since N/L is subnormal.

Hence P/L is an elementary p -group for some p , by 9.13. Let $Q = M \cap P$ so that $L \leq Q \leq M$. Since L is the greatest normal subgroup of G contained in M , and $L \leq P \leq G$, we have $P \leq M$ and so $PM = G$ by the maximality of M . Since P/L is Abelian, $Q \leq P$ and so $Q = PM = G$. But $L \leq Q \leq P$ and P/L is a chief factor of G . Hence $Q = L$.

Since N/L is subnormal and $P \leq G$, we have $L_1 = L[P, N] \leq P$ and $L_1 \leq G$, hence $L_1 = L$. So $N \leq C_G(\mathcal{O}/P/L)$. Let $D = C_G(M)$. Then $[D, P] \leq L \leq D$ and so $D \leq PM = G$. Hence $D = L$ and so $N = P = C$. Thus we obtain $N = C_G(N/L)$, $M \cap N = L$, $MN = G$.

If K/L is any minimal normal subgroup of G/L distinct from N/L , we should have $[R, N] \leq R \cap N = L$ and so $R \leq C_G(N/L) = N$, a contradiction. Thus the Fitting subgroup N/L of G/L is the only minimal normal subgroup of G/L and all is proved.

If $F = D \leq G$, then $L/P/L \cong F/L \cap F$ which is subnormal and so $F \leq N$ and $F' \leq N' \leq L \leq M$. This is true for any maximal subgroup M .

of G . Hence $F' \leq \mathcal{O} = \mathcal{O}(G)$ and the proof of (9) is now also complete.
 Indeed we see that the Sylow subgroups of F/\mathcal{O} are all elementary Abelian.
 9.1 and 9.2 are due to G. H. Hardy (1912-22), G. Frobenius (1849-1937) and W. Burnside.

Theorem 9.3. Let H/K be a chief factor of G and suppose that $M \cap H = 1$. Then (1) $F/\mathcal{O} \text{ mod } K \leq C = C_G(N/K)$ and (2) the subnormal $K = C_G(N/K) \leq G/C$ of N/K in G has no normal subgroup of C .

Proof: (1) The proof that $N = N_G(\mathcal{O} \text{ mod } K)$ as C has already occurred in the course of proving 9.2.

(2) Let B be any subgroup of K . Since B leaves invariant the n -th element of N/K and $H \cap N = 1$, B must also leave invariant further elements of N/K , by 9.1. In the isomorphism of A with G/C , let B correspond to B/C . If $B \leq A$, then $B = 1$ and the set of all elements $B \leq H$ such that $N \cap B$ is invariant under B , or equivalently $[B, B] \leq K$ is also a normal subgroup H_1 of G . Since $H \cap H_1 = 1$, it follows that $H_1 = 1$ and so $[H, B] \leq K$, $D \leq C$, $B = 1$.

(3). Let f be any representation of a group H by automorphisms of another group K , so that $f(g) \in \text{Aut } K$ for $g \in H$ and $f(g_1 g_2) = f(g_1) f(g_2)$. It is convenient to write the ordered pairs (g, k) with $g \in H, k \in K$ as formal products gk , the form of 'formal' being that $gk_1 = k_2 g$ and only if $k_1 = k_2$ and $g = 1$. If we define in the set G of all such formal products a multiplication $(g_1 k_1)(g_2 k_2) = g_2 k_2 g_1 k_1$ by the rule that

$$g_1 k_1 g_2 k_2 = g_2 k_2 g_1 k_1, \quad (4)$$

then G becomes a group. It is clearly the formal product gk with the element g of H and suitably k with the element k of K , then H and K become subgroups of G and that

$$HN = G, \quad H \cap K = 1 \quad (5)$$

is true. In G , the automorphism $f(g)$ of K induced by conjugation

by the element g of H is precisely $f(g)$.

G is called the split (or semidirect) extension of K by H .

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and the representation f^β of H which we obtain by replacing each $g \in H$ by $f^\beta(g) \in H$, is one of those given by (3), i.e. will be $\alpha = \beta_1(K)$ and $\beta = \alpha$. We define

$$\alpha_1 = \alpha_1(H, K, f^\beta) \quad (4)$$

to be the number of classes of conjugates in G into which the subgroups H_i complementary to K split up. If G has no subgroup K_i of K of the kind considered above and if α is arbitrary $q \neq 1$, then we can assume that $G = \langle H, K, f^\beta \rangle = \langle H, K, f^\alpha \rangle$ and only if f^β is related to f by (4), for some choice of α and β . If this simple case occurs for all choices of the representation f , then it is an easy matter to determine in principle how many non-isomorphic split extensions of K by H exist, although in practice this involves a good understanding of the structure of H and $\text{Aut } K$.

(5). We now prove an important result due to H. Zassenhaus. This is attributed by him to Schur. This is

Theorem 9.21. Let K be a normal Sylow-subgroup of G . Then G contains an S_{p-1} -subgroup H , so that $K \cap H = 1$, $HK = G$, $H \in \mathcal{C}/K$. If either H or K is soluble, then every S_{p-1} -subgroup of G is conjugate to H in G .

Proof: (i) Suppose that G is a group of least order for which the result is not true, and let S be a Sylow p -subgroup of K and $N = N_G(S)$. Then $K \cap N = 1$ since $S \leq K \leq N$, and $K \cap N$ is a normal Sylow-subgroup of N . If $N \leq G$, then N is a subgroup H complementary to K in G , by choice of G . Since $K \cap H = 1$, H is an S_{p-1} in G , contrary to hypothesis. Hence $N \not\leq G$. This is true for all p , so that K is insoluble. Let $Z = Z(K)$. Then $Z \leq K$ and so $Z \leq G$. N/Z is a normal Sylow-subgroup of G/Z and, since $Z \neq 1$, G/Z is a normal subgroup of G/Z and $G/Z = (N/Z)Z = N$. Then Z is a normal Sylow-subgroup of G , and, if $G \leq G$, then Z is a complement H to K in G . Since $G = G/K$, H is an S_{p-1} -subgroup of G contrary to hypothesis.

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determined by the representation f , and we shall denote it by

$$G = \langle H, K, f^\beta \rangle. \quad (5)$$

Since $K \in \text{Aut } K$ and $\beta \in \text{Aut } H$, we obtain from f a new representation f^β of H by automorphisms of K , defined by

$$f^\beta(g) = f(g)^\beta \quad (7 \in H, \beta \in K). \quad (6)$$

And we have

$$G = \langle H, K, f^\beta \rangle = \langle H, K, f \rangle, \quad (5)$$

the mapping $f \rightarrow f^\beta$ of G into G being an isomorphism. However, it is possible for there to be an isomorphism of the form (5) even when f and f^β are not related as in (6). The reason for this is the fact that, if we suppose that G has a normal subgroup K_i of K and that $N_i = N_G(K_i)$ and $G/N_i \in \mathcal{C}/K_i$, and such that K_i is complemented in G by a subgroup H_i . Then $H_i \in H$. If $i = 1, 2, 3$, and $f \rightarrow f_i$ are isomorphisms of K_i into K_i , and H_i into H_i , and if we define f^β by the equation $f_i^\beta = (f_i^\beta)^{\beta_i}$, then f^β is a representation of H by automorphisms of K and $\langle H, K, f^\beta \rangle = \langle H, K, f \rangle$. However, this possibility can be excluded if K is a Sylow-subgroup of G i.e. if $|H|$ and $|K|$ are coprime, for then $K_i = K$ by 5.12.

If there is no such subgroup K_i of K , we have only to consider the subgroups H_i which are complementary to K_i in G . Every such H_i contains exactly one element $\eta_i(K_i)$ in the case $\eta_i(K_i) = K_i \eta_i$ of K_i in G . Here η_i is any element of H and $K_i \eta_i \in K$. Since H_i is a subgroup, we have $\eta_i \eta_j^{-1} K_i = \eta_i K_i \eta_j^{-1}$ and so

$$K_i \eta_i \eta_j^{-1} = (\eta_i \eta_j^{-1}) K_i = K_i \eta_j. \quad (8)$$

Any mapping A of H into K satisfying (8) is called a cocycle with respect to f . The subgroups H_i complementary to K_i in G are therefore in one-to-one correspondence with cocycles. Among these cocycles there must be at least one of H in G . Every such conjugate has the form $H_i = \eta_i H$ with $\eta_i \in K$ and for this case the cocycle is defined by

$$A(\eta) = \eta \eta^{-1} \eta_j. \quad (9)$$

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A group G is called π -separable if the composition factors of G are all either π -groups or π' -groups. G is called π -soluble if the composition factors of G are all either π -groups for some $p \in \pi$ or else π' -groups. Note that π -separable and π' -separable, π -soluble and π' -soluble are p -separable. Also, in the above definitions the composition factors could be replaced by chief factors.

π -separable $\Leftrightarrow \pi$ -separable $\Leftrightarrow \pi$ -soluble if π has a series whose factors are all either π -groups or π' -groups. G is π -soluble if and only if it has a series whose factors are all either π -groups with some $p \in \pi$ or else π' -groups. Subgroups, subdirect groups and sections of π -separable groups are π -separable. Similarly for π -soluble.

Lemma 9.52 Let $p \in \pi$, $q \in \pi'$ and let G be π -separable. Then G has S_{π} -subgroups, $S_{\pi'}$ -subgroups and S_{π} -subgroups. Since π -separable $\Leftrightarrow \pi'$ -separable, it follows that G also has $S_{\pi'}$ -subgroups and S_{π} -subgroups.

Proof by induction on $|G|$. We may assume $G \neq 1$. Let M be a minimal normal subgroup of G , and let M/M be an S_{π} -subgroup of G/M with $\psi = \pi$, $\pi'q$ or πq . H exists by the induction hypothesis, and if $H \leq G$ the result follows at once for G . Hence we may assume G/M is a ψ -group.

First let M be a π -group. If $\psi = \pi$ or $\pi'q$, the result follows at once. If S is a Sylow q -subgroup of M and $G_1 = N_G(S) \leq G$, then $MG_1 = G$ and an S_{π} -subgroup of M is an S_{π} -subgroup of G . Hence we may assume $G_1 = G$, $S \leq G$, $M = S$ and again the result is immediate.

If M is not a π -group, it must be a π' -group. When $\psi = \pi'$, M is a Sylow π' -subgroup of G and the result follows from 9.46. When $\psi = \pi'q$ let T be a Sylow q -subgroup of M and $G_2 = N_G(T)$. If $G_2 < G$, an S_{π} -subgroup of G_2 exists by induction and is an S_{π} -subgroup of G since $MG_2 = G$. If $G_2 = G$, then $T \leq G$, $T = M$ and the result is immediate.

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immediate. The case $\psi = p \notin \pi$ is symmetrical as between π and π' and has already been dealt with.

Lemma 9.52 Let G be either π -soluble or π' -soluble, let H be any S_{π} -subgroup of G and let L be any π -subgroup of G . Then $L^H \leq H$ for some $g \in G$. In particular, any two S_{π} -subgroups of G are conjugate in G .

Proof by induction on $|G|$. We can assume $G \neq 1$. Let M be a minimal normal subgroup of G . Then M/M is an S_{π} -subgroup of G/M and M/M is a π -subgroup. By induction, $L^H \leq MH$ for some $g \in G$. The lemma follows by a second induction if $MH = G$, since H is an S_{π} -subgroup of MH . So we may assume $MH \neq G$. Then G/M is a π -group. If M is a π -group, then $H = G$ and there is nothing left to prove. If M is not a π -group, it is a normal S_{π} -subgroup of G , since G is π -separable. Further, $H \cap M = 1$, $H \cong G/M$ and by hypothesis at least one of the groups M and H is soluble. Right? Also $L \cap M = 1$ and if $G_1 = LM$, $G_2 = H \cap G$, then L and L_1 are S_{π} -subgroups of G_1 . Either L or M is soluble. Hence $L_1 = L^G$ for some $g \in M$ by 9.4(3) and the lemma follows.

For soluble groups, we may now state a result analogous to Sylow's Theorem but with the prime p replaced by an arbitrary set of primes. This is

Theorem 9.5 Let G be soluble and let π be any set of primes. Then (i) G contains S_{π} -subgroups.

(ii) Any two S_{π} -subgroups of G are conjugate in G .
 (iii) Every π -subgroup of G is contained in some S_{π} -subgroup of G .
 By 9.52, if G is either π -soluble or π' -soluble, the S_{π} -subgroups of G may be described alternately as the maximal π -subgroups of G . It also follows that any S_{π} -subgroup H of G is π -normal in G and $H = N_G(H)$ is π -normal in G .

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A group G is called π -separable if the composition factors of G are all either π -groups or π' -groups. G is called π -soluble if the composition factors of G are all either π -groups for some $p \in \pi$ or else π' -groups. Note that π -separable and π' -separable, π -soluble and π' -soluble are p -separable. Also, in the above definitions the composition factors could be replaced by chief factors.

π -separable $\Leftrightarrow \pi$ -separable $\Leftrightarrow \pi$ -soluble if π has a series whose factors are all either π -groups or π' -groups. G is π -soluble if and only if it has a series whose factors are all either π -groups with some $p \in \pi$ or else π' -groups. Subgroups, subdirect groups and sections of π -separable groups are π -separable. Similarly for π -soluble.

Lemma 9.52 Let $p \in \pi$, $q \in \pi'$ and let G be π -separable. Then G has S_{π} -subgroups, $S_{\pi'}$ -subgroups and S_{π} -subgroups. Since π -separable $\Leftrightarrow \pi'$ -separable, it follows that G also has $S_{\pi'}$ -subgroups and S_{π} -subgroups.

Proof by induction on $|G|$. We may assume $G \neq 1$. Let M be a minimal normal subgroup of G , and let M/M be an S_{π} -subgroup of G/M with $\psi = \pi$, $\pi'q$ or πq . H exists by the induction hypothesis, and if $H \leq G$ the result follows at once for G . Hence we may assume G/M is a ψ -group.

First let M be a π -group. If $\psi = \pi$ or $\pi'q$, the result follows at once. If S is a Sylow q -subgroup of M and $G_1 = N_G(S) \leq G$, then $MG_1 = G$ and an S_{π} -subgroup of M is an S_{π} -subgroup of G . Hence we may assume $G_1 = G$, $S \leq G$, $M = S$ and again the result is immediate.

If M is not a π -group, it must be a π' -group. When $\psi = \pi'$, M is a Sylow π' -subgroup of G and the result follows from 9.46. When $\psi = \pi'q$ let T be a Sylow q -subgroup of M and $G_2 = N_G(T)$. If $G_2 < G$, an S_{π} -subgroup of G_2 exists by induction and is an S_{π} -subgroup of G since $MG_2 = G$. If $G_2 = G$, then $T \leq G$, $T = M$ and the result is immediate.

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(E) Lemma 9.6. Let H and K be subgroups of G , $H \cap K = 1$ and $L = H \cap K$. Then (i) $|K:L| \leq |G:H|$ and so $|G:L| \leq |G:H| \cdot |G:K|$ and (ii) if H and K are conjugate in G , then $|K:L| \leq |G:H|$ and $|HK:L| \leq |G:H| \cdot m$ and $|G:L| \leq m$, where $|G:L| = m$ and $|HK:L| = m$.

Proof: (i) $|K:L| = |HK:L|$ by 5.5. (ii) If $g \in H, g \in K$, then $H^g \cap K = H^g \cap K = K$. Hence if H and K are conjugate in G , and distinct, then $H \cap K = 1$ and so $|K:L| \leq |G:H|$. (iii) Since $L \leq H \leq G$, $|G:L|$ is a multiple of m . Similarly it is a multiple of n . Since $(m,n) = 1$, $|G:L|$ is a multiple of mn . But $|G:L| \leq mn$ by (i). Hence $|G:L| = mn$ and so $|K:L| = m$ and $|HK:L| = n$.

Theorem 9.7. Suppose that G has exactly three soluble subgroups H_1, H_2, H_3 where H_1, H_2, H_3 in G are coprimely pairs. Then G is soluble.

Proof. If $H_1 = 1$, then $H_2 = H_3 = G$ and there is nothing to prove. So we can assume $H_i \neq 1$. Let M be a minimal normal subgroup of G . Then $|M| = p^m$ for some prime p . Since $(m_1, m_2) = 1$, we may assume that p does not divide m_2 . By 9.6 (ii), if $H_1 \cap H_2 = H_1 \cap H_3$, then $|H_1:L_{12}| = m_2$. Hence L_{12} contains a Sylow p -subgroup of H_1 and so $M \leq L_{12}$. Also $G = H_1 H_2$ and so every element of G has the form xy with $x \in H_1, y \in H_2$. This shows that every conjugate $M^g = M^y$ of M in G is contained in H_2 . Hence $K = \langle M^g, y \in H_2 \rangle$ is normal in G and soluble. In G/K , the subgroups KH_i/K are $H_i/K \cap H_i$ are soluble and their indices n_i are coprime in pairs since $n_i = |G:KH_i|$ divides $n_i = |G:H_i|$. By induction on $|G|$, we may assume that G/K is soluble. Hence G is soluble.

(F). The proof of the following theorem of Burnside depends on the theory of group characters and will be postponed to §.

Theorem 9.8. If the group G has a class of $p^2 > 1$ conjugate elements, then G is not simple.

Corollary. If $|G|$ is divisible by only two primes p and q , then G is soluble.

Proof by induction on $|G|$. We may assume that $G \neq 1$ and that $3 \leq p < q$. Then G is not a p -group. Let Q be a Sylow q -subgroup of G . Then $Q \neq 1$ and so $3 \leq q < p$. Let $z \in Z(Q), z \neq 1$.

Since $z \in G$, we have $|G:C_G(z)| = p^m > 1$. Hence G is not simple by 9.4. So there is a subgroup $K < G$ with $1 < K < G$. By induction both K and G/K are soluble. Hence G is soluble.

Theorem 9.9. A group G is soluble if and only if it has an S_{p^k} -subgroup for every prime p .

Proof: The condition is necessary by 9.5. Let $|G| = p_1^{a_1} p_2^{a_2} \dots p_k^{a_k}$ where $p_1 < p_2 < \dots < p_k$ are primes. If $k \leq 2$, then G is soluble by 9.8. Suppose that G has an $S_{p_i^{a_i}}$ -subgroup H_i for each $i=1,2,\dots,k$. If $k=1$, then H_1 is soluble by 9.8 and so G is soluble by 9.7. Let $k \geq 3$.

By 9.6 (ii), if $i \neq j$, $H_i \cap H_j$ is an $S_{p_i^{a_i}}$ -subgroup of H_i . So we may assume each H_i is soluble by induction on k . It now follows that G is soluble by 9.7 again.

§10. Systematics and Carter Subgroups of Soluble Groups

(A) Let G be a soluble group and let p_1, \dots, p_k be the distinct primes which divide $|G|$. By 9.5, G has for each $i=1, 2, \dots, k$ at least one Sylow subgroup H_i . By 9.6(b), the 2^k intersections of sets of subgroups chosen from H_1, \dots, H_k , including the empty intersection \emptyset , are Sylow subgroups of G for various π . In particular $P_i = \bigcap_{j \neq i} H_j$ is a Sylow p_i -subgroup of G . Since for $i \neq l$

$$|P_i P_l| = |\bigcap_{j \neq i, l} H_j| = |P_l P_i|,$$

we must have $P_i P_l = P_l P_i = \bigcap_{j \neq i, l} H_j = P_i \cap P_l$, so that the Sylow subgroups P_1, P_2, \dots, P_k of G are permutable in pairs. And if Λ is any subset of $1, 2, \dots, k$ and Λ' is the complementary set, we have more from

$$\prod_{i \in \Lambda} P_i = \bigcap_{i \in \Lambda'} H_i$$

that the 2^k intersections of sets of H_j coincide with the 2^k products of sets of P_i , including the empty product 1 . This set of 2^k subgroups is called an Sylow system of G . Given any Sylow system I of G , and any set of primes π , just one member of I is an Sylow-subgroup of G and this will be denoted by T_π . If π and ψ have the same intersection with the set p_1, \dots, p_k , then $T_\pi = T_\psi$. I is determined uniquely by the T_π and equally well by the T_{p_i} . For any two sets of primes π and ψ , we have

$$T_\pi T_\psi = T_\psi T_\pi = T_{\pi \cup \psi}$$

and in particular, $T_\pi T_{p_i} = G$ and $T_\pi \cap T_{p_i} = 1$.

Lemma 10.1 Let S and I be any two Sylow systems of the soluble group G . Then there is an element $g \in G$ such that $S_g = T_\pi$ for all π .

Proof: ~~Let $S = \{S_\pi\}$ and $I = \{T_\pi\}$ be any two Sylow systems of the soluble group G . Then there is an element $g \in G$ such that $S_g = T_\pi$ for all π .~~ Suppose that $S_g = T_\pi$ for $i=1, 2, \dots, r$ ($0 < r \leq k$) while $S_g \neq T_{p_i}$. Since S_g and T_{p_i} are conjugate in G and $S_g S_{p_i} = G$, there is an element $q \in S_g$ such that $S_g^q = T_{p_i}$. But

$S_{p_i} = S_{p_i}^q$ for $i=1, 2, \dots, r$. Hence the Sylow system S_g^q of G coincides with I in its first r Sylow complements T_{p_1}, \dots, T_{p_r} . By induction on r , $S_g^q = I$ for some $g \in G$.

(B). The normalizers in G of the Sylow system I of G will be called the systemizers of G . By definition,

$$N_G(I) = \bigcap_{p_i \in I} N_G(T_{p_i}) = \bigcap_{p_i \in I} N_G(T_{p_i}).$$

If H is any subgroup of G and \mathcal{Y} is an Sylow system of H , then $N_G(\mathcal{Y}) = \bigcap_{p_i \in \mathcal{Y}} N_G(T_{p_i}) = \bigcap_{p_i \in \mathcal{Y}} N_G(T_{p_i})$ will be called a relative systemizer of H in G . Since $H = \prod_{p_i \in \mathcal{Y}} T_{p_i}$, it is clear that $N_G(\mathcal{Y}) = N_G(H)$, where $L = N_G(H)$. Hence $N_G(\mathcal{Y}) = N_G(H)$ is a typical

Lemma 10.2 Let G be a soluble group and let $H \triangleleft G$.

(1) The relative systemizers L of H in G are all conjugate in G and $LH = G$. In particular any two systemizers of G are conjugate in G .

(2) The systemizers of G are nilpotent.

Proof: (1) Let $L = N_G(\mathcal{Y})$ and $M = N_G(\mathcal{Z})$ where \mathcal{Y} and \mathcal{Z} are Sylow systems of H . By 10.1, $\mathcal{Y} = \mathcal{Z}^x$ for some $x \in H$. Hence $M = L^x$. If $y \in G$ then \mathcal{Y}^y is an Sylow system of H since $H \triangleleft G$. Hence $\mathcal{Y}^y = \mathcal{Y}$ for some $z \in L$ by 10.1 and so $zy \in L$, $y \in L^{-1}H$. This is true for all $y \in G$ so $G = L^{-1}H$ and $LH = G$ and let P be a Sylow p -subgroup of L . Then P normalizes \mathcal{Y} which is now a Sylow p -subgroup of G . Hence $P \leq V_p$, $P = L \cap V_p$ and so $P \triangleleft L$ since L normalizes \mathcal{Y} . Thus L is nilpotent.

(2) Now let M be a minimal normal subgroup of the soluble group G so that $|M| = q^m$ for some prime q and let I be an Sylow system of G and $H = N_G(I)$ the corresponding systemizer. The groups $M_{T_{p_i}}/M$ are permutable in pairs and by 5.7, $M_{T_{p_i}}/M$ is an Sylow-subgroup of G/M . Since $M_{T_{p_i}} = \bigcap_{p_j \in I} M_{T_{p_j}}$, the groups $T_{p_i}^x = M_{T_{p_i}}/M$ form an Sylow system I^x of G/M . Let $H^x/M = \bigcap_{p_i \in I^x} M_{T_{p_i}}/M$. Then $H^x = \bigcap_{p_i \in I} M_{T_{p_i}}$, and so $H \triangleleft H^x$. Conversely let $z \in H^x$. If $p \neq q$, then $M_{T_{p_i}} = T_{p_i}$. Also $T_{p_i}^z$ is an Sylow-subgroup of $M_{T_{p_i}}$ and hence it is conjugate to T_{p_i} in $M_{T_{p_i}}$. Thus $T_{p_i}^z = T_{p_i}$ for some $g \in M$. Hence $g \in I$.

for all $p \neq 2$ and $\exists \beta^{-1} \in H, \exists \alpha \in M$. Then $H = H^M$.
 By repeated application of this argument we obtain
Lemma 10.3 Let G be a soluble group. $K \triangleleft G$ and let I be an S-system of G , $T_I^* = K T_I / K$. The set I^* of groups $T_{I^*}^*$ is an S-system of G/K ; and if $L = N_G(I)$, $L^* = N_{G/K}(I^*)$, then $L^* = K L / K$.
 In other words, in any quotient of a soluble group G into another group G^* , the S-systems of G map into the S-systems of G^* and the S-systems of G map into the S-systems of G^* .
 Now let H be any non-trivial normal subgroup of G with $|M| = 2^r$, r prime; and let $C = C_G(M)$. If $C \neq G$, so that $M \leq Z(C)$ it is clear that every S-system of G contains M . If $C \leq G$, let D/K be a chief factor of G . Then by 9.3 (5), D/C is a $2'$ -group. $\Rightarrow D/K$ is a chief factor of G and $L = D \cap T_I^*$. It follows that $L C = D$ by 5.7. Let $H = N_G(I)$ and $N = H \cap M$. Then N normalizes L and we have $[L, N] \leq L \cap M = 1$. Hence L centralizes N . But C also centralizes N . Hence $N \leq Z(D) \triangleleft G$. If $M \neq 1$, it follows that $M \leq Z(D)$ since M is a minimal normal subgroup of G , contrary to the definition of $C = C_G(M)$. Hence $N = 1$.
 Using 10.3, this gives
Theorem 10.4 Let H/K be any chief factor of the soluble group G and let L be any S-system of G . Then $H \leq LK$ or $H \cap L \leq K$ according as H/K is a central factor of G or not.
 We may express this result by saying that the S-systems of G cover every central chief factor of G and avoid every non-central chief factor of G . Now the index $|G:L|$ of a S-system of G in G is equal to the number of distinct S-systems of G . Hence we have
Corollary 10.5 Let $G = G_1 \times G_2 \times \dots \times G_r = I$ be a chief series of the soluble group G . Then the number of distinct S-systems of G is equal to $|T| \cdot |G_1| \cdot |G_2| \dots |G_r|$ where over all the non-central factors $G_{i_1} \times \dots \times G_{i_s}$ of the given chief series.

Subgroups and direct products of nilpotent groups are nilpotent. By 8.5, it follows that every group G has a uniquely determined normal subgroup $G^0 = \gamma_2 G$.
 The limit of the lower central series of G , with the property that a quotient group G/K of G is nilpotent if and only if $K \supseteq G^0$.
Corollary 10.42 If L is a S-system of the soluble group G , then $\{G = L \leq L^2 \leq \dots \leq L^r = 1\}$.
 For every chief factor H/K of G with $K \supseteq G^0$ is a central factor of G and so $H \leq LK$ by 10.4. Hence $G = L \leq G^0$. If M is a minimal normal subgroup of G , then G/M is cyclic of prime order and so $M \supseteq G^0$. It follows that $L \neq M$. Hence G is the normal closure of L in G .
 (C) We consider next the relation between the S-systems of a soluble group G and the abnormal minimal subgroups of G . First we need
Lemma 10.43 Let I be an S-system of the soluble group G . Let $H = N_G(I)$ and let $N = N_G(T_I)$. Then the Sylow p -subgroup P of H is a Sylow p -subgroup of N and $|P|$ is equal to the product of the orders of the central p -factors in any chief series of G .
 Proof: Let P^0 be any Sylow p -subgroup of N . For $2 \nmid p$, P^0 is an Sylow p -subgroup $T_{P^0}^*$ of G which contains P^0 , by 9.5. Let $T_{P^1}^* = T_{P^0}$ and let I^* be the S-system of G determined by the $T_{P^i}^*$. Then $P^0 \leq H^* = N_G(I^*) = \bigcap N_G(T_{P^i}^*)$ so $|N_{P^0}| = |P^0| \leq |H_{P^0}| = |H|_p = |P|$ and $|P| \leq |N_{P^0}|$ since $P \leq N$. Hence $|P| = |P^0|$.
 The second part of 10.43 now follows from 10.4.
 Now let M be an abnormal minimal subgroup of the soluble group G and suppose first that $K_G(M) = \text{fit } M^G = 1$. By 9.2, G has a unique minimal normal subgroup $L = \text{fit } G$ of order p^r , p prime, and $M \cap L = 1$, $M \leq G$. Hence M contains an Sylow p -subgroup T_{P^1} of G . If $2 \nmid p$ and T_{P^1} is any Sylow p -subgroup of G , then $L \leq T_{P^1}$ and so $M \cap T_{P^1} = \text{fit } M$ is an Sylow p -subgroup of M . Taking $S_{P^1} \rightarrow T_{P^1}$, we thus obtain an S-system

(6) In this case $\{L_i : i \in I\} = p^m$ say and if T is any Sylow-subgroup of L , then T is an Sylow-subgroup of G and we have $N_{G_i}(T) = N_{L_i}(T)$ for each i since L_{i-1} is abnormal and maximal in L_i . Hence $N_G(T) = N_G(T)$. But $N_G(T)$ is abnormal in G , since T is p -Syl. Hence L is abnormal in G .

(7) By (1), we have only to show that $H = N_G(\Sigma)$ is subnormal in G , where Σ is any S-system of G . Let $G = G_0 \triangleright G_1 \triangleright \dots \triangleright G_m = 1$ be a chief series of G and let $H_i = N_{G_i}(\Sigma^{(i)})$, where $\Sigma^{(i)} = \Sigma \cap G_i$, so H_i is an S-system of G_i by (5) and H_i is the relative splinterizer. Clearly $H = H_0 \leq \dots \leq H_m = G$. Now $|G_{i-1} : G_i| = p^k$ for some prime $p = p(i)$ and some $k = n(i)$. Hence $S_{G_i}^{(i)} = S_{G_{i-1}}^{(i-1)}$ for $2 \leq i$, and so H_i is the normalizer of $S_{G_i}^{(i)}$ in H_{i-1} . Hence $H_i \cap G_i = N_{G_i}(S_{G_i}^{(i)})$ is a splinterizer of G_i and so $H_i \cap S_{G_i}^{(i)} = P_i$ is the Sylow p -subgroup of the splinterizer group $H_i \cap G_i$. Hence $H_i \cap G_i = N_{G_i}(P_i)$. But $S_{G_i}^{(i)} = S_{G_{i-1}}^{(i-1)}$ since G_{i-1}/G_i is a p -group and $H_i/G_i \cong H_{i-1}/G_{i-1}$. Hence every element of $N_{G_i}(P_i)$ normalizes $S_{G_{i-1}}^{(i-1)}$, and also $S_{G_{i-1}}^{(i-1)}$ for all $i \in I$. It follows that $N_{G_i}(P_i) \leq H_{i-1}$. Consequently $H_{i-1} = N_{G_i}(P_i)$. [Now P_i is a Sylow subgroup of the normal subgroup $H_i \cap G_i$ of H_i , so that P_i is a Sylow subgroup of H_i when H_i is abelian, and since this is true for each i , H is subnormal in G .]

The argument in (7) shows rather more viz.

Lemma (10.1) Let $K \leq G$, let K be soluble and let I be any S-system of K . Then the relative splinterizer $N_G(I)$ is subnormal in G . Note that although splinterizers are subnormal subgroups, not every subgroup which contains a splinterizer of the soluble group G need be subnormal in G . For example, if G is the dihedral group and if H is one of the three elementary non-normal subgroups of order 4, then H contains two of the six splinterizers of G (which are of order 2) but H is not subnormal in G since the only proper subgroup of G which contains H is one of the Sylow 2-subgroups S and $H \not\leq S$. However H is p -Syl.

(10.1) is a special case of the following theorem.

Σ of M and an S-system I of G . Since M is not normal in G , we have $M \not\leq L$ and so L/I is a non-central chief factor of G by 9.2. Since M and G/L meet in the identity, it follows that the product of the orders of the central p -factors in a chief series is the same for M as for G . Hence $N_G(T_p)$ and $N_G(T_p)$ have Sylow p -subgroups of the same order. Therefore $N_G(T_p) \leq N_G(T_p)$ and $H = N_G(I) \leq M$.

In the general case where $K = N_G(M) \neq 1$, we can apply this result to the group G/K . Let Σ and I be S-systems of G/M and G respectively, such that $S_{p_i} = T_{p_i}$. Here as before we write $|G : M| = p^m$. For $2 \leq i$ we have $|T_{p_i} : S_{p_i}| = p^m$ and so $M T_{p_i} = G$, $M \cap T_{p_i} = S_{p_i}$. Since $N_G(T_{p_i}) \leq N_G(K T_{p_i}) = N_M(K T_{p_i})$ by the above particular case, the result $N_G(T_{p_i}) = N_M(T_{p_i})$ holds in general. Hence again $H = N_G(I) \leq M$ and therefore $H \leq N_G(\Sigma)$.

A subgroup H of a group G is called subnormal in G if there is a chain of subgroups H_i such that $H = H_0$, where $H_1 \leq H_2 \leq \dots \leq H_n = H_n = G$. Since every subgroup containing an abnormal subgroup is abnormal, we may also say that H is subnormal in G if and only if H can be linked to G by a chain of subgroups each of which is normal in the next.

We can now state Theorem 10.5. Let G be a soluble group and let L be a subnormal subgroup of G . Then

- (1) every splinterizer of L contains a splinterizer of G .
 - (2) If $|G : L| = p^m$, p prime, then L contains the normalizer in G of some Sylow-subgroup of G and L is abnormal in G .
 - (3) The splinterizers of G are the minimal subnormal subgroups of G .
- Proof:* (1) $L = L_0 \leq L_1 \leq \dots \leq L_n = G$ where each L_{i-1} is an abnormal normal subgroup of L_i . As we have seen, this implies that every splinterizer of L_{i-1} contains a splinterizer of L_i .

(D) A group G is called metaindecomposable if it has a normal subgroup K such that both K and G/K are nilpotent. For such a K , clearly $G^p \leq K \leq \Omega(G)$. G is metaindecomposable if and only if $G/\Omega(G)$ is nilpotent or equivalently if and only if G^p is nilpotent.

Theorem 10.16. Let G be metaindecomposable and let H be a p -subgroup of G . Then H is a p -subgroup of every subgroup L of G which contains H . Also H is abnormal in G .

Proof. Let $F = \Omega(G)$ be the Fitting subgroup of G . Since G is metaindecomposable and so $HF = G$ by 10.12. The group $L \cap F$ is nilpotent and so also is H by 10.2 (B). Let R_p and S_p be the Sylow p -subgroups of $L \cap F$ and H respectively. Then R_p also $L \cap F \trianglelefteq L$ and so $R_p \trianglelefteq L$. Hence $T_p = R_p S_p$ is a p -subgroup of L . Since $HF = G$, we have $H(L \cap F) = L$ and so T_p is a Sylow p -subgroup of L . Since $R_p \trianglelefteq L$ and $S_p \trianglelefteq H$, we have $T_p T_p = T_p$ for any two primes p and l . Hence the groups T_p and their products form an S -system \mathcal{I} of L and $H \trianglelefteq N_G(\mathcal{I})$.

Now let C/D be any chief factor of G such that $C \leq F$. By 7.3 (U), F contains C/D . ~~Therefore $N_G(C/D) \trianglelefteq N_G(C/D)$~~ Let $N = N_G(C/D)$. If $H < N$, there exists such a chief factor C/D with $H \cap C \leq D$ and $D(N \cap C) > D$, by 9.4. Then $[H, N \cap C] \leq H \cap C \leq D$ and since $HF = G$, it follows that $[G, N \cap C] \leq D$. Hence $D(N \cap C) \trianglelefteq G$ and consequently $C = D(N \cap C)$ and C/D is a central factor of G . But in that case $C \leq HD$ by 9.4, contrary to $H \cap C \leq D$. This contradiction shows that $H = N$ is abnormal in G . It now follows that $H = H^*$ is a p -subgroup of L , since H^* is nilpotent by 10.2 (B).

Finally, let $H \trianglelefteq L \cap L^g$, where $g \in G$. Then $\{H, H^g\}$ and H are two p -subgroups of L and hence they are conjugate in L . Hence there is an element j of L such that $H^j = H$. Then $\{H, H^j\} = H$ and so $j \in H$. Since L is any subgroup of G containing H , it follows that H is abnormal in G .

H_i is the normalizer of $S_p^{(i)}$ in H_i . Since

$$G_{i-1} = G_i(S_p^{(i)} \cap H_i), \text{ it follows that}$$

$$|S_p^{(i-1)} \cap H_i| = |G_{i-1} : G_i| |S_p^{(i)} \cap H_i|. \text{ If } P^* \text{ is a}$$

Sylow p -subgroup of $G_{i-1} \cap H_i$, then $P^* = (G_{i-1} : G_i) |P^* \cap H_i|$.

Since $P^* \cap G_i$ is a p -subgroup of G_i , it follows that

$$|P^*| \leq |S_p^{(i-1)} \cap H_i|. \text{ Thus, } S_p^{(i-1)} \cap H_i \text{ is a}$$

Sylow p -subgroup of $G_{i-1} \cap H_i$. Since $S_p^{(i-1)} = S_p^{(i)} (S_p^{(i)} \cap H_i)$,

it follows that $H_{i-1} = H_i (S_p^{(i)} \cap H_i)$. Since $S_p^{(i-1)}$

is a Sylow p -subgroup of $G_{i-1} \cap H_i \trianglelefteq H_i$, it follows

that $S_p^{(i-1)} \cap H_i$ is H_{i-1} , so H_{i-1} is H_i .

This theorem like the next is due to R. Carter.

In any group G , a nilpotent subgroup which is diagonal in G will be called a Carter subgroup of G .

Theorem 10.7. Let G be a soluble group. Then G contains at least one Carter subgroup.

(1) The Carter subgroups of G are all conjugate in G .

(2) If H is a Carter subgroup of G and if L is any subgroup of G containing H , then H is a Carter subgroup of L .

(3) The Carter subgroups of G are abnormal in G .

(4) If $K \triangleleft G$, and if H is a Carter subgroup of G , then KH/K is a Carter subgroup of G/K .

(5) The Carter subgroups of G coincide with the systemizers of G if and only if the latter are diagonal in G , and this is the case in particular when G is metanilpotent.

Proof by induction on $|G|$. ~~Base case: $|G| = 1$. Then G is trivially metanilpotent and the result follows.~~
 The only Carter subgroup of G is G itself and the result (1)-(5) are immediate. Hence we suppose that G is not nilpotent. Let M be a minimal normal subgroup of G , $|M| = p^m$.

(1) By induction G/M has a Carter subgroup L/M . Since L/M is nilpotent, L is metanilpotent. Let H be a systemizer of L and let $\bar{H} \leq N_G(H)$. Since $M \triangleleft G$ and $HM = L$, we have $\bar{H} \leq N_G(L) = L$ and so $\bar{H} \leq N_G(H) = H$ by 10.6. But H is nilpotent. Hence H is a Carter subgroup of G .

(2) Let H^* be any Carter subgroup of G and let $L^* = H^*M$. Then L^*/M is a Carter subgroup of G/M and so L^* is metanilpotent. If $L^* \triangleleft G$, it follows from (1) by induction that H^* is a systemizer of L^* , since H^* is a Carter subgroup of L^* and the systemizers of metanilpotent groups are also Carter subgroups. If $L^* = G$, then $H^*M = G$ and so is either 1 or M . But G is not nilpotent. Hence $H^*M = 1$. H^* normalizes

$M \leq S_{p^2}$ where S_{p^2} is the unique Sylow p -subgroup of H^* for each prime $q \neq p$ and H^* also normalizes $S_{p^2} = T_{p^2}$. Since T_{p^2} is an Sylow p -subgroup of $G = H^*M$, it follows that $H^* \leq N_G(T_{p^2})$ where T_{p^2} is the S -system of G determined by the T_{p^2} . But M is a non-central chief factor of G and so $N_G(T_{p^2}) \cap M = 1$ by 9.1c. Since $G = H^*M$, it follows that $H^* = N_G(T_{p^2})$ is a systemizer of G in this case also.

Let $\bar{H} \in N_G(L^*)$. Then $H^* \bar{H}$ is also a systemizer of L^* , hence conjugate to H^* in L^* and so $H^* \bar{H} = H^*$ for some $q \in L^*$. Therefore $\bar{H} \in N_G(H^*) = H^*$ and so $\bar{H} \in L^*$. Thus $N_G(L^*) \leq L^*$ and since L^*/M is nilpotent it follows that L^* is a Carter subgroup of G/M . By induction, L^* is conjugate to L in G and hence H^* is conjugate to H . (iii) is clear.

(iv) Let $H \leq L \cap L^*$, where $L \in \mathcal{L}$ and L^* is a subgroup of G containing the Carter subgroup H of G . Then H and L^*M are Carter subgroups of L , hence conjugate in L by (ii), so that $\bar{H} \in N_G(H) = H$ for some $q \in L$. It follows that $\bar{H} \in L$. Thus H is abnormal in G .

(v) This has been shown in proving (i) and (ii) for the case $K = M$, a minimal normal subgroup of G . The general result follows at once. (vi) Since Carter subgroups are diagonal by definition and systemizers are nilpotent by 10.2 (ii), this is clear.

§ 11. Subnormal Subgroups (Wielandt)

(A). If H is any subgroup of G , we denote by $H^{(n)}$ the n -th subnormal class of H in G , i.e. the intersection of all the subnormal subgroups of G which contain H . By 6.3(B), we have $H^{(n)} \triangleleft_n G$.

The normal closure of H in $H^{(n)}$ is $H^{(n)}$ itself. Define $H_0 = G$ and $H_{i+1} = \langle H, H_i \rangle$ inductively for $i \geq 0$. Then for some $r \geq 0$, we have $G = H_0 \triangleright H_1 \triangleright \dots \triangleright H_r = H_{r+1}$.

Since $H_{i+1} \triangleleft H_i$ for all i , we have $H_i \triangleleft_n G$ and so $H^{(n)} \triangleleft H_i$. By 6.11, $H^{(n)} \triangleleft_n H_i$. But $H_i = \langle H, H_i \rangle = \langle H^{(n)}, H_i \rangle$. Hence $H_i = H^{(n)}$.

$H \triangleleft_n G$ if and only if $H = H^{(n)}$. In this case we call $G = H_0 \triangleright H_1 \triangleright \dots \triangleright H_r = H$ (4)

the successor series from G to H and write $r = m(G, H)$. If we have any series $G = G_0 \triangleright G_1 \triangleright \dots \triangleright G_n = H$ from G to H , then we obtain $m(G, H) \leq n$ for all i by induction on i , and so $m(G, H) \leq n(G, H)$.

$m(G, H)$ is the minimal length of a series from G to H .

Theorem 11.1. Let H and K be subnormal in G . Then $J = \langle H, K \rangle$ is subnormal in G .

Proof: Suppose first that $H \triangleleft J$. Then K normalizes not only H but also, by induction on i , every term H_i of the canonical series (4). Hence $K_i = H_i \cap K$ is a subgroup of G . Since $H \triangleleft_n G$, we have $H_i \triangleleft_n K_i$ by 6.3. But $H_i \triangleleft_n H_i$ and K normalizes H_i . Hence $H_i \triangleleft_n K_i$. Now the product of a normal subgroup with a subnormal subgroup is a subnormal subgroup, since spinoriforms preserve the normality relation. Hence $K_i = H_i \cap K \triangleleft_n K_i$. By 6.3(B), it follows that $J = \langle H, K \rangle = \langle H, K \rangle \triangleleft_n G$ as required.

We prove the general result by induction on $m(G, H) = r$. If $r = 1$, then $H \triangleleft G$ and so $J = \langle H, K \rangle \triangleleft_n G$. Suppose $r \geq 2$, and let $H = \langle H^{(1)}, H^{(2)} \rangle$. For any $J \triangleleft J$, we have $H^{(1)} \triangleleft_n H^{(2)}$ and $m(H^{(1)}, H^{(2)}) = r - 1$.

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(B) Following Wielandt, we call a subgroup H of G intrinsically in G if $H^{(n)}$ is conjugate to H in G for every $n \in \mathbb{N}$. Equivalently, H is intrinsically in G if and only if the class of conjugates to which H belongs is a characteristic class of conjugates in G , i.e. is invariant under $\text{Aut } G$. A normal subgroup is intrinsically in G if and only if it is characteristic.

Lemma 10.1. (1) If H is intrinsically in K and K is intrinsically in G , then H is intrinsically in G .

(2) If H is intrinsically in G , so also are $N_G(H)$ and $C_G(H)$.

(3) If H is intrinsically in K and $K \triangleleft G$, then $K N_G(H) = G$.

(4) Sylow subgroups are intrinsically in G .

(5) In a soluble group, $S_{p'}$ -subgroups, Sylow subgroups and Carter subgroups are intrinsically in G .

The proofs are immediate for (1), (2), (3). (4) follows from Sylow's Theorem and (5) from 9.5, 10.1 and 10.7.

By the induction on r , all the groups $M = (M^1, M^2, \dots, M^r)$ with M^1, \dots, M^r subnormal in H , and hence in G . This follows by induction on n . But for a suitable choice of the M^i , $M = \bar{H}$. Hence \bar{H} sub G . And $J = \bar{H}K$ and $\bar{H} = J$, K sub G . So by the special case already considered we obtain J sub G as required.

In view of 6.3 (iv), we may now state as a corollary that the subnormal subgroups of any group G form a sublattice of the lattice of all subgroups of G .

(E) By 7.61, every subnormal nilpotent subgroup of G is contained in the Fitting subgroup $F(G)$ of G . Conversely, if H is any subgroup of $F(G)$, then H sub $F(G)$ by 6.18 (ii). Since $F(G)$ char G , it follows that H sub G . Thus the subnormal nilpotent subgroups of G are precisely the subgroups of the Fitting subgroup of G .

Theorem 11.2 (i) Let H sub G and let P be a Sylow p -subgroup of G . Then $H \cap P$ is a Sylow p -subgroup of H .

Proof: (i) is a corollary of 5.6. (ii) Assuming $H \not\leq P$, let M be a minimal normal subgroup of H . Since H is soluble, $|M| = p^k$ for some prime p and every Sylow p -subgroup of H contains M . Hence every Sylow p -subgroup of G contains M and $M \leq \bar{H}$, the maximal normal p -subgroup of G . Let $L = H\bar{H}$ and let Q be a Sylow q -subgroup of L where $q \neq p$. Then Q sub Q , where Q is some Sylow q -subgroup of G and we have $H \cap Q \leq L \cap Q = Q$. By hypothesis, $H \cap Q$ is a Sylow q -subgroup of H . But $\bar{H} \leq L$ and $Q \cap \bar{H} = 1$, since $q \neq p$. Since $L = H\bar{H}$, it follows that $H \cap Q$ is a Sylow q -subgroup of L . Hence $H \cap Q = Q \leq H$. Thus H contains all p -elements of L . Hence from a normal subgroup K of L and L/K is a p -group. Since $K \leq H \leq L$ it follows that

H sub L , (since L/K is nilpotent by 6.19.) The Sylow subgroups of G/\bar{H} have the form M_i/\bar{H} where S is a Sylow subgroup of G , by 5.6. Since $H \cap S$ is a Sylow subgroup of H by hypothesis, it follows that $(L \cap M_i)/\bar{H}$ is a Sylow subgroup of L/\bar{H} . This is true for every Sylow subgroup M_i/\bar{H} of G/\bar{H} . By induction on $|G|$, we may therefore assume that L/\bar{H} sub G/\bar{H} , since $L/\bar{H} \cong H/\bar{H}$ is soluble. Thus L sub G . Since H sub L , we now have H sub G by 6.3 (i).

Problem: let p be a fixed prime. What can be said of those (nil) subgroups H of G which intersect every Sylow p -subgroup of G in a Sylow p -subgroup of H ?

(C) Theorem 11.3 (i) Let H sub G , let P be a Sylow p -subgroup of H and let L be any subgroup of G . Then $\langle P, L \rangle$ contains a Sylow p -subgroup of $\langle H, L \rangle$.

(ii) Let H_1, H_2, \dots, H_n be subnormal subgroups of G and let P be a Sylow p -subgroup of H_1 . Then $Q = \langle P, H_2, \dots, H_n \rangle$ contains a Sylow p -subgroup of $K = \langle H_1, H_2, \dots, H_n \rangle$.

(iii) The mapping $H \rightarrow H \cap P$ (H sub G), where P is a fixed Sylow subgroup of G is a lattice homomorphism in particular, if H_1 and H_2 are subnormal in G , then $\langle H_1, H_2 \rangle \cap P = \langle H_1 \cap P, H_2 \cap P \rangle$.

Proof: (i) is immediate if $H = G$ since then P is a Sylow p -subgroup of G . If $H < G$, then $\bar{H} = \langle H^g; g \in L \rangle < G$ and H^g sub \bar{H} for all $g \in L$ by 6.2. Also P^g is a Sylow p -subgroup of H^g . By induction on $|G|$, we may assume (ii) holds in \bar{H} and so $\bar{P} = \langle P^g; g \in L \rangle$ contains a Sylow p -subgroup of \bar{H} . Also $\langle H, L \rangle = \bar{H}L = \langle \bar{P}, L \rangle$ and by 6.3 (i) $\bar{H}L$ sub $\langle \bar{P}, L \rangle = \langle \bar{H}, P \rangle = \langle H, P \rangle$.

To prove (ii), first suppose $n=2$. Then we have

By the induction on r , all the groups $M = (M^1, M^2, \dots, M^r)$ with M^1, \dots, M^r subnormal in H , and hence in G . This follows by induction on n . But for a suitable choice of the M^i , $M = \bar{H}$. Hence \bar{H} sub G . And $J = \bar{H}K$ and $\bar{H} = J$, K sub G . So by the special case already considered we obtain J sub G as required.

In view of 6.3 (iv), we may now state as a corollary that the subnormal subgroups of any group G form a sublattice of the lattice of all subgroups of G .

(E) By 7.61, every subnormal nilpotent subgroup of G is contained in the Fitting subgroup $F(G)$ of G . Conversely, if H is any subgroup of $F(G)$, then H sub $F(G)$ by 6.18 (ii). Since $F(G)$ char G , it follows that H sub G . Thus the subnormal nilpotent subgroups of G are precisely the subgroups of the Fitting subgroup of G .

Theorem 11.2 (i) Let H sub G and let P be a Sylow p -subgroup of G . Then $H \cap P$ is a Sylow p -subgroup of H .

Proof: (i) is a corollary of 5.6. (ii) Assuming $H \not\leq P$, let M be a minimal normal subgroup of H . Since H is soluble, $|M| = p^k$ for some prime p and every Sylow p -subgroup of H contains M . Hence every Sylow p -subgroup of G contains M and $M \leq \bar{H}$, the maximal normal p -subgroup of G . Let $L = H\bar{H}$ and let Q be a Sylow q -subgroup of L where $q \neq p$. Then Q sub Q , where Q is some Sylow q -subgroup of G and we have $H \cap Q \leq L \cap Q = Q$. By hypothesis, $H \cap Q$ is a Sylow q -subgroup of H . But $\bar{H} \leq L$ and $Q \cap \bar{H} = 1$, since $q \neq p$. Since $L = H\bar{H}$, it follows that $H \cap Q$ is a Sylow q -subgroup of L . Hence $H \cap Q = Q \leq H$. Thus H contains all p -elements of L . Hence from a normal subgroup K of L and L/K is a p -group. Since $K \leq H \leq L$ it follows that

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(D). Let $M = \langle H \rangle$ be the semisimple radical of G and let H be the direct product of H_i for $i=1, \dots, r$.
 Then $M_i = \langle H_i \rangle$ is a direct factor of M by (B) and we may suppose that M_i is the direct product of $M_{i,j}$ and $M_{i,k}$ for each $j=1, \dots, s$. Since H normalizes both M_i and M_{i-1} , it must also normalize $M_{i,j}$. But $[H_i, M_{i,j}] \leq [H_i, M_i] \leq H_{i-1}$ and so $[M_i, M_{i,j}] \leq H_{i-1} \cap M_{i-1} = 1$. If $N = M_i M_{i-1} \dots M_1$, we then have $[N, M_i] = 1$ and $M = \langle N \rangle$ is the direct product of M_i and N . Since M_i is semisimple, it is clear that $N = N_i \cap C_G(H_i)$.

Consider next the case in which $M = \langle H \rangle$ is the intersection of the Sylow p -subgroups of G . Then $M_i = M \cap H_i = C_G(H_i)$ by (D). Let $K = \langle H_i \rangle$ be the subgroup generated by the p -elements of H_i . Now $M = M_p$ and $[M_i, H_i] \leq M_{i-1}$ for each i , as in the semisimple case. Hence $[M_i, K] \leq M_{i-1}$. Suppose that for some $i > 0$, we have $[M_i, K] \leq M_{i-1}$ but $[M_i, K] \not\leq M_{i-1}$. Then we can choose $u \in M_i$ so that $[u, K] \not\leq M_{i-1}$. If $z, y \in K$, we have $[z, y] \in [M_i, K] \leq M_{i-1}$ and so $[z, y] \in M_{i-1}$. Now $[z, y] \in M_i$ by hypothesis, and so $[z, y, i] \in M_{i-1}$. Hence the mapping $\phi: M_i/M_{i-1} \rightarrow [z, y, i] M_{i-1}$ is a homomorphism of K into the p -group M_i/M_{i-1} . But $K = H_i^p$ and so K has no faithful p -group other than K/K . Therefore the kernel of this homomorphism must be K and so $[K, u] \leq M_{i-1}$, which is a contradiction. It follows that $[M_i, K] \leq M_{i-1}$.

Thus we have proved

- Theorem 11.3 Let H be a subnormal subgroup of G . Then
 (1) $\langle H \rangle$ is the direct product of $\langle H_i \rangle$ with $\langle H_i \rangle \cap C_G(H_i)$.
 (2) For every prime p , $[C_G(H), H^p] \leq C_G(H)$.

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$[K, \Omega] = \langle [K, \Omega_1], [K, \Omega_2], \dots, [K, \Omega_r] \rangle$ and both these factors are prime to p .
 By (1) since H_i and H_j are subnormal in G , then let $n > 2$. By induction on n , we may assume that $[K^n, \Omega^n]$ is prime to p , if $n \leq r$. Hence we have $[K^n, \Omega^n] = \langle [K^n, \Omega_1^n], \dots, [K^n, \Omega_r^n] \rangle$. But K^n is prime to p by (1). Hence we have $[K^n, \Omega^n] \leq [K^n, \Omega^n]$ is prime to p by the same argument, since Ω^n contains a Sylow p -subgroup of K^n . Then (3) is proved.

(ii) $(H_1, H_2) \cap (H_2, H_3) \cap \dots \cap (H_{r-1}, H_r) \cap H_1$ is the mapping $H \rightarrow H \cap P$ is homomorphic with respect to intersections. If H_1 and H_2 are subnormal in G , then $H_1, H_2 \cap P$ are respectively Sylow p -subgroups of H_1 and H_2 and so $Q = (H_1, H_2) \cap P$ contains a Sylow p -subgroup of $H_1 \cap H_2$ and so $Q = (H_1, H_2) \cap P$ is a p -group. Hence Q is a Sylow p -subgroup of K . But K is prime to p by (1) and so $K \cap P = 1$ and so a Sylow p -subgroup of K . Since $Q \leq K \cap P$, it follows that $Q = K \cap P$ and so the mapping $H \rightarrow H \cap P$ is also homomorphic with respect to joins.

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§14 Direct Products. Burnside's Transfer Theorem. Z-groups, A-groups, Supersoluble

(A) Let H and K be subgroups of $Z(K)$ and $Z(Y)$ respectively, and let $Z = X \times Y$ be the set of all ordered pairs (x, y) with $x \in X, y \in Y$. For $\xi \in H, \eta \in Y$ and $\gamma \in K$ define the permutations ξ_γ and η_γ of Z by $(x, y)\xi_\gamma = (x\xi, y)$; $(x, y)\eta_\gamma = (x, y\eta)$ if $x \in X$;

The group $H \times K$ generated by all these permutations is called the regular product of H with K . It is isomorphic to identity $\bar{\gamma}$ with γ , so that K becomes a subgroup of $W = H \times K$. For fixed $\gamma \in Y$, the mapping $\xi \rightarrow \xi_\gamma$ ($\xi \in H$) is an isomorphism of H onto a subgroup H_γ of W , and the product

$$\bar{H} = \prod_{\gamma \in Y} H_\gamma$$

is direct. \bar{H} is called the base group of W . K is faithfully represented by the automorphisms of \bar{H} according to the law

$$\eta^\gamma \xi_\gamma \eta^{-\gamma} = \xi_{\eta\gamma} \quad (\eta \in K, \gamma \in Y, \xi \in H),$$

i.e. K permutes the $|Y|$ direct factors of \bar{H} . If this representation of K is f , then

$$H \times K = \langle K, \bar{H} \rangle$$

is the corresponding split extension. $W = \bar{H} \rtimes K$; $\bar{H} \cap K = 1$; $\bar{H} \cong K$.

If $|K| = l, |Y| = n$, then W is a permutation group of degree ln and order $|H|^n |K|$.

If L is a subgroup of $Z(Y)$, since $|Y| = n$ and if we make the natural identifications $(x, y) \mapsto (x, (y, v)) = (x, y, v)$, then the two groups $Z(H \times K) \times L$ and $H \times (K \times L)$ coincide. This is the isomorphism law for wreath products, and permits us to write brackets in a repeated wreath product.

If H or K or both are not given explicitly as permutation groups, then $H \times K$ is to be interpreted as being formed from the appropriate regular representation. We will use this the regularity convention. For example we have

Lemma 14.1 If C is a cyclic group of order p , then the group

$$V^C = \langle C, C, \dots, C \rangle$$

is a Sylow p -subgroup of $Z(C^n)$.

For $\{C^n\} = \{C^n\}$, C^n being the set of all ordered n -tuples of elements of C . Hence $|Z(C^n)|_p = (p^n)^n = p^{n^2}$, $|V^C|_p = |C^n|_p = p^{n^2}$ by induction on n . This result is due to Higman.

Note that $H \times K$ is determined to within isomorphism by the group H and the permutation group K , and this set depend on a group on the permutation representation chosen for H .

(B) Suppose H and K are given groups and $W = H \times K$ (with the regularity convention), then every subgroup K_1 of W which is transverse to the base group \bar{H} in W is conjugate to K in W .

Proof. Here we may identify $\xi \in H$ with $\xi_1 \in H_1$, so that $H = H_1$ and $H_2 = H^1, H_3 = \xi_2^{-1} \xi_1 \xi_2, \dots$ for $\alpha \in K$. \bar{H} is the direct product of the H_i , $\alpha \in K$. Every element $u \in \bar{H}$ is uniquely expressible as a product $\prod_{\alpha \in K} \alpha_i^{\epsilon_\alpha}$, with factors $\alpha_i \in H_i$. Since K_1 is transverse to \bar{H} in W , the word $u \bar{H}$ contains exactly one element $\alpha \in K$ of K and since K_1 is a subgroup of W , we have $u(\alpha) \alpha(p) = \alpha(\alpha) \alpha(p)$ for all $\alpha, \beta \in K$. Factoring, this gives

$$\alpha(\alpha) \beta = \alpha(\alpha) \beta \alpha(\beta) \quad (1)$$

for all $\alpha, \beta \in K$. Let $v \in \bar{H}$ have factors $v_p = \alpha(\beta^1) \alpha(\beta^2) \dots \alpha(\beta^n)$ so that $v = \prod_{\alpha \in K} \alpha(\beta^i)$. The p -factor of v is therefore $v_p \alpha^1 = \alpha(\beta^1) \alpha^1$, and this is precisely the p -factor of $v \alpha^1$. Hence $v \alpha^1 = \alpha(\alpha) v$ for all $\alpha \in K$, and so with $\alpha \in K$, $v \alpha^1 = v \alpha^1$. Then $K = K_1$ is conjugate to K in W as stated.

Proof: Since f is homomorphic, we have for all i and all $\tau, \sigma \in G$,
 $f(\sigma\tau\sigma^{-1}) = f(\sigma)f(\tau)f(\sigma^{-1}) + f(\sigma\tau\sigma^{-1}) = f(\tau) + f(\sigma)$
 and f° is homomorphic. If T is any other transversal to H in G ,
 with elements $\tau_i = \tau_i\sigma_i$, $\tau_i \in H$, then $\tau_i\sigma_i\tau_i^{-1} = \tau_i(\sigma_i\tau_i\sigma_i^{-1})\tau_i^{-1}$
 and $f(\tau_i\sigma_i\tau_i^{-1}) = f(\tau_i) + f(\sigma_i) + f(\tau_i\sigma_i\tau_i^{-1})$. Summing for $i=1, \dots, m$
 we obtain $\sum_{i=1}^m f(\tau_i\sigma_i\tau_i^{-1}) = \sum_{i=1}^m f(\tau_i) + \sum_{i=1}^m f(\sigma_i) + \sum_{i=1}^m f(\tau_i\sigma_i\tau_i^{-1}) = \sum_{i=1}^m f(\tau_i\sigma_i\tau_i^{-1})$.
 Thus f° is independent of the choice of transversal.

The contribution to $f^\circ(S)$ of the cycle of $\tau_i\sigma_i$ which contains $H\sigma_i$
 is $\sum_{\tau \in \tau_i\sigma_i} f(\tau_i\sigma_i\tau_i^{-1}) = f(\tau_i\sigma_i\tau_i^{-1})$ since $\tau_i\sigma_i\tau_i^{-1} = \tau_i$ and f is
 homomorphic. Thus 12.4 is proved.

We call f° the transfer of f to G . This is the terminology
 suggested by elementary theory. However, if f is the natural
 homomorphism $\gamma \rightarrow H/\gamma$ ($\gamma \in H$) of H onto $A = H/H'$, then f° is
 the transfer of G into A . It is a homomorphism of G into H/H'
 and, for $\xi \in G$, we have in this case

$$f^\circ(\xi) = H' \prod_{\tau \in \xi} (\tau\sigma\tau^{-1}) = H' \prod_{\tau \in \xi} (\tau\sigma\tau^{-1}). \quad (1)$$

Note that $\tau\sigma\tau^{-1}$ is the first positive power of $\sigma\tau\sigma^{-1}$ which lies in H .

(2) Theorem 12.5. Let the group G have an Abelian Sylow p -subgroup S
 let $N = N_G(S)$, $C = C_G(S) \cap H$, $D = [H, N]$. Let ξ be the identity
 mapping of H , ξ° the transfer of ξ to G and K the kernel of ξ° .
 Then H is the direct product of C and D . Also $C = \xi^\circ(G) = \xi^\circ(H)$
 and $D = G \cap N = K \cap H$. ξ is the split extension of K by C
 and $G/K \cong C$ is the maximal p -quotient group of G .

Proof: Since $H' \leq S$, we have $C^\circ(G) = \prod_{\tau \in \xi} \tau\sigma\tau^{-1}$ by (1)
 where, for any $\tau \in G$, if $\tau \in H$, then the elements $\tau\sigma\tau^{-1}$ and
 $\sigma\tau\sigma^{-1}$ both lie in H and are conjugate in G . By (1.13), it follows
 that these two elements are conjugate in N , since $H = \xi(H)$ is Abelian.
 Since $\sum_{\tau \in \xi} \tau\sigma\tau^{-1} = m \cdot (G \cap H)$, we may therefore write $\tau\sigma\tau^{-1} = \tau\sigma\tau^{-1}$.

(C) Lemma 12.3. Let H be a subgroup of G and let S be any
 transversal to H in G , with elements $\sigma_1, \sigma_2, \dots, \sigma_m$. Since $m = |G:H|$
 for $\xi \in G$ and $\xi \in S$, ξ is in the coset $H\sigma_i$ for some $\sigma_i \in S$, and
 let X be the set whose elements are the m cosets of H in G , so that
 $\xi \in H\sigma_i$ is the permutation $H\sigma_i \rightarrow H\sigma_j$ in $H\sigma_j$ ($i=1, \dots, m$) of X .
 Define $\eta_H(\xi)$ to be the permutation

$$(\tau, H\sigma_i) \rightarrow (\tau\sigma_i\tau^{-1}, H\sigma_j) \quad (g \in H, 1 \leq i, j \leq m)$$

of $H \times X$. Then η_H is an isomorphism mapping G into $H \times X$ of G .
 Here H is in its regular representation in the regular permutation
 operators for H . Since $(\xi\sigma_i)\tau = \tau(\sigma_i\xi)$ for $\tau \in H$, $\sigma_i \in G$, we have

$\eta_H(\xi\sigma_i) = \eta_H(\xi)\eta_H(\sigma_i)$. $\eta_H(\xi)$ is the identity on $H \times X$ only if
 $\xi\sigma_i = \sigma_i$ for all i and in addition $\tau\sigma_i\tau^{-1} = \tau$, which implies $\tau = 1$.
 Hence η_H is an isomorphism mapping G into X which we identify
 with $\tau\sigma_i$, and $\eta_H(\xi)\tau\sigma_i = (\tau, H\sigma_j)$ which we identify
 with $\tau\sigma_j$, and $\eta_H(\xi)\tau\sigma_i = \tau\sigma_j$ maps $(\tau, H\sigma_i)$ into $(\tau\sigma_j\tau^{-1}, H\sigma_j)$.
 This shows that $\eta_H(\xi)\tau\sigma_i$ belongs to the same group H of X and
 to $\eta_H(\xi) \in X = H \times X$. This proves 12.3.

It is clear that the representation η_H of G depends not only on
 the subgroup H but also on the choice of the transversal S . However,
 preserving the notation of 12.3, we have

Lemma 12.4. Let f be a homomorphism of the subgroup H of
 G into an Abelian group A , which we write in additive notation.
 Then the mapping f° of G into A defined by

$$f^\circ(\xi) = \sum_{\tau \in \xi} f(\tau\sigma\tau^{-1})$$

is also homomorphic and is independent of the choice of the transversal S .
 Moreover

$$f^\circ(\xi) = \sum_{\tau \in \xi} f(\tau\sigma\tau^{-1})$$

where the cosets $H\sigma_i$ are chosen one from each cycle of $\tau\sigma\tau^{-1}$,
 and τ is the order of the cycle to which $H\sigma_i$ belongs, so that
 $\sum_{\tau \in \xi} \tau = |G:H| = m$.

with $\gamma_j \in N$, and so

$$i^*(S) = \sum_{j=0}^m \gamma_j^{-1} \gamma_j \quad (2)$$
 for all $\gamma \in H$, where of course $\gamma_j \gamma_j^{-1}$ is trivial. If $\gamma \in C$, then $\gamma \gamma^{-1}$ commutes with γ_j and so $i^*(S) = \gamma^m$. But C is a p -group and $(m, p) = 1$. Hence $i^*(C) = C$.

If $\gamma \in N$ and $\gamma \in G$, then $\gamma \in \sum_{j=0}^m \gamma_j^{-1} \gamma_j \in H$ and only if $\gamma_j \gamma_j^{-1} \in H$. Hence $\gamma \in S$ is also transposed to H in G , and replacing S by γS does not affect $i^*(S)$ by 12.4. Hence $\gamma i^*(S) \gamma^{-1} = i^*(S)$ for all $\gamma \in G$, $\gamma \in H$. It follows that $i^*(G) = i^*(H) = i^*(C) = C$. Hence $K \cap C = 1$ and G is a split extension of K by C . Since n is prime to $|H|$, every element of H has the form γ^m with $\gamma \in H$ and (2) now shows that $H = CD$. But $D = [H, N] \leq G' \leq K$ since $G/K \cong C$ is Abelian. Since $K \cap C = 1$, this shows that the product $H = CD$ is direct, and that $G' \cap H = K \cap H = D$.

Finally, let $K_1 \leq G$ and let G/K_1 be a p -group. Then $K_1 H = G$ by 5.7 and so $G/K_1 \cong H/K_1 \cap H$ is Abelian. Hence K_1 contains G' and $|G:K_1| \leq |H:G' \cap H| = |C| = |G:K|$. Since $G/K \cap K_1$ is also a p -group, it follows that $K_1 \geq K$ and so G/K is the maximal p -quotient group of G .

Corollary 12.51 If the group G has a Sylow p -subgroup H which is contained in the centre of its normaliser in G , then G has a normal Syl_p -subgroup.

For in this case $C = H$ and G is the split extension of K by H .

Hence K is a normal Syl_p -subgroup of G .

(E) A soluble group whose Sylow subgroups are all Abelian is called an A -group. The following result is due to D.R. Tamek.

Theorem 12.6 Let G be an A -group. Then

- (i) $\exists G \cap G' = 1$; (ii) if S is any Sylowiser of G , then G is the split extension of G' by S ; (iii) more generally, if T is a relative Sylowiser of K in G , where $K \trianglelefteq G$, then $K'T = G$ and $T \cap K' = 1$; (iv) the length of the derived series of G cannot exceed the number of distinct primes which divide $|G|$.

Proof: (i) Suppose if possible that $\exists G \cap G'$ contains an element γ of prime order p , let H be a Sylow p -subgroup of G containing γ . Since H is Abelian, we can use the notation of 12.5. Then $\exists \gamma \in M \cap H = C$ and $\exists \gamma \in G' \cap H = D$. But $C \cap D = 1$ so $\gamma = 1$, a contradiction. Hence $\exists G \cap G' = 1$.

(ii) Let K/L be a chief factor of G with $K \leq G'$. Then G/L is a p -group by 5.6. The derived group in G/L . It follows from (i) that K/L is not a central factor of G . Hence $S \cap K \leq L$ by 10.4. Since this is true for all such K/L , we have $S \cap G' = 1$. By 10.42, $S \cap G'$ and this proves (ii).

(iii) Here K is an A -group by 5.6 and so $T \cap K' = 1$ by (i), while by 10.8 (ii) and (v), $T \cap K = G$. Since $K \leq TK'$ by 10.42, we have $TK' = K$ and this proves (iii).

(iv) Suppose that $|G|$ has just n different prime factors. If $n=1$, G is Abelian and $G' = 1$ by definition. Let $n > 1$ and let M be a minimal normal subgroup of G . If $|M| = p^m$, then every Sylow p -subgroup H contains M . Since H is Abelian, $H \leq C_G(M)$ and hence the autmorphism $G/C_G(M)$ is a p' -group. More generally, if K/L is any chief factor of G and if $|K:L| = p^k$, $C = C_G(K/L)$ is of index prime to p in G . Hence $|G/C|$ has at most $n-1$ different prime factors. By induction on n , we may assume that the $n-1$ -st derived group of G/C is the unit subgroup. Since $(G/C)^{(n-1)} \leq C \leq G^{(n-1)}/C$, this means that $G^{(n-1)} \leq C$. Thus $G^{(n-1)}$ centralises every chief factor of G . Hence $G^{(n-1)}$ has a central series and so is nilpotent. But $G^{(n-1)}$ is also an A -group and so it is the direct product of Abelian groups via its Sylow subgroups. Hence $G^{(n-1)}$ is itself Abelian and $G^{(n-1)} = 1$, as required.

(F) Note in passing the following (forgotten)

Lemma 12.7 (i) In any group G , the Fitting subgroup is the intersection of the centralizers of the chief factors of G .

(ii) In any soluble group G , the hypercenter is the intersection of the centralizers of G .

Proof: (i) Let $F = F(G)$, and let $C = \cap C_G(H/K)$ over all chief factors H/K of G . By 9.3 (i), we have $F \leq R(G \text{ mod } K) \leq C_G(H/K)$ and so $F \leq C$. On the other hand, C has a central series viz., the part of any chief series of G from 1 to C . Hence C is nilpotent, and so $C \leq F$ by 7.6i, since $C \leq G$. Thus $C = F$.

(ii) Let $H = Z^*G$ and let K be the intersection of the centralizers of G . If L/M is any chief factor of G with $L \leq K$, then $H \leq L \leq MS$ for any systemizer S and so, by 10.4, L/M is a central factor of G . Hence $[L, G] \leq M$ and so $K \leq H$. Conversely, if L/M is any chief factor of G then L/M is a central factor of G and so $L \leq H$. This is true for all such L/M and so $H \leq S$. This is true for all systemizers S of G and so $H \leq K$. Hence $H = K$.

(G) Let C_n denote a cyclic group of order n and let $\Phi_n = \text{Aut } C_n$. If $C_n = \langle y \rangle$ and $x \in \Phi_n$, then $C_n = \langle y^x \rangle$ and so $y^x = y^a$ where $(a, n) = 1$, by 2.5. Conversely, if $(a, n) = 1$, then $y^2 \rightarrow y^{2a}$ ($s = 0, 1, \dots, n-1$) is an automorphism of C_n . If $\beta \in \Phi_n$ and $y^\beta = y^b$, then $y^{a\beta} = y^{ab} = y^{\beta a}$. Hence Φ_n is Abelian. The integer a is determined by the automorphism of only modulo n . Thus $\Phi_n \cong$ isomorphic with the multiplicative group of residue classes mod n , i.e. of all those residue classes (a) and n for which $(a, n) = 1$.

The number $\varphi(n) = |\Phi_n|$ is called Euler's function. If $(a, n) = 1$, then Φ_n is the direct product of subgroups isomorphic with Φ_m and Φ_k respectively by 8.2. Thus, if $(m, n) = 1$,

$$\Phi_{mn} \cong \Phi_m \times \Phi_n \quad \text{and} \quad \varphi(mn) = \varphi(m)\varphi(n)$$

Now we merely need to record

Lemma 12.8 (i) The group of automorphisms of a cyclic group of order n is the an Abelian group of order $\varphi(n)$.

(ii) If p is a prime, $\varphi(p^m) = p^{m-1}(p-1)$.

(iii) Let H be a Sylow p -subgroup of G , where p is the smallest prime dividing $|G|$. If H is cyclic, then G has a normal S_p -subgroup.

Proof: (i) has already been shown and (ii) is clear since C_{p^m} has only one maximal subgroup viz. $\langle y^p \rangle$ of order p^{m-1} .

(iii) Since H is Abelian, we have $H \leq C_G(H) \leq N_G(H) = H$ and so the automizer $A_G(H) \cong N_G(H)/C_G(H)$ has order prime to p . Any prime q which divides $|A_G(H)|$ is therefore greater than p , by definition of p . But $q \leq |G|$ by (ii), so no such prime exists. Hence $H \leq ZN$ and the result follows from 12.51.

(H) A group G is called a Z -group if all its Sylow subgroups are cyclic. A group G is called supersoluble if all its chief factors are cyclic.

Theorem 12.9. (i) Z -groups are supersoluble.

(ii) If G is supersoluble, then G' is nilpotent; every maximal subgroup of G has index a prime; and if \mathfrak{G} consists of all primes $\geq p$, then G has a normal $S_{\mathfrak{G}}$ -subgroup. Here p is any prime.

(iii) If G is a Z -group, then G' is cyclic and so are the systemizers S of G' ; moreover $|G'|$ and $|S|$ are coprime and G is the split extension of G' by S . Further, for each divisor d of $|G|$, G contains one and only one class of conjugate subgroups of order d .

Proof: (i) Let G be a Z -group. By 12.8, G has a normal S_p -subgroup G_1 where p is the smallest prime dividing $|G|$. By induction on $|G|$, we may assume G_1 soluble. G/G_1 is a p -group and therefore soluble. Hence G is soluble. By 5.6 all sections of Z -groups are Z -groups. In particular, a chief factor of G must be cyclic of prime order, since it is an Abelian prime-power group by the solubility

(i) Let G be supersoluble. By 12.7 (i), $F = \mathcal{N}G = \cap C_G(H/K)$ over all chief factors H/K of G . By definition, each H/K is cyclic and so, by 12.8 (i), $G/C_G(H/K)$ is Abelian. All these centralizers contain G' by 7. Hence $G' \leq F$ and so G' is nilpotent.

By 9.2, if M is any maximal subgroup of a soluble group G , then $|G:M| = |H:K|$ for some chief factor H/K of G . If G is supersoluble, $|H:K|$ is a prime and so every maximal subgroup of G is of index a prime. Let $G = G_0 > G_1 > \dots > G_n = 1$ be a chief series of G . Since G is supersoluble, each $|G_{i-1}:G_i| = p_i$ is a prime. Choose a chief series for which $\sum_{i=1}^n p_i = S$ is a maximum and suppose if possible that $p_2 > p_{1n}$ for some i . A Sylow p_i -subgroup G_i^*/G_{i+1} of G_{i-1}/G_{i+1} is normal in G_{i-1}/G_{i+1} by 5.4 (v). G_i^*/G_{i+1} is therefore characteristic in G_{i-1}/G_{i+1} so $G_i^* \triangleleft G$ and we may obtain a new chief series with larger p_i by replacing $G_i^* \triangleleft G$. This contradicts our choice of chief series. Hence $p_i \leq p_{1n}$ for all $i=1,2,\dots,n-1$ and one of the terms G_i will be an S -subgroup of G with the given set of primes π .

(ii) Let G be a Z -group. By (i) and (b), G' is nilpotent. G' is abelian, hence G' is cyclic by 8.2 (i) and (ii). Since G is an A -group we have $G = G'S$, $G' \cap S = 1$, by 12.6 (b); and so $S \cong G/G'$ is Abelian. Since S is also a Z -group, S is cyclic by 8.2 again.

Let \mathcal{Q} be the greatest prime dividing $|G|$. By (i) and (b) the Sylow \mathcal{Q} -subgroup of G is normal in G . It is also cyclic. Hence G has one and only one subgroup \mathcal{Q} of order \mathcal{Q} . If \mathcal{Q} divides d , every subgroup H of G of order d must contain \mathcal{Q} . By induction on $|G|$, we may assume that the Z -group G/\mathcal{Q} has one and only one class of conjugate subgroups of order d/\mathcal{Q} . The result for G now follows.

If \mathcal{Q} does not divide d , and if H is any subgroup of G of order d then $\mathcal{Q} \cap H = 1$ and $\mathcal{Q}H/\mathcal{Q}$ is a subgroup of order d of G/\mathcal{Q} , and so H is an $S_{\mathcal{Q}}$ -subgroup of $\mathcal{Q}H$. By induction, we may assume that

subgroups G/\mathcal{Q} has a single class of conjugate subgroups K/\mathcal{Q} of order d/\mathcal{Q} . By 9.5, each such K has a single class of conjugate subgroups H of order d and $K = H\mathcal{Q}$. Hence in this case also the result follows for G .

This completes the proof of 12.9. Perhaps we should add Lemma 12.91. Let A be a maximal normal Abelian subgroup of the supersoluble group G . Then $A = C_G(A)$.

Proof: Let $C = C_G(A)$. Since A is Abelian, we have $A \leq C$. Since $A \leq G$ we have $C \leq G$. Hence if $A < C$, there is a chief factor B/A of G such that $B \leq C$. Since G is supersoluble, B/A is cyclic of order a prime. Since $B \leq C_G(A)$, we have $A \leq \langle B \rangle$, and so $B/\langle B \rangle$ is cyclic. Hence $B = \langle B \rangle$ is Abelian by 7.2 (ii), contrary to the definition of A . This contradiction shows that $A = C$.

Note that nilpotent groups are supersoluble. Hence 12.91 applies. Then G is a p -group.

Corollary 12.92. Let H be a Sylow p -subgroup of any group G and let A be a maximal normal Abelian subgroup of H . Then A is a Sylow p -subgroup of $C = C_G(A)$.

Proof: Since $A \triangleleft H$, we have $H \leq N = N_G(A)$. H is therefore a Sylow p -subgroup of N . Since $C \triangleleft N$, it follows that $H \cap C$ is a Sylow p -subgroup of C by 5.6. But $H \cap C = A$ by 12.91.

Since $(y)^\xi = y\xi + w(y)\xi = y^\xi + w(y)\xi = y^\xi + w(y^\xi) = t(y^\xi)$ by (3), we see that T is Γ -invariant. This gives

Theorem 13.1 Let the p -group Γ be represented by automorphisms of the additive Abelian p -group V . If Γ leaves invariant a direct summand X of V , then there is a Γ -invariant subgroup T such that $V = X \oplus T$.

Note that the number of subgroups Y of V such that $V = X \oplus Y$ is equal to $|Hom(V/X, X)|$, which is a power of p . Hence 13.1 is immediate in the case $|T| = \xi^2, \xi$ prime.

V is called Γ -indecomposable if it cannot be expressed as the direct sum of two proper Γ -invariant subgroups. Obviously, we can always express V as a direct sum $V_1 \oplus V_2 \oplus \dots \oplus V_r$ ($r \geq 0$) of Γ -invariant subgroups $V_i \neq 0$ each of which is Γ -indecomposable. The V_i are Γ -indecomposable components of V .

An Abelian p -group is called homocyclic if all its invariants are equal. Lemma Corollary 13.11. The Γ -indecomposable components of V are all homocyclic.

Proof: We may assume that V is Γ -indecomposable. Suppose if possible that V is not homocyclic and let ℓ be the largest invariant of V . Then $\ell > 1$ and if $W = \Omega_1 V$ and $X = W \cap \Omega_{\ell-1} V$, we have $0 < X < W$. Since W is elementary, X is a direct summand of W . Since X and W are characteristic in V , they are Γ -invariant. Hence there is a Fitting subgroup T_ℓ such that $W = X \oplus T_\ell$ by 13.1. Since $X \cap T_\ell = 0$, the largest invariant of $V_1 = V/T_\ell$ is still ℓ . If V_1 is not homocyclic, we can proceed similarly to the existence of a Γ -invariant $T_2 > T_1$ such that $T_2 \cap X + T_1 = T_1$ and so $T_2 \cap X = 0$, and the largest invariant of $V_2 = V/T_2$ is therefore still ℓ . Continuing in this way, we eventually find a Γ -invariant subgroup T of V such that $X \cap T = 0$ and V/T is homocyclic. If $|X| = p^r$, this implies that V/T is of type (ξ, ξ, \dots, ξ) . Here r is the multiplicity of ℓ as an invariant of V , and V contains subgroups $U \cong V/T$. For any such U , $\Omega_1(U) = \Omega_1(V) = X$ and so $V = T \oplus$

13. Automorphisms of p -groups.

(A) Let $V = X \oplus Y$ be an additive Abelian p -group and let Γ be a p -group operating on V and leaving the direct summand X invariant; so $v(x) = (v\xi)\xi$ for $v \in V$, ξ and η in Γ ; $(v + w)\xi = v\xi + w\xi$ for $v, w \in V$, $\xi \in \Gamma$; and $X\Gamma = X$.

Every $v \in V$ is uniquely expressible in the form $x + y$ with $x \in X, y \in Y$. For $y \in Y, \xi \in \Gamma$ we can therefore define $y\xi \in Y$ and $y\xi \in X$ by the equation

$$y\xi = y^\xi + y_1 \tag{1}$$

Expressing that $y(\xi\eta) = (y\xi)\eta$, we then obtain

$$y^{\xi\eta} = (y^\xi)^\eta \quad \text{and} \quad y_1\eta = y^\xi\eta + y_2\eta$$

for all $\xi, \eta \in \Gamma$.

Let $|T| = q$. Since $(y, p) = 1$, we can choose an integer k such that $qk \equiv 1 \pmod{|V|}$ and so $g^{qk} = v$ for all $v \in V$.

Define

$$\omega(y) = k \sum_{i=0}^{q-1} y^i g^{i-1} \tag{2}$$

so that $y \rightarrow \omega(y)$ is a mapping of Y into X . Then for $\xi \in \Gamma$,

$\omega(y^\xi) = k \sum_{i=0}^{q-1} y^{\xi i} g^{i-1} = k \sum_{i=0}^{q-1} y^i g^{i-1}$ by the first equation of (1) and so, by the second equation of (1), we have

$$\omega(y^\xi) = k \sum_{i=0}^{q-1} (y_2 - y_1 g^i) g^{i-1} = y_2 - (k \sum_{i=0}^{q-1} y_1 g^i) g^{q-1}$$

But $y_1 g^q = y_1$ and so $y_2 - y_1 = 0$ if 1 is the unit element of Γ . So (1) for $y_1 g^i = -y_1 g^{i-1}$ and we obtain

$$\omega(y^\xi) = y_2 + \omega(y) \xi \tag{3}$$

For y_1, y_2 in Y we have $(y_1 + y_2)\xi = y_1\xi + y_2\xi$ and hence

$$(y_1 + y_2)^\xi = y_1^\xi + y_2^\xi = (y_1)_\xi + (y_2)_\xi \quad \text{and so} \tag{4}$$

$$\omega(y_1 + y_2) = \omega(y_1) + \omega(y_2)$$

So $y \rightarrow \omega(y)$ is a homomorphism of Y into X and the set T of all $t(y) = y + \omega(y)$ ($y \in Y$)

It follows from 10.1 that U may be chosen to be Γ -invariant. This contradicts the Γ -incompatibility of V and we conclude that V must in fact be Γ -invariant.

(B) Theorem 13.2 Let the p -group \mathcal{B} be represented by automorphisms of the p -group G , let $H = \langle G, \Gamma \rangle$ and let $C = C_G(\Gamma)$. Then (i) $G = HC$; (ii) $[H, \Gamma] = H$ and (iii) if $H \leq \mathcal{B}$, then $H = 1$.

Here C is the subgroup of all $z \in G$ which are left invariant by every $\gamma \in \Gamma$. We can form the split extension $G\Gamma$ of G by Γ determined by the given representation of Γ . By 7.1(2), $H \leq G\Gamma$.

Proof. (ii) Let $\mathcal{B} = \langle \mathcal{B} \rangle$. By 5.2(ii), G/\mathcal{B} is an elementary Abelian p -group. Let $\mathcal{B} = \langle \mathcal{B}_1, \dots, \mathcal{B}_r \rangle$ be a basis of G/\mathcal{B} . By 9.1(i), $G = \langle \mathcal{B}_1, \dots, \mathcal{B}_r \rangle$. Let A_i be the group of all automorphisms $\alpha \in \text{Aut } G$ such that $[G, \alpha] \leq \mathcal{B}_i$. Then $A_i \leq A = \text{Aut } G$ and A/A_i is isomorphic with a subgroup of $\text{Aut } G/\mathcal{B}_i$, which has order $(p^{r-1})(p-1)$ by 8.81. If $\alpha \in A_i$, then $[\mathcal{B}_i, \alpha] \in \mathcal{B}$ for each i ; and since the \mathcal{B}_i generate G , α is uniquely determined by the r elements $[\mathcal{B}_i, \alpha]$. More generally, if $\mathcal{B}_i, \mathcal{B}_j \in \mathcal{B}$ for each i, j , then $G = \langle \mathcal{B}_1, \dots, \mathcal{B}_r \rangle$ and so no element $\alpha \neq 1$ in A_i leaves any such ordered set $\langle \mathcal{B}_1, \dots, \mathcal{B}_r \rangle$ invariant. The total number of such ordered sets is $|\mathcal{B}|^r$. Hence $|A_i|$ divides $|\mathcal{B}|^r$ and thus A_i is a p -group. If $H \leq \mathcal{B}$, the automorphisms of G which represent Γ all belong to A_i , and since Γ is by hypothesis a p -group, it follows that $H = 1$ and $C = G$. Thus (ii) is proved.

The principle used here may be worth stating as characteristic Lemma 13.2.1 Let G be any group and K any normal subgroup of G contained in $\mathcal{B}(G)$. Let $A = \text{Aut } G$ and let A_i consist of all $\alpha \in A$ such that $[G, \alpha] \leq K$, so that $A_i \leq A$ and A/A_i is isomorphic with the subgroup of $\text{Aut } G/K$ induced by elements of A . Then $|A_i|$ divides $|K|^r$ where r is the minimum number of generators of G .

Note that r is also the minimum number of generators of G/K and equally of $G/\mathcal{B}(G)$.

(i) Suppose first that $H \leq \mathcal{B}(G)$. Then the mapping $\mathcal{B} \rightarrow [K, \mathcal{B}]$, $(\mathcal{B} \in G)$, is homomorphic by 7.1(i) for each $\gamma \in \Gamma$. Since $[G, \mathcal{B}] \leq H$, which is Abelian, the kernel of this homomorphism contains G' , by 7.1(ii). This is true for each $\gamma \in \Gamma$. Hence $G'/G' \leq C$. G/G' is a normal Abelian Sylow p -subgroup of $G\Gamma/G'$ and so, by 11.8, it is the direct product of \mathcal{B}/G' and \mathcal{B}/G' where $\mathcal{B} = H/G'$ and \mathcal{B} consists of all $z \in G$ such that $[K, z] \leq G'$. Hence $C \leq \mathcal{B}$. If $\mathcal{B} = G'$, we have $H = 1$ by (ii) and $C = G$ and the result follows. If $\mathcal{B} > G'$, then $\mathcal{B} < G$ and we may assume by induction on (i) that $\mathcal{B} = [C, \Gamma]C$. But $[C, \Gamma] \leq \mathcal{B} \cap \mathcal{B} = G' \leq C$ and so $\mathcal{B} = C$. Since $H \leq G$, CH is a subgroup of G and we have $CHG' = \mathcal{B}H = G$. Hence $CH = G'$ by 9.1(i) and 5.2(ii).

If $H \not\leq \mathcal{B}(G)$, let $K = H \cap \mathcal{B}(G)$. Then $1 < K < G\Gamma$. Let C_i consist of all $\gamma \in G$ such that $[K, \gamma] \leq K$. Then $C \leq C_i$ and by induction we may assume that $H C_i = G$. Since $H \not\leq \mathcal{B}(G)$, we have $C_i < G$. By induction again we may assume that $[C_i, \Gamma]C = C_i$. This gives $HC = H[C_i, \Gamma]C = H C_i$ since $[C_i, \Gamma] \leq H$. Thus (i) is proved.

(ii) Let $H_i = [H, \Gamma]$. By (i), $H \leq H_i C$ and so $G = H_i C$. By 9.1(ii) we have $G = CH_i$ and every element of G has the form γz , with $z \in C$, $\gamma \in H_i$. Therefore H is generated by the elements $[K, \gamma]$ with $\gamma \in H_i$ and $z \in \Gamma$. But H and H_i are normalised by Γ , so $[K, \gamma] \in H_i$ and $H_i \leq H$. Hence $H_i = H$.

Suppose first that D/C is a p -group. Then $|D/C| = p$ by hypothesis and the $C_M(D) \neq 1$ by 5.1. Since $C_M(D) \leq G$, it follows that $[M, D] = 1$. Let $D = \langle C, \xi \rangle$ and $L = \langle M, \eta \rangle$. Since $[L, C] = 1$, $[L, D]$ is generated by $\eta = [\xi, \eta]$ and its conjugates in D . But $[L, D] \leq M$ by 5.1 and so $[L, D] = 1$. Hence $[L, D] = \{1\}$ is cyclic. Since $[L, D] \leq G$, this contradicts the hypothesis that M is a minimal normal subgroup of G .

We conclude that D/C must be a p' -group Γ . This is represented by automorphisms of the elementary Abelian p -group L . By 13.1, we have L as the direct product $L_0 L_1 \dots L_r$ of a certain number of minimal normal subgroups L_i of D , where we may assume that $M = L_1 L_2 \dots L_r$, so that $|L_0| = p$. Joining together those L_i which are F -isomorphic to a given one, we obtain L as the direct product $W_0 W_1 \dots W_s$, where each W_i is a W -elementary component of L with respect to Γ . Let W_0 contain L_0 . Then $W_1 W_2 \dots W_s \leq M$. Since $D \triangleleft G$, the mapping $W_i \rightarrow W_i^g$ is a permutation of W_0, \dots, W_s , as in Lemma 8.9. Since $M \triangleleft G$, it follows that $W_0 \triangleleft G$ and since M is minimal normal, we must have $W_0 \cap M = 1$ and so $W_0 = L_0$. This contradicts the hypothesis that M is the only minimal normal subgroup of G . So this case also is impossible and we conclude that L/M has to be a p' -group.

This concludes the proof of 13.3.

Theorem 13.4 \textcircled{A} Let M be a minimal subgroup of G . Then
 (1) If $|G/M|$ is either a prime or the square of a prime for all M , then G is soluble.
 (2) If $|G/M|$ is a prime for all M , then G is supersoluble.

Note that (2) ~~is a consequence of (1)~~ is the converse of part of 12.9(1). It allows us to characterize supersoluble groups as those groups in which every maximal subgroup is of prime index. This result is due to B. Huppert.

Proof: (1). Let p be the largest prime divisor of $|G|$, let Z be a Sylow p -subgroup of G . If $Z \triangleleft G$, we may assume that G/Z is soluble.

Lemma 13.3 Let G be a p -soluble group, let M be a minimal normal subgroup of G and suppose that $|M| = p^n$ with $n > 1$, but that every chief p -factor of G/M has order p . Then G is the split extension of M by a minimal subgroup H of index p^n in G .
 Proof: If $K \triangleleft G$ and $K \cap M = 1$, then G/K satisfies the conditions prescribed for G . If H/K is a subgroup transversal to KM/K in G/K , then $H \cap M = 1$ and $HM = G$. By induction on $|G|$, we may therefore assume that M is the only minimal normal subgroup of G .

Let L/M be a chief factor of G , and suppose first that L/M is a p' -group. By 7.4, L has a single class of conjugate Sp -subgroups H and these are invariant in L . If $N = N_G(M)$, it follows that $NL = G$ by 10.8(iii). Since $HM = L$, we have $NM = G$. Since M is Abelian and normal in G , $N \cap M$ is normal in $NM = G$. Since M is a minimal normal subgroup of G , either $N \cap M = 1$ or $M \leq N$. In the second case, $H \triangleleft L$ and, as a normal Sp -subgroup of L , H is even characteristic in L . This would imply $H \triangleleft G$ contrary to the assumption that M is the only minimal normal subgroup of G . Hence $N \cap M = 1$ and the theorem follows.

Since G is p -soluble, L/M if not a p' -group must be a p -group and therefore, by the assumption about G , we have $|L/M| = p$. Then L is a p -group and ~~split extension~~ $Z \cap M \neq 1$ by 5.2(1). Since $Z \cap M \triangleleft G$ it follows that $M \leq Z$. But L/M is cyclic and hence $L = Z$ is Abelian by 7.2(2). Since M is elementary of order p^n , L must be elementary of order p^{n+1} ; for the only alternative is for L to be of type (p^{n-2}) and since $n > 1$ this would imply $M = D_2(L) > U_2(L) > 1$, $U_2(L) \triangleleft G$, contrary to the hypothesis that M is a minimal normal subgroup of G .

Now let $C = C_G(L)$. Then $C \leq G$. Let D/C be a chief factor of G .

by induction on $|G|$. Hence G is soluble, since P is a p -group and therefore certainly soluble. If $M \cap N_G(P) < G$, let M be a maximal subgroup of G containing N . Then $|G:M| = 2$ or 3 for some prime $2 < p$, by hypothesis. But $|G:M| \equiv 1 \pmod{p}$ and p by 5.4.3. Since $(p, 2-1) = 1$, we must have $|G:M| = p$ divides $q+1$. Hence $p=3$, $q=2$ and G is soluble by 7.8, accordingly.

(ii) By (i), G is soluble. Let $G = G_0 > G_1 > \dots > G_m = 1$ be a chief series of G . If G is not supersoluble, let i be the least integer for which $|G_{i-1} : G_i|$ is not a prime. Then G_{i-1}/G_i is an elementary Abelian p -group of order p^m , $m > 1$, for some prime p ; and the group G/G_i satisfies the hypothesis of 13.5 with $G_{i-1}/G_i = M$. Hence there is a maximal subgroup H of G such that $H \cap G_{i-1} = G_i$ and $H/G_{i-1} = G$. Then $|G:H| = p^m$, contrary to the assumption that all maximal subgroups of G are of prime index. We conclude that G is supersoluble.

(D) Theorem 13.5 If every maximal subgroup of G is nilpotent, then G is soluble.

This theorem is due to Otto Schreier and was proved later independently by Lüneburg.

Proof: Suppose there are two distinct maximal subgroups M_1 and M_2 of G such that $D = M_1 \cap M_2 \neq 1$. Choose M_1 and M_2 so that D is as large as possible, and let $N = N_G(D)$. By 6.8 (iii), $D < M_1 \cap N = N_G(D)$ and similarly $D < M_2 \cap N$. If $N < G$, let M be a maximal subgroup of G containing N . By the maximality of D , we should then have $M_1, M_2 = M$, contrary to $M_1 \neq M_2$. Hence necessarily $N = G$ and $D = G$. By induction on $|G|$ we may assume that G/D is soluble. Since D is nilpotent, it follows that G is soluble.

We may now assume that $M_1 \cap M_2 = 1$ for every pair of maximal subgroups M_1 and M_2 of G . If $M_1 \trianglelefteq G$, we obtain the solubility of G by induction on $|G|$, since $M_1 = 1$ implies that G is cyclic of order a prime and in any case M_1 is supposed to be nilpotent. We may therefore suppose

that no maximal subgroup M_i of G is normal in G . Let $|M_1| = m_1$ and $|G:M_1| = m_2$. Then M_1 has m_1 conjugates in G and together they contain exactly $1 + m_1(m_2 - 1)$ elements of G . This number is less than $|G| = m_1 m_2$, and so G must have a maximal subgroup M_2 of order m_2 and index m_1 in G , which is not conjugate to M_1 in G . Since M_2 and its conjugates contain besides the unit-element exactly $m_2(m_2 - 1)$ further elements of G , we obtain $|G| = m_1 m_2 \geq 1 + m_1(m_2 - 1) + m_2(m_2 - 1)$ and so $m_2 - 1 \geq m_1(m_2 - 1) \geq m_1$. Similarly, $m_1 - 1 \geq m_2$, which is a contradiction. Thus 13.5 is proved.

(E). Suppose that G is not nilpotent but that all proper subgroups of G are nilpotent. By 13.5, G is soluble. Hence G has a maximal normal subgroup M of index a prime p and so $G = \langle M, x \rangle$ where x is of order p for some x and $x^p \in M$. Since M is nilpotent but not G , we have $M = \Omega_1(G)$ the Fitting subgroup of G . For some $z \neq 1$, the Sylow z -subgroup Q of M must contain an element q which does not commute with x ; for otherwise G would be nilpotent. Then $G = \langle x, q \rangle$ since $\langle x, q \rangle$ is not nilpotent, and so $Q = \langle q \rangle$ and M is the direct product of Q with $\langle x \rangle$. Thus G is the split extension of Q by the cyclic p -group $\langle x \rangle$. Moreover every proper subgroup of Q which is normalized by x must be centralized by x , for otherwise $\langle Q, x \rangle = G$ would be a proper subgroup of G but not nilpotent.

We consider a slightly more general situation in Theorem 13.6B. Let the q -group Γ be represented by automorphisms of the z -group Q and suppose that there is an element $y \in \Gamma$ which centralizes every proper subgroup of Q which is normal in the split extension $Q\Gamma$ but which does not centralize Q . Then $Q' \leq \langle y \rangle Q$ and Q/Q' is a chief factor of $Q\Gamma$. Moreover Filler Q is elementary; or Q is of class 2 with $Q/Q' = q(Q)$ and Q' both elementary and $Q/Q' \cong Q$.

Proof: Since $Q' \leq q(Q)$ and y has order prime to z and hence centralizes Q , it follows from 13.21 that y does not centralize Q/Q' .

(F) Let Γ be any group, let X be a cyclic group of order p and let $G = X \rtimes \Gamma$. If $|\Gamma| = n$, the base group V of G is elementary Abelian of order p^n , with a basis x_i ($i \in \Gamma$) such that $x_i^p = x_{ip}$ for all $i, p \in \Gamma$. Thus Γ is represented faithfully by automorphisms of V and this is called the regular representation of Γ mod p .

Let $1 = V_0 < V_1 < \dots < V_n = V$ be part of a chief series of V . Then Γ is represented, with kernel Γ_i say, by automorphisms of V_i/V_{i-1} . If $\Delta = \bigcup_{i=1}^n \Gamma_i$, we have $[V, \Delta] \leq V_1$ for all i and so Δ is a p -group by 7.9 (i).

A group Γ is called noncyclic if it has only one minimal normal subgroup M . This implies $\Gamma \neq 1$.

Theorem 13.7 (i) Let Γ be a noncyclic group such that $\mathcal{O}_p \Gamma = 1$. Then Γ has a faithful irreducible representation of mod p .

(ii) If p_1, p_2, \dots, p_n are primes such that $p_i \nmid p_j$ ($i=2, \dots, n$), then there exist groups G_i with one and only one chief series $G_i = G_{i0} > G_{i1} > \dots > G_{i\alpha_i} = 1$

and such that G_{i-1}/G_i is an elementary Abelian p_i -group for each $i=1, \dots, n$.

Proof: (i) Since $\mathcal{O}_p \Gamma = 1$, we have $\Delta = 1$; and this implies that $\Gamma_i = 1$ for some i , since Γ is noncyclic. Hence Γ is represented faithfully and irreducibly by automorphisms of V_i/V_{i-1} .

(ii) When $n=1$, we can take G to be cyclic of order p . Let $n > 1$. By induction we may assume the existence of a group Γ with only one chief series $\Gamma = \Gamma_0 > \Gamma_1 > \dots > \Gamma_{n-1} = 1$ and such that Γ_{i-1}/Γ_i is a p_i -group for $i=1, \dots, n-1$. Then Γ is noncyclic. The unique minimal normal subgroup Γ_{n-2} is a p_{n-1} -group. Since $p_{n-1} \nmid p_n$, we have $\mathcal{O}_{p_n} \Gamma = 1$.

By (i), Γ has a faithful irreducible representation f by automorphisms of an elementary Abelian p_n -group G_{n-1} . Let $G = \langle \Gamma, G_{n-1} \rangle$ be the corresponding split extension. Since f is irreducible, G_{n-1} is a minimal normal subgroup of G ; it is the only one by 6.9, since f is faithful. Hence G is noncyclic. Since $G/\mathcal{O}_{p_n} G \cong \Gamma$, G has only one chief series, with term $G_{n-2} = 1$, $G_{n-1} = G_{n-1}$, Γ_{n-1} ($k=1, 2, \dots, n$) and $G_{n-1}/G_i \cong \Gamma_{i-1}/\Gamma_i$ is a p_i -group, as required.

Hence $\mathcal{O}_p \mathcal{Q}'$ is Γ -incomparable; for otherwise, there would be a Γ -invariant subgroup \mathcal{Q}_1 containing \mathcal{Q}' and such that $[\mathcal{Q}_1, \mathcal{Q}'] \neq \mathcal{Q}'$, contrary to hypothesis. By 12.5, $\mathcal{Q}_1/\mathcal{Q}'$ is the direct product of $\mathcal{C}_p(\mathcal{Q}_1/\mathcal{Q}')$ with another group. Since \mathcal{Q}' does not centralise $\mathcal{Q}_1/\mathcal{Q}'$, it follows that \mathcal{Q}' does not centralise $\mathcal{C}_p(\mathcal{Q}_1/\mathcal{Q}')$ either. But $\mathcal{C}_p(\mathcal{Q}_1/\mathcal{Q}') = \mathcal{C}_p(\mathcal{Q}_1/\mathcal{Q}')$ with \mathcal{Q}_2 characteristic in \mathcal{Q}_1 . Hence $\mathcal{Q}_1/\mathcal{Q}'$ is elementary. Since it is Γ -incomparable, it is therefore a chief factor of \mathcal{Q}' .

Let $\Gamma_0 = \langle \mathcal{Q}' \rangle$. Then $\Gamma_0 \triangleleft \Gamma$, and Γ_0 centralises \mathcal{Q}' because \mathcal{Q}' does so. $[\mathcal{Q}_1, \Gamma_0] \triangleleft \mathcal{Q}'$ and $[\mathcal{Q}_1, \Gamma_0]$ is not contained in \mathcal{Q}' since \mathcal{Q}' does not centralise $\mathcal{Q}_1/\mathcal{Q}'$. Since the latter is a chief factor of \mathcal{Q}' , it follows that $\mathcal{Q}'[\mathcal{Q}_1, \Gamma_0] = \mathcal{Q}'$ and hence $[\mathcal{Q}_1, \Gamma_0] = \mathcal{Q}'$ by 9.1 (i). Since $[\mathcal{Q}', \Gamma_0] = 1$, both $[\mathcal{Q}', \mathcal{Q}_1, \Gamma_0]$ and $[\Gamma_0, \mathcal{Q}', \mathcal{Q}_1]$ are equal to 1. Hence $[\mathcal{Q}', \mathcal{Q}_1] = [\mathcal{Q}_1, \Gamma_0, \mathcal{Q}'] = 1$ by 7.7 (ii). Thus $\mathcal{Q}' \leq \mathcal{Z} \mathcal{Q}$.

If \mathcal{Q} is not elementary, we have $\mathcal{Q}' \neq 1$ and so $\mathcal{Q}' \leq \mathcal{Z} \mathcal{Q} < \mathcal{Q}$. But $\mathcal{Z} \mathcal{Q} \triangleleft \mathcal{O}_p \Gamma$ and $\mathcal{O}_p \mathcal{Q}'$ is a chief factor of $\mathcal{O}_p \mathcal{Q}'$, so in this case $\mathcal{Q}' = \mathcal{Z} \mathcal{Q}$. Let \mathcal{Y}, \mathcal{Z} be in \mathcal{Q} and let $\mathcal{Y} = [\mathcal{Z}, \mathcal{Y}]$. Then $\mathcal{Y} \in \mathcal{Z} \mathcal{Q}$ and $\mathcal{Y}^2 \in \mathcal{Z} \mathcal{Q}$ and so $\mathcal{Y}^2 \in \mathcal{Z} \mathcal{Q}$. But $\mathcal{Y}^2 \in \mathcal{Q}' = \mathcal{Z} \mathcal{Q}$ and so \mathcal{Y}^2 commutes with \mathcal{Y} . Hence $\mathcal{Y}^2 = 1$. The Abelian group \mathcal{Q}' is generated by elements \mathcal{Y} of order 2. Thus \mathcal{Q}' is elementary and 13.8a is proved.

Lemma 13.8E Let \mathcal{Q} be a 2-group such that $\mathcal{O}_2 \mathcal{Q} = \mathcal{Z} \mathcal{Q}$ is of order q a prime. Then \mathcal{Q} is the central product of a certain number t of non-Abelian groups of order q^3 . For each prime l and each integer $r=1, 2, \dots$ there are exactly two non-isomorphic groups \mathcal{Q} of this kind and $\mathcal{O}_2 \mathcal{Q}$ is elementary of order q^2 .

Proof: Let \mathcal{S}, \mathcal{Y} be in \mathcal{Q} and let $[\mathcal{S}, \mathcal{Y}] = \mathcal{Z}$. By hypothesis $\mathcal{Z} = \mathcal{O}_2 \mathcal{Q}$ and we have just seen, $\mathcal{Y}^2 \in \mathcal{Z} = \mathcal{O}_2 \mathcal{Q}$ since $\mathcal{Z} \in \mathcal{Z} \mathcal{Q}$. Hence \mathcal{Y}^2 commutes with \mathcal{Y} . This is true for all \mathcal{S}, \mathcal{Y} in \mathcal{Q} . Hence $\mathcal{Y}^2 \in \mathcal{O}_2 \mathcal{Q} = \mathcal{Z} \mathcal{Q}$ and $\mathcal{O}_2 \mathcal{Q}$ is elementary.

Let $[\mathcal{O}_2 \mathcal{Q}] = \mathcal{P} \mathcal{Q}'$. Then $\mathcal{P} \mathcal{Q}' = 1$, since $\mathcal{O}_2 \mathcal{Q}$ cannot be cyclic. Since $\mathcal{O}_2 \mathcal{Q} \neq 1$, we can choose \mathcal{S}, \mathcal{Y} in \mathcal{Q} such that $\mathcal{S} = [\mathcal{Y}, \mathcal{Y}] \neq 1$.

$\eta \in H$ and $\rho \in M$. Hence \bar{M} is one of the normal subgroups of \bar{E} described in 14-1. Hence we have

Lemma 14-12. If $G = HK$ with $K \triangleleft G$, then $G \cong \bar{E}/\bar{M}$ where $\bar{E} = \langle H, K, f \rangle$, f is the representation of H by the automorphisms of K induced by transforming in G and \bar{M} consists of all pairs (ρ, ρ') with $\rho \in M = H \cap K$, then

The conditions (1) of 14-1 are therefore necessary and sufficient for the existence of a semi-normal product $G = HK$ obtained with a given representation f of H by automorphisms of K and a given identification θ of a subgroup M of H with a subgroup \bar{M} of K , and such that $M = H \cap K$.

The simplest particular case is that of cyclic extensions. Here we have

Lemma 14-13 Let α be an automorphism of the group K which leaves invariant a certain element ξ of K and suppose that α^n is the inner automorphism $f(\xi)$ of K . Then there is a group $G = \langle K, \eta \rangle$ such that $\eta^n = \xi$, $\eta^p \eta = \eta$ for all $\rho \in K$ and $|G:K| = n$.

If ξ is of order m , we take $H = \langle \eta \rangle$ to be of order mn , $M = \langle \eta^n \rangle$ and $(\eta^n)^m = \xi$. The conditions (1) of 14-1 are then fulfilled.

Since the composition factors of a soluble group are all cyclic of prime order, 14-13 gives by repeated application a method of constructing all soluble groups, at least in principle. In simple enough cases, this method is usable and we shall illustrate it by discussing some special types of p -groups.

§ 14 Some Special p -Groups.

(A) Lemma 14-1 Let f be a representation of the group H by automorphisms of the group K and let $\bar{E} = \langle H, K, f \rangle$. Let θ be an isomorphic mapping of a subgroup M of H into K such that for all $\rho \in M, \rho \in H$ and $\xi \in K$ we have

$$\rho \theta f(\rho) = \rho \theta \rho \quad \text{and} \quad f(\rho \theta) = \xi \rho \theta \quad (1)$$

Then the set of all pairs (ρ', ρ'') with $\rho \in M$ forms a normal subgroup \bar{M} of \bar{E} such that $\bar{H} \cap \bar{M} = \bar{K} \cap \bar{M} = 1$ and $\bar{M} \cong M$.

Here necessarily \bar{H} is the subgroup of \bar{E} with the set of all pairs $(\eta, 1)$, $\eta \in H$, and similarly for \bar{K} .

Proof: Let ρ_1, ρ_2 be in M . Then $\rho_1 \theta f(\rho_2) \rho_2 = \rho_1 \theta \rho_2 = (\rho_1 \rho_2) \theta$ and so $(\rho_1^{-1} \rho_2^{-1}) (\rho_1 \theta f(\rho_2) \rho_2) = (\rho_1 \theta \rho_2) \theta = (\rho_1 \rho_2) \theta = (\rho_1 \rho_2) \theta$.

Thus \bar{M} is a subgroup of \bar{E} . The transform of (ρ', ρ'') by $(\eta, 1)$ is $(\eta^{-1} \rho' \eta, \rho' \theta f(\eta)) = (\rho', \rho' \theta f(\eta))$ which is in \bar{M} . The transform of (ρ', ρ'') by $(1, \xi)$ is $(\rho', \xi f(\rho') \rho'')$ which is also in \bar{M} . Thus $\bar{M} \triangleleft \bar{E} = \langle \rho', \rho' \theta f(\rho'') \rho' \theta \xi \rangle = \langle \rho', \rho' \theta \rangle$ which is also in \bar{M} . Thus $\bar{M} \triangleleft \bar{E} = HK$. It is clear that $\bar{H} \cap \bar{M} = \bar{K} \cap \bar{M} = 1$ since θ is an isomorphism. The mapping $(\rho', \rho'') \rightarrow \rho'$ is an isomorphism of \bar{M} onto M . Thus 14-1 is proved.

Clearly, $G = \bar{E}/\bar{M} = H, K$, where $H_1 = \bar{H}/\bar{M} \cong H$ and $K_1 = \bar{K}/\bar{M} \cong K$. Moreover $K_1 \triangleleft G$ and $H_1 \cap K_1 = M_1 \cong M$.

Conversely, let $G = HK$ with $K \triangleleft G$ and let $M = H \cap K$.

Let $f(\eta)$, $\eta \in H$, be the automorphism of K induced by transforming with η .

Then f is a representation of K . Since $(\eta, \xi) (\eta, \xi) = \eta, \xi$ where $\eta = \eta, \xi$ and $\xi = \xi, f(\eta) \xi$ for all $\eta \in H, \xi \in K$, the mapping $(\eta, \xi) \rightarrow \eta$ of $\bar{E} = \langle H, K, f \rangle$ onto G is a homomorphism, with kernel \bar{M} such that $\bar{H} \cap \bar{M} = \bar{K} \cap \bar{M} = 1$. Moreover \bar{M} consists of all pairs (ρ, ρ') with $\rho \in M$. If θ is the identity mapping of M considered as a subgroup of H into itself considered as a subgroup of K , the relations $\rho \theta f(\rho') = \rho \theta \rho'$ and $f(\rho \theta) = \xi \rho \theta$ hold for all $\rho \in K$.

order 4.
 (ii) $\text{Aut } O \cong O$; $\text{Aut } Q \cong \Sigma_4$. The automorphisms $\alpha: \mathcal{S} \rightarrow \mathcal{S} \gamma$ and $\beta: \mathcal{S} \rightarrow \mathcal{S} \gamma^{-1}$ of Q generate a group isomorphic with Σ_3 . The split extension $\{O, \alpha, \beta\}$ of order 48 is called the binary octahedral group; $\{O, \alpha\}$ of order 24 is the binary tetrahedral group.
 (iv) If $p=2$ and $n>3$, then G is of class either 2 or $n-1$. In the former case, $G = \langle \mathcal{S}, \gamma \rangle$ is determined to within isomorphism by the relation $\gamma^2=1, \gamma^2 \mathcal{S} \gamma = \mathcal{S}^{1+2^{n-1}}$. G has exactly three involutions, forming with 1 the characteristic subgroup $\langle \mathcal{S}^{2^{n-2}}, \gamma \rangle$.
 (v) If $p=2, n>3$ and G has class $n-1$, then there are three distinct types of group G to within isomorphism, ~~which are distinguished~~ which will be denoted by O_{2^n}, P_{2^n} and Q_{2^n} . We have $G = \langle \mathcal{S}, \gamma \rangle$ where $\gamma^2=1, \gamma^2 \mathcal{S} \gamma = \mathcal{S}^{-1}$ in the dicentral group O_{2^n}
 $\gamma^2=1, \gamma^2 \mathcal{S} \gamma = \mathcal{S}^{1+2^{n-1}}$ in the intermediate group P_{2^n}
 $\gamma^2 = \mathcal{S}^{2^{n-2}}, \gamma^2 \mathcal{S} \gamma = \mathcal{S}^{-1}$ in the generalized quaternion group Q_{2^n} .
 Note that $O = O_{2^n}, Q = Q_{2^n}$. Besides the cyclic subgroup $H = \langle \mathcal{S} \rangle, G$ has two other subgroups of index 2. In O_{2^n} , these are both of type $O_{2^{n-1}}$. In P_{2^n} , one is of type $O_{2^{n-1}}$ and the other of type $O_{2^{n-1}}$. In Q_{2^n} , both are of type $Q_{2^{n-1}}$. G has centre $\langle \mathcal{S}^{2^{n-1}} \rangle = Z$ of order 2 and in all these groups, G/Z is of type $O_{2^{n-1}}$.

(c) We shall now consider the structure of a non-Abelian p -group G which has a cyclic subgroup of index p . By 7.2(vi), $|G| = p^n$ with $n > 2$.
Lemma 14.31. Let $C = \langle \mathcal{S} \rangle$ be a cyclic group of order $p^n, n > 2$.
 (i) If $p=2, n=2$, then $A = \text{Aut } C$ is of order 2 and is generated by the automorphism $\beta: \mathcal{S} \rightarrow \mathcal{S}^{-1}$.
 (ii) If $p=2, n>2$, then A is an Abelian 2-group of type $(1-2, 1)$ with a basis α, β defined by $\alpha: \mathcal{S} \rightarrow \mathcal{S}^5$ and $\beta: \mathcal{S} \rightarrow \mathcal{S}^{-1}$.
 (iii) If p is odd, the Sylow p -subgroup of the Abelian group A is generated by the automorphism $\alpha: \mathcal{S} \rightarrow \mathcal{S}^{1+p}$ and has order p^{n-1} .
Proof: (i) is clear.
 (ii) $\mathcal{S}^{2^{n-1}} = 1$ mod 2^{n-2} but $\neq 1$ mod 2^{n-3} . Hence α has order 2^{n-2} . Obviously $\beta \notin \langle \alpha \rangle$. Hence $A = \langle \alpha, \beta \rangle$ with basis α, β , since $|A| = \varphi(p^n) = 2^{n-1}$.
 (iii) Here $|A| = \varphi(p^n) = p^{n-1}(p-1)$ and $(1+p)^{p^n} \equiv 1$ mod p^{n+1} but $\neq 1$ mod p^n . Hence α generates the Sylow p -subgroup of A .
Corollary 14.32. If $C = \langle \mathcal{S} \rangle$ has order $2^n, n > 2$, then C has exactly three involutory automorphisms viz.
 $\alpha: \mathcal{S} \rightarrow \mathcal{S}^{1+2^{n-1}}; \beta: \mathcal{S} \rightarrow \mathcal{S}^{-1}; \gamma: \mathcal{S} \rightarrow \mathcal{S}^{1+2^{n-1}}$.
 We can now prove
Theorem 14.3. Let G be a group of order p^n with a cyclic subgroup $H = \langle \mathcal{S} \rangle$ of index p . Suppose that $n > 2$ and that G is not Abelian.
 (i) If p is odd, G is determined to within isomorphism and $G = \langle H, \gamma \rangle$ where $\gamma^2=1$ and $\gamma^2 \mathcal{S} \gamma = \mathcal{S}^{1+p}$. G is of class 2 and $\Omega_1 G = \langle \mathcal{S}^{p^{n-1}}, \gamma \rangle$ is of order p^2 .
 (ii) If $p=2$ and $n=3$, there are to within isomorphism two distinct types of group $G = \langle H, \gamma \rangle$: the cyclic group O is determined by the equation $\gamma^2=1, \gamma^2 \mathcal{S} \gamma = \mathcal{S}^{-1}$; the quaternion group Q by $\gamma^2=\mathcal{S}^5, \gamma^2 \mathcal{S} \gamma = \mathcal{S}^{-1}$. H is a characteristic subgroup of O and the remaining two subgroups of order 4 in O are elementary. In Q , every element of order 4 is not in $\Omega_1 G$, has

(iv). Here we have three possible choices for A_0 by 14-32, and $G = \langle \xi, \eta \rangle$ where $\eta^2\xi\eta$ is either ξ^{1+2p-2} , ξ^{-1} or ξ^{1+2p-1} . Let $\gamma = \xi^{2p-1}$. In the first case, as in (i) for p odd, we have $G' = \langle \gamma \rangle \leq \langle \xi \rangle = \langle \xi^2 \rangle$ and G is of class 2, and again the $\langle \xi \rangle$ subgroups of G are all Abelian. If $\langle \xi \rangle$ is cyclic, we may suppose $\eta^2 = \xi^2$ and obtain $(\eta\xi^{-1})^{2p} = 1$. If $\langle \xi \rangle$ is not cyclic, it is of type $(p-2, 2)$. In any case we may choose $\eta^2 = 1$. Thus G is determined to within isomorphism; and in fact just two of the three subgroups of index 2 are cyclic. The third is a characteristic subgroup of type $(p-2, 2)$. Hence $\Omega_2(G)$ contains all the involutions of G and is an elementary group of order 4, characteristic in G .

(v) In the remaining two cases, η transforms ξ^2 into its inverse and $\langle \xi \rangle = \langle \xi^2 \rangle$ is of order 2. $\langle \eta, \xi \rangle = \langle \xi^2 \rangle$ or $\langle \xi^2 \rangle$ and successive commutations will give $\xi^{2p}, \xi^4, \xi^{6p}, \dots$. Hence G is of class $p-1$ and $\Omega_2(G) = \langle \xi^2 \rangle$. For p^2 only two values are available, 1 and 3. However, when $\eta^2\xi\eta = \xi^{-1}$ we have $(\eta\xi^{-1})^{2p} = \eta^2$ and the two choices for η^2 lead to the same group Ω_2 , of intermediate type. Taking $\eta^2 = 1$, $\langle \xi^2, \eta \rangle$ is then of type Ω_2^{p-1} , and $\langle \xi^2, \eta \rangle$ of type Ω_2^{p-1} . When $\eta^2\xi\eta = \xi^{-1}$, we have $(\eta\xi^{-1})^{2p} = \eta^2$ for all η . When $\eta^2 = 1$, we have the group Ω_2 in which every element outside $\langle \xi \rangle$ is of order 2. When $\eta^2 = \xi$, we have Ω_2 in which every element outside $\langle \xi \rangle$ is of order 4. The remaining statements of (v) are clear.

Corollary 14.33 If the p -group G has no elementary subgroup of order p^2 , then either G is cyclic, or else $p=2$ and $G = \Omega_2$ is the p -group for some $n \geq 3$. (Including the case $\Omega = \Omega_2$, $n=3$, the ordinary quaternion group)

Proof: Suppose G is not cyclic. Then it contains a p -subgroup E , with a cyclic subgroup of index p . E cannot be Abelian, for then $\Omega_2(G)$ would be elementary of order p^2 . By 14-3 (v), it follows that $p=2$. By 14-3 (v), $\Omega_2(G)$ must be of type Ω_2 for some n . By induction on n , we may assume $|\Omega_2(G)| = 2$. We may assume that

Note that $H \rightarrow G$ by 5.22(v). Since $\Omega_2(G) = 1$, the automiser $A_0 = A_0(H)$ is of order p . *Proof:* (i) By 14-31 (ii), there is only one choice for A_0 viz. $A_0 = \langle \alpha^p, \beta^p \rangle$. Here $\gamma = \alpha^{1+2p-2} = \xi \rightarrow \xi^{1+2p-1} = \xi^2$. So $\langle \xi \rangle = \langle \xi^2 \rangle$ is of index p^2 and $G' = \langle \xi \rangle$ is of order p and G is of class 2. Also $G = \langle \xi, \eta \rangle$ where $\eta^2\xi\eta = \xi^2$. Then $\eta^2 = \xi^2$ for some η , since η^2 commutes with η . Since $\langle \xi^2, \eta \rangle = \langle \xi^2 \rangle$ and $\langle \xi^2 \rangle = 1$, we have $(\eta\xi^{-1})^{2p} = 1$ because p is odd. Then $G = \langle \xi, \eta \rangle$ with $\eta^2\xi\eta = \xi^{1+2p-1}$, $\eta^2 = 1$. $K = \langle \eta, \xi \rangle$ is normal in G and elementary; $G = \langle K, \xi \rangle$ is G/K is cyclic. No element outside of K can have order p , so $K = \Omega_2(G)$.

(ii) By 14-31 (i), the choice of A_0 is again unique and $G = \langle \xi, \eta \rangle$ with $\eta^2\xi\eta = \xi^{-1}$. Here $G' = \langle \xi \rangle = \langle \xi^2 \rangle$ is of order 2 and there are only two possibilities for η^2 viz. $\eta^2 = 1$ giving Ω_2 or $\eta^2 = \xi^2$ giving Ω_2 . Since $(\eta\xi^{-1})^{2p} = \eta^2$, $\eta^2\xi\eta = \xi^{-1}$, and η^2 transforms ξ^2 into its inverse for all choices of η . These two cases are distinct. In Ω_2 , $H = \langle \xi \rangle$ is the only cyclic subgroup of order 4 is characteristic; $|\text{Aut } \Omega_2| = 8$ because each of the mappings $\xi \rightarrow \xi^{2p-1}$, $\eta \rightarrow \eta\xi^2$ ($\tau = 0, 1, 2, 3$) defines an automorphism of Ω_2 . If $\eta^2 = \xi^2$, $\eta \rightarrow \eta$ and $\xi^2 \rightarrow \xi^2$, $\eta \rightarrow \eta\xi^2$. Then $\text{Aut } \Omega_2 = \langle \xi^2, \eta^2 \rangle$ where ξ^2 is of order 4, η^2 of order 2 and η^2 transforms ξ^2 into its inverse. Then $\text{Aut } \Omega_2 \cong \Omega_2$.

(iii) In the case of Ω_2 , all three subgroups of index 2 are cyclic. Hence if β, γ, δ are any two non-commuting elements of Ω_2 , we have $\Omega = \langle \xi, \eta \rangle$ and $\xi \rightarrow \xi, \eta \rightarrow \eta$, is an automorphism. Hence $|\text{Aut } \Omega| = (p-2)(p-1) = 26$. If α is $\xi \rightarrow \eta \rightarrow \xi\eta$ and β is $\xi \rightarrow \eta^{-1}$, then $\alpha^2 = \beta^2 = 1$ and $\beta\alpha\beta = \alpha^{-1}$ so $S = \langle \alpha, \beta \rangle \cong \Sigma_3$. S permutes the three subgroups of index 2 in Ω faithfully. These subgroups are normal in Ω and so $\text{Aut } \Omega = S \cdot T$, where $S \cap T = 1$ and T is the group of inner automorphisms of Ω . $T \cong \Omega/\Omega'$ is elementary of order 4, so $|\text{Aut } T| = 6$. But $C_2(T) = 1$. Hence $\text{Aut } \Omega$ is isomorphic with the holomorph of T . The same applies to Σ_4 . So $\text{Aut } \Omega \cong \Sigma_4$.

(E) We consider now the non-Abelian groups of order p^3 . If $p=2$, then 14-3(6) shows that the odd and quaternion groups are the only possible types. For odd p there are also exactly two distinct types of group.

For let $|G|=p^3$, $G' \neq 1$, p odd. If G has an element ξ of order p^2 , then $G = \langle \xi, \eta \rangle$ with $\eta^p = 1$, $\eta^{-1}\xi\eta = \xi^{1+p}$ by 14-3(6). On the other hand, if G is of exponent p , then $G = \langle \xi, \eta \rangle$ with $\xi^p = \eta^p = 1$ and $\xi = [\xi, \eta]$. Here $\langle \xi \rangle, \langle \eta \rangle = G'$, and G is the split extension of the elementary group $\langle \xi, \eta \rangle$ by $\langle \eta \rangle$, with $\eta^{-1}\xi\eta = \xi^p$ and of course $\eta^{-1}\eta\eta = \eta$. We shall denote these two groups by P^{p^3} and T^{p^3} respectively. We state

Lemma 14-35 For given p , there are exactly two non-Abelian groups of order p^3 . Moreover for both of them, $G' = Z$ and G/G' is of order p and G/G' is elementary. When $p=2$, they are the odd and quaternion groups. When p is odd, they are the groups P^{p^3} and $T^{p^3} = P^{p^3} \cdot P^{p^3}$. $p-1$ subgroups of index p are all elementary, while in P^{p^3} there is a characteristic elementary subgroup $\Omega_1 P$ of order p^2 and the remaining p subgroups of index p are all cyclic.

It is easy to see that $|\text{Aut } P| = (p^2 - p)(p^2 - p)$ and $|\text{Aut } T| = p(p^2 - p^2)$.

(F) To form a central product of two groups H and K , we have to establish an isomorphism identifying a subgroup Z of the center of H with a subgroup of the center of K . If $G = HK$ is the central product determined in this way, then $|G| = |H| \cdot |K| / |Z|$. The choice $Z = 1$ is always possible and then G is simply the direct product of H and K . However in speaking of a central product it will always be tacitly assumed that the amalgamated subgroup Z is $\neq 1$. If $Z = H$ and $Z = K$ are of order p a prime, this leaves only one possible choice $Z = Z = H = K$. But $|\text{Aut } Z| = p-1$ so that there are $p-1$ possible choices for the identifying

$G_1 = \langle \xi, \eta \rangle$ with $\langle \xi \rangle$ of order 2^{n-1} and normal in $G = \langle G_1, \tau \rangle$; for $\langle \xi \rangle$ char G_1 , if $n > 3$ and if $n=3$, G_1 has three cyclic subgroups of order 4. In $\text{Aut } \langle \xi \rangle$, the automorphism $\xi \rightarrow \xi^3$ induced by η is not a square. Hence $\tau^2 \in \langle \xi \rangle$, and $G/\langle \xi \rangle$ is elementary of order 4. Since the automorphism $\xi \rightarrow \xi^{1+2^{n-2}}$ cannot be induced in G , 14-32 shows that G has an Abelian subgroup H of index 2, viz. $H = C_2 \langle \xi \rangle$. Then H must be cyclic and so G is of type Q_{2^n} .

(D). *Theorem 14-34b* Let G be a group of order 2^n , $n > 3$. Then the following conditions are equivalent.

- (i) $|G : G'| = 4$.
- (ii) G is of class $n-1$.
- (iii) G is one of the groups Q_8, P_8, Q_{16}, Q_{32} .

Proof: (ii) \Rightarrow (i) is clear. For a nilpotent group G with cyclic G/G' must itself be cyclic, by 9-1(1) and 6-8(1). [This should have been made explicit in §9 (A) somewhere]. Hence $|G : G'| = 4$. But $|G : G'| = 4$ implies $Q_8, G = 1$, so that the class of G would be less than $n-1$.

(i) \Rightarrow (iii). By 14-2.1 and 14-3(8), O and Q are the only non-Abelian groups of order 8. Suppose let $Z = O$, $|Z| = 2$, $Z \leq G'$. Such a subgroup Z exists by 5-2.4(1), and by induction we may assume that G/Z has a cyclic subgroup H/Z of index 2. Let $Z = \langle \xi \rangle$ and $H = \langle \xi, \eta \rangle$. If $H = \langle \xi \rangle$, the result follows from 14-3(8) and (8). If H is not cyclic, it is Abelian of type $(n-2, 1)$ and if $\eta = \xi^{2^{n-1}}$, then $\eta = \langle \eta \rangle$ is a characteristic subgroup of H . So $\eta \in G'$. Also $\eta \in G'$ since G/G' is elementary. By induction, G/η also has a cyclic subgroup of index 2. But H/η is Abelian of type $(n-3, 1)$. Hence $H = \langle \eta, \xi \rangle$ and $G = \langle H, L \rangle$ where $G = \langle \eta, \xi \rangle$ is of index 4. Hence G is of class 2 and $G' = \langle \eta, \xi \rangle$ whose $G' = \langle \eta, \xi \rangle$. Hence $\eta = [\eta, \xi]$ is of order 2 since $\eta^2 \in \langle \xi \rangle$. Then $|G'| = 2$, contradicting $n > 3$.

(iii) \Rightarrow (i) is contained in 14-3(8), (9).

have $\mathfrak{Z}G_i = G_i$ also. Hence $\mathfrak{Z}G_i = G_i$ is of order p and (i) follows by induction on $|G|$.

(ii) It will be sufficient to prove $Q^2 \cong O^2$. Let $G = Q^2$. Then $G = \langle \mathfrak{Z}, \eta, \mathfrak{Z}_2, \eta_2 \rangle$ and $G' = \langle \mathfrak{Z} \rangle$, where $\eta^2 = 1$ and $\mathfrak{Z}_2^2 = \eta^2 = \langle \mathfrak{Z}, \eta \rangle = 1$ ($i=1,2$), while $[\mathfrak{Z}, \mathfrak{Z}_2] = \langle \mathfrak{Z}, \eta \rangle = \langle \mathfrak{Z}, \eta_2 \rangle = \langle \mathfrak{Z}, \eta \rangle = 1$. Hence $G = G_1 G_2$ where $G_1 = \langle \mathfrak{Z}, \eta, \mathfrak{Z}_2 \rangle$ and $G_2 = \langle \mathfrak{Z}, \eta_2 \rangle$ are two quaternion groups of order 8. Let $G_1' = \langle \mathfrak{Z}, \eta, \mathfrak{Z}_2 \rangle$ and $G_2' = \langle \mathfrak{Z}, \eta_2 \rangle$. Then $G = G_1' G_2'$ and $[G_1', G_2'] = 1$ and here G_1', G_2' are cyclic groups.

(iii) The group Q^2 contains 6 cyclic subgroups of order 4, three in G_1 and three in G_2 , while OQ contains 10. More generally, Q^r contains $3r + 3^r(\mathfrak{Z}) + 3^r(\eta) + \dots = \frac{1}{2}((43)^r - (1-3)^r) = 2^{r-1}(2^r - 1)^r$ cyclic subgroups of order 4; while OQ^{r-1} contains $2^{r-2} + 2^{r-1}(2^{r-1} - 1)^r = 2^{r-2}(2^r + (-1)^r) = 2^{r-2}(2^r + (-1)^r)$ cyclic subgroups of order 4. So Q^r and OQ^{r-1} cannot be isomorphic.

(iv) Let p be odd. If G is of exponent p , then all the central factors G_i must be of type $P = P_1^{p_i}$ and so $G = P^p$. It is sufficient to prove $(P^p)^2 \cong P^{2p}$. Let $G = G_1 G_2$ where $G_i = \langle \mathfrak{Z}_i, \eta_i \rangle$ and $[G_1, G_2] = 1$ and $\mathfrak{Z}_i^2 = \langle \mathfrak{Z}_i, \eta_i \rangle$ generates G_i' , while $\eta_i^2 = 1, i=1,2$. So G is of type $(P^p)^2$. We may assume that $\mathfrak{Z}_1^2 = \mathfrak{Z}_2^2$, so that $\mathfrak{Z}_1 \mathfrak{Z}_2$ is of order p . Then $G_1' = \langle \mathfrak{Z}_1, \eta_1 \rangle$ is of type I and if G_2' is of type II, then G is the central product $G_1' G_2'$. Here G_1' is necessarily of type P^p , for otherwise G would be of exponent p , by (iv).

If $G = G_1 G_2 \dots G_r$ is of type $P^p P^{p^2} \dots$ with G_1, \dots, G_r of type P and $G_i = \langle \mathfrak{Z}_i, \eta_i \rangle$ of type P^p , where $\eta_i^2 = 1$, then $\mathfrak{Z}_i G_i = \langle \mathfrak{Z}_i, G_i \rangle = G_i$ of order p . It is the direct product of $\langle \eta_i \rangle$ with $\mathfrak{Z}_i G_i = G_i$ if type P^p .

Lemma 14.41. If G is a p -group and that $G' \cong G$ is of order p , then every automorphism of G which transforms G/G' identically is an inner automorphism.

Proof. Let $|G:G'| = p^2$. Since $|G'| = p$, the number of automorphisms of G transforming G/G' identically is at most p^2 . This is also equal to $|G:G'|$, the number of inner automorphisms of G and the latter all transform G/G' identically. Hence the result.

(isomorphism automorphism). Nevertheless, if $\mathfrak{Z}G$ is the automizer of $Z = \mathfrak{Z}H$ in $\text{Aut } H$ is the full group $\text{Aut } Z$, then the central product $G = H \times K$ will be determined by within isomorphism. For by definition $G = \mathfrak{Z}/ZG$, where $\mathfrak{Z} = H \times K$ is the Cartesian product and ZG consists of all pairs (Y, Y') with $Y \in Z$, and where θ is an isomorphism of $Z = \mathfrak{Z}H$ onto $\mathfrak{Z}K$. Under the assumption mentioned, the $p-1$ subgroups ZG of \mathfrak{Z} are all conjugate under $\text{Aut } \mathfrak{Z}$. When H is one of the non-Abelian groups of order p^2 , then this assumption is in fact correct, as is easy to see. Hence there is no ambiguity in the definition of the central product of any number of such groups.

Theorem 14.40(i) Let G be a p -group such that $G' = \mathfrak{Z}G$ is of order p . Then $|G| = p^{2r+1}$ for some $r=1,2,\dots$; G/G' is elementary; and G is expressible (in many ways if $r > 1$) as the central product $G_1 G_2 \dots G_r$ of r non-Abelian groups of order p^2 .

(ii) when $p=2$, we have the central product isomorphism $Q^r \cong O^r Q^{r-2} \cong O^r Q^{r-4} \cong \dots$ and $OQ^{r-1} \cong O^3 Q^{r-3} \cong \dots$ but Q^r and OQ^{r-1} are not isomorphic.

(iii) when p is odd, we have the central product isomorphism $P^p P^{p^2} \dots \cong (P^p)^2 P^{p^2-2} \cong \dots \cong (P^p)^r$ but $P^p P^{p^2}$ and P^p are not isomorphic. Here $P = P_1^{p^2}$ and $P^p = P_1^p$.

Proof. From 7.1 (i), and $G' = \mathfrak{Z}G$ of order p , it follows that G/G' is elementary viz. $\mathfrak{Z}^p \in \mathfrak{Z}G$ for all $\mathfrak{Z} \in G$. In fact, in a group of class 2, the mapping $\mathfrak{Z} \rightarrow [\mathfrak{Z}, \mathfrak{Z}]$ is homomorphic for any fixed \mathfrak{Z} .

Since G is not Abelian, we can choose $\mathfrak{X}, \mathfrak{Y}$ so that $[\mathfrak{X}, \mathfrak{Y}] \neq 1$. Then $G_1 = \langle \mathfrak{X}, \mathfrak{Y} \rangle$ is non-Abelian of order p^2 , let $\mathfrak{Z}_1 = \mathfrak{Z}_1(G_1)$. Then $\mathfrak{Z}_1 = \mathfrak{X}, \mathfrak{Y}$, where $\mathfrak{X}_1 = \mathfrak{Z}_1(\mathfrak{X})$ and $\mathfrak{Y}_1 = \mathfrak{Z}_1(\mathfrak{Y})$. But \mathfrak{Z}_1 and η_1 have each exactly p conjugates in G . Hence $|G:G_1| = |G:\mathfrak{Z}_1| = p$. Since \mathfrak{Z}_1 and η_1 do not commute, $\mathfrak{X}_1 + \mathfrak{Y}_1$ and so $|G:\mathfrak{Z}_1| = p^2$. Also $G_1 \cap \mathfrak{Z}_1 = \mathfrak{Z}_1 G_1 = G_1$ and so the elementary group G/G' is the direct product of G/G_1' and \mathfrak{Z}_1/G_1' . Since $G = G_1 G_2$ is a central product of G_1 and G_2 , since $\mathfrak{Z}_1 G_1 = G_1$, we

(G) Let H be a p -group with $|H|$ of order p and let Z be cyclic of order p^n , $Z = \langle z \rangle$. Then the central product HZ is defined to be isomorphism, because the identified subgroup can only be $Z_1 = \langle z \rangle H$ with $Z_1 = \langle z^{p^{n-1}} \rangle$ and the automorphism of Z_1 in $\text{Aut } Z_1 = \text{Aut } Z_1$.

Theorem 14.5 Let G be a non-Abelian p -group such that every characteristic Abelian subgroup of G is cyclic and let $Z = \langle z \rangle$ of order p^n . Then

- (i) for $n \geq 2$, G is the central product ZP^r for some $r \geq 1, 2, \dots$ where $P = P_2$ and the factor Z may be supposed of order $|G| = p^{2r}$.
 - (ii) If $p=2$ and $n=1$, G is a central product of one of the forms Q_8^r, O_8^{r-1} ($r > 0$) or $Q_8^r Q_8^s, P_8^r Q_8^s, Q_8^r Q_8^s$ ($k > 3, r \geq 0$) and no two of them are isomorphic. We have the central product isomorphisms $Q_8^r Q_8^s \cong Q_8^r Q_8^t \cong Q_8^s Q_8^t \cong Q_8^r Q_8^{s+t}$ and $P_8^r Q_8^s \cong P_8^r Q_8^t \cong P_8^s Q_8^t \cong P_8^r Q_8^{s+t}$.
 - (iii) If $p=2$ and $n > 1$, G is the central product ZQ^r for some $r > 0$ and we have the central product isomorphisms $ZQ^r \cong ZQ^s \cong ZQ^{r+s}$.
- We need first

Lemma 14.51 Let G be a p -group such that zG is cyclic. Then G' is cyclic.

Proof: Let $Z = \langle z \rangle$. Since $Z \trianglelefteq G$, $Z \leq G'$, we can choose a normal cyclic subgroup H of G which is maximal subject to $Z \leq H \leq G'$. If G' is not cyclic, then $H < G'$ and by 5.2 (i), there is a normal subgroup K of G such that $H < K \leq G'$ and $|K:H| = p$. Then K is not cyclic. If K is Abelian, it is of type $(m, 1)$ where $|H| = p^m$, and $|K|$ is of order p^2 . If K is not Abelian, $|K|$ is still of order p^2 for all p , by 14.3 (i), and also for $p=2$ provided K is of class 2, by 14.3 (ii). The only alternative is for K to be one of the groups $Q_8, P_8, Q_8^2, Q_8^3, Q_8^4, Q_8^5, Q_8^6, Q_8^7, Q_8^8, Q_8^9, Q_8^{10}, Q_8^{11}, Q_8^{12}, Q_8^{13}, Q_8^{14}, Q_8^{15}, Q_8^{16}, Q_8^{17}, Q_8^{18}, Q_8^{19}, Q_8^{20}, Q_8^{21}, Q_8^{22}, Q_8^{23}, Q_8^{24}, Q_8^{25}, Q_8^{26}, Q_8^{27}, Q_8^{28}, Q_8^{29}, Q_8^{30}, Q_8^{31}, Q_8^{32}, Q_8^{33}, Q_8^{34}, Q_8^{35}, Q_8^{36}, Q_8^{37}, Q_8^{38}, Q_8^{39}, Q_8^{40}, Q_8^{41}, Q_8^{42}, Q_8^{43}, Q_8^{44}, Q_8^{45}, Q_8^{46}, Q_8^{47}, Q_8^{48}, Q_8^{49}, Q_8^{50}, Q_8^{51}, Q_8^{52}, Q_8^{53}, Q_8^{54}, Q_8^{55}, Q_8^{56}, Q_8^{57}, Q_8^{58}, Q_8^{59}, Q_8^{60}, Q_8^{61}, Q_8^{62}, Q_8^{63}, Q_8^{64}, Q_8^{65}, Q_8^{66}, Q_8^{67}, Q_8^{68}, Q_8^{69}, Q_8^{70}, Q_8^{71}, Q_8^{72}, Q_8^{73}, Q_8^{74}, Q_8^{75}, Q_8^{76}, Q_8^{77}, Q_8^{78}, Q_8^{79}, Q_8^{80}, Q_8^{81}, Q_8^{82}, Q_8^{83}, Q_8^{84}, Q_8^{85}, Q_8^{86}, Q_8^{87}, Q_8^{88}, Q_8^{89}, Q_8^{90}, Q_8^{91}, Q_8^{92}, Q_8^{93}, Q_8^{94}, Q_8^{95}, Q_8^{96}, Q_8^{97}, 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Q_8^{189}, Q_8^{190}, Q_8^{191}, Q_8^{192}, Q_8^{193}, Q_8^{194}, Q_8^{195}, Q_8^{196}, Q_8^{197}, Q_8^{198}, Q_8^{199}, Q_8^{200}, Q_8^{201}, Q_8^{202}, Q_8^{203}, Q_8^{204}, Q_8^{205}, Q_8^{206}, Q_8^{207}, Q_8^{208}, Q_8^{209}, Q_8^{210}, Q_8^{211}, Q_8^{212}, Q_8^{213}, Q_8^{214}, Q_8^{215}, Q_8^{216}, Q_8^{217}, Q_8^{218}, Q_8^{219}, Q_8^{220}, Q_8^{221}, Q_8^{222}, Q_8^{223}, Q_8^{224}, Q_8^{225}, Q_8^{226}, Q_8^{227}, Q_8^{228}, Q_8^{229}, Q_8^{230}, Q_8^{231}, Q_8^{232}, Q_8^{233}, Q_8^{234}, Q_8^{235}, Q_8^{236}, Q_8^{237}, Q_8^{238}, Q_8^{239}, Q_8^{240}, Q_8^{241}, Q_8^{242}, Q_8^{243}, Q_8^{244}, Q_8^{245}, Q_8^{246}, Q_8^{247}, Q_8^{248}, Q_8^{249}, Q_8^{250}, Q_8^{251}, Q_8^{252}, Q_8^{253}, Q_8^{254}, Q_8^{255}, Q_8^{256}, Q_8^{257}, Q_8^{258}, Q_8^{259}, Q_8^{260}, Q_8^{261}, Q_8^{262}, Q_8^{263}, Q_8^{264}, Q_8^{265}, Q_8^{266}, Q_8^{267}, Q_8^{268}, Q_8^{269}, Q_8^{270}, Q_8^{271}, Q_8^{272}, Q_8^{273}, Q_8^{274}, Q_8^{275}, Q_8^{276}, Q_8^{277}, Q_8^{278}, Q_8^{279}, 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Q_8^{371}, Q_8^{372}, Q_8^{373}, Q_8^{374}, Q_8^{375}, Q_8^{376}, Q_8^{377}, Q_8^{378}, Q_8^{379}, Q_8^{380}, Q_8^{381}, Q_8^{382}, Q_8^{383}, Q_8^{384}, Q_8^{385}, Q_8^{386}, Q_8^{387}, Q_8^{388}, Q_8^{389}, Q_8^{390}, Q_8^{391}, Q_8^{392}, Q_8^{393}, Q_8^{394}, Q_8^{395}, Q_8^{396}, Q_8^{397}, Q_8^{398}, Q_8^{399}, Q_8^{400}, Q_8^{401}, Q_8^{402}, Q_8^{403}, Q_8^{404}, Q_8^{405}, Q_8^{406}, Q_8^{407}, Q_8^{408}, Q_8^{409}, Q_8^{410}, Q_8^{411}, Q_8^{412}, Q_8^{413}, Q_8^{414}, Q_8^{415}, Q_8^{416}, Q_8^{417}, Q_8^{418}, Q_8^{419}, Q_8^{420}, Q_8^{421}, Q_8^{422}, Q_8^{423}, Q_8^{424}, Q_8^{425}, Q_8^{426}, Q_8^{427}, Q_8^{428}, Q_8^{429}, Q_8^{430}, Q_8^{431}, Q_8^{432}, Q_8^{433}, Q_8^{434}, Q_8^{435}, Q_8^{436}, Q_8^{437}, Q_8^{438}, Q_8^{439}, Q_8^{440}, Q_8^{441}, Q_8^{442}, Q_8^{443}, Q_8^{444}, Q_8^{445}, Q_8^{446}, Q_8^{447}, Q_8^{448}, Q_8^{449}, Q_8^{450}, Q_8^{451}, Q_8^{452}, Q_8^{453}, Q_8^{454}, Q_8^{455}, Q_8^{456}, Q_8^{457}, Q_8^{458}, Q_8^{459}, Q_8^{460}, Q_8^{461}, 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Q_8^{735}, Q_8^{736}, Q_8^{737}, Q_8^{738}, Q_8^{739}, Q_8^{740}, Q_8^{741}, Q_8^{742}, Q_8^{743}, Q_8^{744}, Q_8^{745}, Q_8^{746}, Q_8^{747}, Q_8^{748}, Q_8^{749}, Q_8^{750}, Q_8^{751}, Q_8^{752}, Q_8^{753}, Q_8^{754}, Q_8^{755}, Q_8^{756}, Q_8^{757}, Q_8^{758}, Q_8^{759}, Q_8^{760}, Q_8^{761}, Q_8^{762}, Q_8^{763}, Q_8^{764}, Q_8^{765}, Q_8^{766}, Q_8^{767}, Q_8^{768}, Q_8^{769}, Q_8^{770}, Q_8^{771}, Q_8^{772}, Q_8^{773}, Q_8^{774}, Q_8^{775}, Q_8^{776}, Q_8^{777}, Q_8^{778}, Q_8^{779}, Q_8^{780}, Q_8^{781}, Q_8^{782}, Q_8^{783}, Q_8^{784}, Q_8^{785}, Q_8^{786}, Q_8^{787}, Q_8^{788}, Q_8^{789}, Q_8^{790}, Q_8^{791}, Q_8^{792}, Q_8^{793}, Q_8^{794}, Q_8^{795}, Q_8^{796}, Q_8^{797}, Q_8^{798}, Q_8^{799}, Q_8^{800}, Q_8^{801}, Q_8^{802}, Q_8^{803}, Q_8^{804}, Q_8^{805}, Q_8^{806}, Q_8^{807}, Q_8^{808}, Q_8^{809}, Q_8^{810}, Q_8^{811}, Q_8^{812}, Q_8^{813}, Q_8^{814}, Q_8^{815}, Q_8^{816}, Q_8^{817}, Q_8^{818}, Q_8^{819}, Q_8^{820}, Q_8^{821}, Q_8^{822}, Q_8^{823}, Q_8^{824}, Q_8^{825}, Q_8^{826}, Q_8^{827}, Q_8^{828}, Q_8^{829}, Q_8^{830}, Q_8^{831}, Q_8^{832}, Q_8^{833}, Q_8^{834}, Q_8^{835}, Q_8^{836}, Q_8^{837}, Q_8^{838}, Q_8^{839}, Q_8^{840}, Q_8^{841}, Q_8^{842}, Q_8^{843}, Q_8^{844}, Q_8^{845}, Q_8^{846}, Q_8^{847}, Q_8^{848}, Q_8^{849}, Q_8^{850}, Q_8^{851}, Q_8^{852}, Q_8^{853}, Q_8^{854}, Q_8^{855}, Q_8^{856}, Q_8^{857}, Q_8^{858}, Q_8^{859}, Q_8^{860}, Q_8^{861}, Q_8^{862}, Q_8^{863}, Q_8^{864}, Q_8^{865}, Q_8^{866}, Q_8^{867}, Q_8^{868}, Q_8^{869}, Q_8^{870}, Q_8^{871}, Q_8^{872}, Q_8^{873}, Q_8^{874}, Q_8^{875}, Q_8^{876}, Q_8^{877}, Q_8^{878}, Q_8^{879}, Q_8^{880}, Q_8^{881}, Q_8^{882}, Q_8^{883}, Q_8^{884}, Q_8^{885}, Q_8^{886}, Q_8^{887}, Q_8^{888}, Q_8^{889}, Q_8^{890}, Q_8^{891}, Q_8^{892}, Q_8^{893}, Q_8^{894}, Q_8^{895}, Q_8^{896}, Q_8^{897}, Q_8^{898}, Q_8^{899}, Q_8^{900}, Q_8^{901}, Q_8^{902}, Q_8^{903}, Q_8^{904}, Q_8^{905}, Q_8^{906}, Q_8^{907}, Q_8^{908}, Q_8^{909}, Q_8^{910}, Q_8^{911}, Q_8^{912}, Q_8^{913}, Q_8^{914}, Q_8^{915}, Q_8^{916}, Q_8^{917}, Q_8^{918}, Q_8^{919}, Q_8^{920}, Q_8^{921}, Q_8^{922}, Q_8^{923}, Q_8^{924}, Q_8^{925}, Q_8^{926}, Q_8^{927}, Q_8^{928}, Q_8^{929}, Q_8^{930}, Q_8^{931}, Q_8^{932}, Q_8^{933}, Q_8^{934}, Q_8^{935}, Q_8^{936}, Q_8^{937}, Q_8^{938}, Q_8^{939}, Q_8^{940}, Q_8^{941}, Q_8^{942}, Q_8^{943}, Q_8^{944}, Q_8^{945}, Q_8^{946}, Q_8^{947}, Q_8^{948}, Q_8^{949}, Q_8^{950}, Q_8^{951}, Q_8^{952}, Q_8^{953}, Q_8^{954}, Q_8^{955}, Q_8^{956}, Q_8^{957}, Q_8^{958}, Q_8^{959}, Q_8^{960}, Q_8^{961}, Q_8^{962}, Q_8^{963}, Q_8^{964}, Q_8^{965}, Q_8^{966}, Q_8^{967}, Q_8^{968}, Q_8^{969}, Q_8^{970}, Q_8^{971}, Q_8^{972}, Q_8^{973}, Q_8^{974}, Q_8^{975}, Q_8^{976}, Q_8^{977}, Q_8^{978}, Q_8^{979}, Q_8^{980}, Q_8^{981}, Q_8^{982}, Q_8^{983}, Q_8^{984}, Q_8^{985}, Q_8^{986}, Q_8^{987}, Q_8^{988}, Q_8^{989}, Q_8^{990}, Q_8^{991}, Q_8^{992}, Q_8^{993}, Q_8^{994}, Q_8^{995}, Q_8^{996}, Q_8^{997}, Q_8^{998}, Q_8^{999}, Q_8^{1000}, Q_8^{1001}, Q_8^{1002}, Q_8^{1003}, Q_8^{1004}, Q_8^{1005}, Q_8^{1006}, Q_8^{1007}, Q_8^{1008}, Q_8^{1009}, Q_8^{1010}, Q_8^{1011}, Q_8^{1012}, Q_8^{1013}, Q_8^{1014}, Q_8^{1015}, Q_8^{1016}, Q_8^{1017}, Q_8^{1018}, Q_8^{1019}, Q_8^{1020}, Q_8^{1021}, Q_8^{1022}, Q_8^{1023}, Q_8^{1024}, Q_8^{1025}, Q_8^{1026}, Q_8^{1027}, Q_8^{1028}, Q_8^{1029}, Q_8^{1030}, Q_8^{1031}, Q_8^{1032}, Q_8^{1033}, Q_8^{1034}, Q_8^{1035}, Q_8^{1036}, Q_8^{1037}, Q_8^{1038}, Q_8^{1039}, Q_8^{1040}, Q_8^{1041}, Q_8^{1042}, Q_8^{1043}, Q_8^{1044}, Q_8^{1045}, Q_8^{1046}, Q_8^{1047}, Q_8^{1048}, Q_8^{1049}, Q_8^{1050}, Q_8^{1051}, Q_8^{1052}, Q_8^{1053}, Q_8^{1054}, Q_8^{1055}, Q_8^{1056}, Q_8^{1057}, Q_8^{1058}, Q_8^{1059}, Q_8^{1060}, Q_8^{1061}, Q_8^{1062}, Q_8^{1063}, Q_8^{1064}, Q_8^{1065}, Q_8^{1066}, Q_8^{1067}, Q_8^{1068}, Q_8^{1069}, Q_8^{1070}, Q_8^{1071}, Q_8^{1072}, Q_8^{1073}, Q_8^{1074}, Q_8^{1075}, Q_8^{1076}, Q_8^{1077}, Q_8^{1078}, Q_8^{1079}, Q_8^{1080}, Q_8^{1081}, Q_8^{1082}, Q_8^{1083}, Q_8^{1084}, Q_8^{1085}, Q_8^{1086}, Q_8^{1087}, Q_8^{1088}, Q_8^{1089}, Q_8^{1090}, Q_8^{1091}, Q_8^{1092}, Q_8^{1093}, Q_8^{1094}, Q_8^{1095}, Q_8^{1096}, Q_8^{1097}, Q_8^{1098}, Q_8^{1099}, Q_8^{1100}, Q_8^{1101}, Q_8^{1102}, Q_8^{1103}, Q_8^{1104}, Q_8^{1105}, Q_8^{1106}, Q_8^{1107}, Q_8^{1108}, Q_8^{1109}, Q_8^{1110}, Q_8^{1111}, Q_8^{1112}, Q_8^{1113}, Q_8^{1114}, Q_8^{1115}, Q_8^{1116}, Q_8^{1117}, Q_8^{1118}, Q_8^{1119}, Q_8^{1120}, Q_8^{1121}, Q_8^{1122}, Q_8^{1123}, Q_8^{1124}, Q_8^{1125}, Q_8^{1126}, Q_8^{1127}, Q_8^{1128}, Q_8^{1129}, Q_8^{1130}, Q_8^{1131}, Q_8^{1132}, Q_8^{1133}, Q_8^{1134}, Q_8^{1135}, Q_8^{1136}, Q_8^{1137}, Q_8^{1138}, Q_8^{1139}, Q_8^{1140}, Q_8^{1141}, Q_8^{1142}, Q_8^{1143}, Q_8^{1144}, Q_8^{1145}, Q_8^{1146}, Q_8^{1147}, Q_8^{1148}, Q_8^{1149}, Q_8^{1150}, Q_8^{1151}, Q_8^{1152}, Q_8^{1153}, Q_8^{1154}, Q_8^{1155}, Q_8^{1156}, Q_8^{1157}, Q_8^{1158}, Q_8^{1159}, Q_8^{1160}, Q_8^{1161}, Q_8^{1162}, Q_8^{1163}, Q_8^{1164}, Q_8^{1165}, Q_8^{1166}, Q_8^{1167}, Q_8^{1168}, Q_8^{1169}, Q_8^{1170}, Q_8^{1171}, Q_8^{1172}, Q_8^{1173}, Q_8^{1174}, Q_8^{1175}, Q_8^{1176}, Q_8^{1177}, Q_8^{1178}, Q_8^{1179}, Q_8^{1180}, Q_8^{1181}, Q_8^{1182}, Q_8^{1183}, Q_8^{1184}, Q_8^{1185}, Q_8^{1186}, Q_8^{1187}, Q_8^{1188}, Q_8^{1189}, Q_8^{1190}, Q_8^{1191}, Q_8^{1192}, Q_8^{1193}, Q_8^{1194}, Q_8^{1195}, Q_8^{1196}, Q_8^{1197}, Q_8^{1198}, Q_8^{1199}, Q_8^{1200}, Q_8^{1201}, Q_8^{1202}, Q_8^{1203}, Q_8^{1204}, Q_8^{1205}, Q_8^{1206}, Q_8^{1207}, Q_8^{1208}, Q_8^{1209}, Q_8^{1210}, Q_8^{1211}, Q_8^{1212}, Q_8^{1213}, Q_8^{1214}, Q_8^{1215}, Q_8^{1216}, Q_8^{1217}, Q_8^{1218}, Q_8^{1219}, Q_8^{1220}, Q_8^{1221}, Q_8^{1222}, Q_8^{1223}, Q_8^{1224}, Q_8^{1225}, Q_8^{1226}, Q_8^{1227}, Q_8^{1228}, Q_8^{1229}, Q_8^{1230}, Q_8^{1231}, Q_8^{1232}, Q_8^{1233}, Q_8^{1234}, Q_8^{1235}, Q_8^{1236}, Q_8^{1237}, Q_8^{1238}, Q_8^{1239}, Q_8^{1240}, Q_8^{1241}, Q_8^{1242}, Q_8^{1243}, Q_8^{1244}, Q_8^{1245}, Q_8^{1246}, Q_8^{1247}, Q_8^{1248}, Q_8^{1249}, Q_8^{1250}, Q_8^{1251}, Q_8^{1252}, Q_8^{1253}, Q_8^{1254}, Q_8^{1255}, Q_8^{1256}, Q_8^{1257}, Q_8^{1258}, Q_8^{1259}, Q_8^{1260}, Q_8^{1261}, Q_8^{1262}, Q_8^{1263}, Q_8^{1264}, Q_8^{1265}, Q_8^{1266}, Q_8^{1267}, Q_8^{1268}, Q_8^{1269}, Q_8^{1270}, Q_8^{1271}, Q_8^{1272}, Q_8^{1273}, Q_8^{1274}, Q_8^{1275}, Q_8^{1276}, Q_8^{1277}, Q_8^{1278}, Q_8^{1279}, Q_8^{1280}, Q_8^{1281}, Q_8^{1282}, Q_8^{1283}, Q_8^{1284}, Q_8^{1285}, Q_8^{1286}, Q_8^{1287}, Q_8^{1288}, Q_8^{1289}, Q_8^{1290}, Q_8^{1291}, Q_8^{1292}, Q_8^{1293}, Q_8^{1294}, Q_8^{1295}, Q_8^{1296}, Q_8^{1297}, Q_8^{1298}, Q_8^{1299}, Q_8^{1300}, Q_8^{1301}, Q_8^{1302}, Q_8^{1303}, Q_8^{1304}, Q_8^{1305}, Q_8^{1306}, Q_8^{1307}, Q_8^{1308}, Q_8^{1309}, Q_8^{1310}, Q_8^{1311}, Q_8^{1312}, Q_8^{1313}, Q_8^{1314}, Q_8^{1315}, Q_8^{1316}, Q_8^{1317}, Q_8^{1318}, Q_8^{1319}, Q_8^{1320}, Q_8^{1321}, Q_8^{1322}, Q_8^{1323}, Q_8^{1324}, Q_8^{1325}, Q_8^{1326}, Q_8^{1327}, Q_8^{1328}, Q_8^{1329}, Q_8^{1330}, Q_8^{1331}, Q_8^{1332}, Q_8^{1333}, Q_8^{1334}, Q_8^{1335}, Q_8^{1336}, Q_8^{1337}, Q_8^{1338}, Q_8^{1339}, Q_8^{1340}, Q_8^{1341}, Q_8^{1342}, Q_8^{1343}, Q_8^{1344}, Q_8^{134$

15. (and replace 16 by 15 throughout) Closure Properties of Classes of Groups. Groups of p -length 1.

(A) An in §8 (M) and §9.2, if \mathcal{X} is any class of groups, it is to be understood that (i) all unit groups belong to \mathcal{X} and (ii) if $G \in \mathcal{X}$ and if $G \cong G$, then $G \in \mathcal{X}$.

In considering a given class \mathcal{X} , it is usually desirable as a matter of routine to examine the closure properties of \mathcal{X} . The most useful of these closure properties will be denoted by the small capitals

D, E, N, P, A, R, S

and, if X is one of these, the following propositions state that \mathcal{X} has X or, as we shall also say, that \mathcal{X} is X -closed.

D. The direct product of two \mathcal{X} -groups is an \mathcal{X} -group.

E. Extensions of \mathcal{X} -groups by \mathcal{X} -groups are \mathcal{X} -groups i.e. $K \triangleleft G$, $K \in \mathcal{X}$ and $G/K \in \mathcal{X}$ together imply $G \in \mathcal{X}$.

N. Normal subgroups of \mathcal{X} -groups are \mathcal{X} -groups.

P. The product of two normal \mathcal{X} -subgroups of any group is an \mathcal{X} -group.

A. Radical groups of \mathcal{X} -groups are \mathcal{X} -groups.

R. A residual product of two \mathcal{X} -groups is an \mathcal{X} -group i.e. $K_1 \triangleleft G$, $G/K_1 \in \mathcal{X}$ (i.e. 1, 2) implies that $G/K_1 \cap K_2 \in \mathcal{X}$.

S. Subgroups of \mathcal{X} -groups are \mathcal{X} -groups.

It is clear that, if X is any one of these closure properties, then the intersection of any number of X -closed classes of groups is also X -closed. Hence, for an arbitrary class \mathcal{X} , there is a uniquely defined smallest X -closed class containing \mathcal{X} . This is called the X -closure of \mathcal{X} and will be denoted by $X\mathcal{X}$. The equation $\mathcal{X} = X\mathcal{X}$ simply states that \mathcal{X} is X -closed. The following facts are clear.

Lemma 16.11. Let \mathcal{X} be any class of groups and let G be any group. Then (i) $G \in D\mathcal{X} \Leftrightarrow G$ is a direct product of \mathcal{X} -subgroups.

(ii) $G \in E\mathcal{X} \Leftrightarrow G$ has a series whose factors are all \mathcal{X} -groups.

(iii) $G \in N\mathcal{X} \Leftrightarrow G$ can be subnormally embedded in an \mathcal{X} -group.

(iv) $G \in P\mathcal{X} \Leftrightarrow G$ is generated by subnormal \mathcal{X} -subgroups.

groups Q_{2^i} , ... have order of order 2, so K cannot be one of them. We conclude that K has a characteristic subgroup $M = \Omega_1(K)$ of order p^2 . Then $M \triangleleft G$ and so, as we have just seen, $[M, G'] = 1$. This contradicts the assumption that ZG' is cyclic.

Proof of 14.5. Since ZG' is a characteristic Abelian subgroup of G , it is cyclic, by hypothesis. Hence G' is cyclic by 14.51, and ZG' is a characteristic Abelian subgroup of G . So ZG' is cyclic too. Let H be a maximal cyclic subgroup of G containing ZG' . Then H is maximal and if $C = C_G(H)$, then ZC is a characteristic Abelian subgroup of G containing H , hence ZC is cyclic, and so $ZC = H$. Also $C' \cong G' \leq H = ZC$. If $C > H$, then C is of class 2 and $[C', C'] = [C', H]$ for all $\xi, \eta \in C$, by 7.1(1), (2). Let the Abelian group C/H be of type (τ_1, τ_2, \dots) , where $\tau_1 \geq \tau_2 \geq \dots$, and let H_1, H_2, \dots be a basis of C/H . Then we have $[x_i^{\tau_i}, x_j] = [x_i, x_j^{\tau_j}] = 1$ for $i, j = 2, 3, \dots$ and hence $[x_i^{\tau_i}, x_j] = 1$ for all i, j . If $\tau_1 > 1$, we have $[x_1^{\tau_1}, x_2^{\tau_2}] = [x_1, x_2^{\tau_2}] = 1$ for all i, j . If $L/H = \Omega_{\tau_1}(C/H)$, it follows that L has C , $L' = L$, $L > H$, hence L is cyclic, contrary to the choice of H . Hence $\tau_1 = 1$ and C/H is elementary, and so C' is of order p .

also holds in H .
 Lemma 16.12. (1) Let G be any group, and let \mathcal{C} be the class consisting of all groups of order 1 together with all groups isomorphic with them. $V(G) = V(\mathcal{C})$.

(2) A class \mathcal{C} of groups is v -closed if and only if $V(G, X, G_1) \subseteq \mathcal{C}$ for all $G, X, G_1 \in \mathcal{C}$.

Proof: (1) Obviously $V(\mathcal{C})$ is a v - and s -closed, hence v -closed. Hence $V(G) \subseteq V(\mathcal{C})$. Suppose conversely that $H = \langle Y, \dots, Z_n \rangle \in V(\mathcal{C})$. Consider the set F of all functions f of the form $f(x) = z_n$, where the arguments z_1, \dots, z_n range over G . Two such functions f, g are the same if and only if $f(z_1, \dots, z_n) = g(z_1, \dots, z_n)$ for all choices of the z_i 's in G . Hence $|F| \leq |G|^n$, where $n = |G|$. Moreover, F is a group, if we define the multiplication and inversion of functions in the obvious way; in fact, it is a subgroup of a certain Cartesian power of G , the factors being in one-to-one correspondence with the ordered multiplicity of elements of G . Hence $F \in \mathcal{C}$. Since $H = \langle Y, \dots, Z_n \rangle$, every element of H has the form $f(y_1, \dots, y_n)$ for some word f . Since $H \in V(\mathcal{C})$, for each word f , f gives the same value of H whenever $f(z_1, \dots, z_n) = f(z'_1, \dots, z'_n)$ for all choices of the z_i 's in G . Hence we have a homomorphism of H into F and so $H \in \mathcal{C}$. Thus $V(\mathcal{C}) = V(G)$. The v -closedness of $V(\mathcal{C})$ is proved.

(2) If $X = v\mathcal{C}$, then $V(G, X, G_1) \subseteq \mathcal{C}$ for all $G, X, G_1 \in \mathcal{C}$ by (1). Conversely, suppose this condition is satisfied. By 16.11(1), every $G \in v\mathcal{C}$ is isomorphic with a section of some group $H = G_1 X \dots X G_n$ with $G_i \in \mathcal{C}$, $i=1, \dots, n$. Hence $G \in V(H) = V(\mathcal{C})$ and by hypothesis $V(H) \subseteq \mathcal{C}$. Hence $G \in \mathcal{C}$ and so $v\mathcal{C} = \mathcal{C}$.

(3) Let f_1, f_2, \dots be any set of words. The class of all (f_i) 's for which the relation $f_1 = f_2 = \dots = 1$ holds identically will be called a variety of groups. Thus $V(\mathcal{C})$ is the smallest variety containing \mathcal{C} . Every variety is v -closed, but the converse is not true. We may define

(4) $G \in \mathcal{C} \iff G$ is a homomorphic image of an \mathcal{C} -group.
 (5) $G \in \mathcal{C} \iff G$ is a residual product of \mathcal{C} -groups in the normal subgroup K of G such that $G/K \in \mathcal{C}$ and $K \in \mathcal{C}$.

(6) $G \in \mathcal{C} \iff G$ can be embedded in an \mathcal{C} -group.
 (7) $G \in \mathcal{C} \iff G$ is isomorphic with a section of an \mathcal{C} -group.
 (8) $G \in \mathcal{C} \iff G$ can be embedded in the direct product of \mathcal{C} -groups.
 (9) $G \in \mathcal{C} \iff G$ is isomorphic with a section of a Cartesian product of \mathcal{C} -groups.

(10) $G \in \mathcal{C} \iff$ Every composition factor of G is a composition factor of some \mathcal{C} -group.

In the last four cases, $\mathcal{C} \iff \mathcal{C}$ means $x \in \mathcal{C}$ of course. This class is x -closed by definition, but need not be y -closed: in general, the mapping $x \rightarrow xy$ is not a closure operation in \mathcal{C} , considered as an operator on classes of groups, will not be idempotent. The smallest class which is both x - and y -closed and contains \mathcal{C} is in any case $\overline{\mathcal{C}}$.

But then we simplify in special cases. For example, (10) shows that $\mathcal{C} \iff \mathcal{C}$ is idempotent and is $\mathcal{C} \iff \mathcal{C}$ is both x - and y -closed. If we write $x \leq y$ to mean that $x \iff y$ and all classes \mathcal{C} , then we have $\mathcal{C} \iff \mathcal{C}$. Hence $\mathcal{C} \iff \mathcal{C}$. Similarly $\mathcal{C} \iff \mathcal{C}$, so that $\mathcal{C} \iff \mathcal{C}$ is also a closure operator. More important, so is

$v = \mathcal{C} \iff \mathcal{C}$.
 If G is any group, we define $V(G)$

to consist of all groups H with the following property: if $f(z_1, \dots, z_n) = 1$ for all choices of the z_i 's in G , then $f(z_1, \dots, z_n) = 1$ for all choices of the z_i 's in H . In other words, $H \in V(G)$ if and only if every law (or identical relation) holding in G

(C) Lemma 16.31 (1) If $\mathfrak{K} = p\mathfrak{K}$, then every group G has a uniquely determined maximal normal \mathfrak{K} -subgroup $\mathfrak{K}G$. If in addition $\mathfrak{K} = N\mathfrak{K}$, then $\mathfrak{K}H = H \cap \mathfrak{K}G$ for every subnormal subgroup H of G .

(V) If $\mathfrak{K} = N\mathfrak{K}$, then every group G has a uniquely determined normal subgroup $G^{\mathfrak{K}}$ such that $G/G^{\mathfrak{K}} \in \mathfrak{K}$ and that $G/K \in \mathfrak{K}$ implies $G^{\mathfrak{K}} \leq K$. If in addition $\mathfrak{K} = \alpha\mathfrak{K}$, then $(G/K)^{\mathfrak{K}} = KG^{\mathfrak{K}}/K$ for every $K \trianglelefteq G$.

(VI) If $\mathfrak{K} = \alpha\mathfrak{K} = \varepsilon\mathfrak{K}$, then $\mathfrak{K} = p\mathfrak{K}$ and every \mathfrak{K} -subgroup of G is contained in $\mathfrak{K}G$.

(VII) If $\mathfrak{K} = p\mathfrak{K} = \varepsilon\mathfrak{K}$, then $\mathfrak{K} = (G \text{ mod } \mathfrak{K}G) = \mathfrak{K}G$.

(VIII) If $\mathfrak{K} = N\mathfrak{K} = \varepsilon\mathfrak{K}$, then $(G^{\mathfrak{K}})^{\mathfrak{K}} = G^{\mathfrak{K}}$.

This is clear.

More interesting is

Theorem 16.32 Suppose that $\mathfrak{K} = p\mathfrak{K} = N\mathfrak{K}$ and that \mathfrak{K} contains some p -group $G \neq 1$. Then $G_p \leq \mathfrak{K}$ i.e. every p -group belongs to \mathfrak{K} .

Proof: G contains a subgroup C of order p and $C \leq \mathfrak{K}G$. Hence $C \in \mathfrak{K}$ since $\mathfrak{K} = N\mathfrak{K}$. But $\mathfrak{K} = p\mathfrak{K}$ and so every elementary p -group belongs to \mathfrak{K} . Now let H be any p -group and let $K = C \cup H$. Then the base group \mathfrak{T} of H is an elementary p -group with a basis u_i ($\alpha \in \mathbb{F}_p$) such that $\beta^u u_i \beta^{-1} = u_i \alpha \beta$ for all $\alpha, \beta \in H$. Then H is represented faithfully by automorphisms of \mathfrak{T} . Let $A = \text{Aut } \mathfrak{T}$ and let S be a Sylow p -subgroup of A . Then $H \cong H$, where H is some subgroup of S . Since $H, \text{Aut } S$ and $\mathfrak{K} = N\mathfrak{K}$, it will be sufficient to show that $S \in \mathfrak{K}$; for then it will follow that $H \in \mathfrak{K}$.

Let $|H| = n$ and let u_1, \dots, u_n be any basis of \mathfrak{T} . Let $C_2 = (u_1, \dots, u_n)$ so that $\mathfrak{T} = C_2 \triangleright C_1 \triangleright \dots \triangleright C_n = 0$, where \mathfrak{T} is written additively. If α is any automorphism of \mathfrak{T} which centralizes each C_i/C_{i+1} then $\alpha^{\mathfrak{T}} = u_i + t_i$ where $t_i \in C_{i+1}, \dots, u_n = u_i + t_i + \dots + u_n + t_n = u_n$ from choice of the form t_1, \dots, t_n , the elements $u_1 + t_1, \dots, u_n + t_n = u_n$ form a basis of \mathfrak{T} and so $u_i \rightarrow u_i + t_i$ ($i=1, 2, \dots, n$) defines an automorphism α of \mathfrak{T} of the kind described. Hence the group S of all such automorphisms

$V(\mathfrak{K})$ for any class \mathfrak{K} to be the smallest variety containing \mathfrak{K} . Then $G \in V(\mathfrak{K})$ if and only if every law which holds in all \mathfrak{K} -groups also holds in G .

Theorem 16.2 $V(\mathfrak{K}_p)$ contains all (finite) groups.

This seems to be due to Lazard. We prove it by showing that if f is given by (1), where $n > 0$ and $u_i = 1$ whenever $u_i = u_i$, i.e. whenever f is a non-trivial reduced word, then there exists a finite p -group in which the law $f = 1$ does not hold. We may write f in the form

$$f = x_1^{m_1} x_2^{m_2} \dots x_r^{m_r} \quad (r > 0)$$

where $x_i \neq 1$ and m_1, m_2, \dots, m_r are integers $\neq 0$. Let

$$m_i = p^{k_i} q_i, \quad k_i \geq 0, \quad (p \nmid q_i) = 1.$$

Let $k = \sum_{i=1}^r k_i$ and let R be an additive elementary Abelian p -group with a basis consisting of all the formal products

$$v = u_1^{a_1} u_2^{a_2} \dots u_r^{a_r} \quad (0 \leq a_i \leq k),$$

including as the case $s=0$ the empty product 1, where the indices i range from 1 to $r = \max k_i$. If $v' = u_1^{a_1} u_2^{a_2} \dots u_r^{a_r}$ we define $v v' = 0$ if $s+t > k$ and otherwise $v v' = u_1^{a_1+a_1'} u_2^{a_2+a_2'} \dots u_r^{a_r+a_r'}$. Extending this multiplication forms the basis elements of R by the distributive law to arbitrary elements, R becomes a ring. The elements of R of the form $1 + \sum \lambda_i v_i$, with integers λ_i , form a multiplicative group G since we have, for any $w = \sum \lambda_i v_i, w' = \sum \lambda_i' v_i$, $w w' = 0$ and so

$$(1+w)(1+w'+v) = 1 + (w+w') + wv = 1.$$

G is a group of order p^k where $g = 1 + \sum_{i=1}^r \lambda_i v_i + 1 + \lambda_i v_i$ and contains the elements $u_i = 1 + u_i$ ($i=1, 2, \dots, r$). Let $y = \sum_{i=1}^r \lambda_i v_i$, $y_1 = \sum_{i=1}^r \lambda_i v_i$. The coefficient of $u_1^{a_1} u_2^{a_2} \dots u_r^{a_r}$ in y is then precisely

$$\binom{m_1}{a_1} \binom{m_2}{a_2} \dots \binom{m_r}{a_r}$$

which is not divisible by p . Hence $y \neq 1$ and the law $f = 1$ does not hold in

This 16.2 is proved.

Note that $\mathfrak{K} \rightarrow V(\mathfrak{K})$ is a closure operation. But it lacks the finishing character of the forming operations D, E, \dots or V .

of order $p^{(3)} = |A|_p$, and so S is a Sylow p -subgroup of A .
 Let S_i be the group of all $\alpha \in S$ which centralize both C_i and Z/C_i .
 Clearly $S_i \leq S$ and S_i is an abelian subgroup of order $p^{i(i-1)}$.
 Thus $S_i \in \mathcal{K}$. Given $\alpha \in S$, define $\beta \in S_i$ by the condition $\alpha\beta^{-1} \in \alpha_i$.
 Then $\alpha\beta^{-1}$ leaves α_i invariant and induces on C_i an automorphism centralizing
 each C_{i-1}/C_i ($i=2,3,\dots,n$). By induction on n , we may assume that
 $\alpha\beta^{-1} \in S_{i-1}S_{i-2}\dots S_1$. Then $\alpha \in S_i S_{i-1} \dots S_1$. Hence $S = S_i S_{i-1} \dots S_1$.
 Since each $S_i \leq S$ and belongs to $\mathcal{K} = \mathcal{P}\mathcal{K}$, it follows that $S \in \mathcal{P}\mathcal{K}$ as stated.

(D) By 16.11 (ii), any class \mathcal{Y} of the form $\mathcal{E}\mathcal{A}\mathcal{N}\mathcal{Z}$ consists of all groups
 whose composition factors belong to a specified list of simple groups.
 Every such class \mathcal{Y} is closed with respect to all seven primary operations
 $\mathcal{D}, \mathcal{E}, \dots$ with the possible exception of \mathcal{E} . The most important classes
 of this kind are:

- $\mathcal{D}\mathcal{A}$ = all π -groups;
- $\mathcal{A}\mathcal{L}$ = all semisimple groups;
- $\mathcal{A}\mathcal{L}$ = all solvable groups.

In addition, the class of all π -separable groups \mathcal{S} and the class of all
 π -solvable groups are of this kind. And all these classes, with the exception
 of \mathcal{A} , are \mathcal{E} -closed.

The class

\mathcal{R} = all nilpotent groups

is closed with respect to all primary operations with the exception of \mathcal{E} ; with
 $\mathcal{R}\mathcal{L}$ = all Abelian groups
 is a typical variety and is closed with respect to all the primary operations
 except \mathcal{P} and \mathcal{E} .

Lemma 16.42. Let \mathcal{R} be any set of primes, let \mathcal{K} be any class of
 groups and let \mathcal{Y} be the class of all π -solvable groups whose Sylow-subgroups
 belong to \mathcal{K} . Then any of the seven primary closure properties which belong
 to \mathcal{K} are inherited by \mathcal{Y} .

For example, suppose that $G = HK$ is the product of two normal \mathcal{Y} -subgroups
 H and K . Then G is π -solvable and if S is any Sylow-subgroup of G ,
 then $S_i = S \cap H$ and $S_i = S \cap K$ are Sylow-subgroups of H and K respectively.
 Hence S_i and S_i belong to \mathcal{K} . But $S = S_i S_i$ and $S_i \leq S_i$ ($i=1,2$). Hence
 ~~$S \in \mathcal{K}$~~ if $\mathcal{K} = \mathcal{P}\mathcal{K}$, we have $S \in \mathcal{K}$ and so $G \in \mathcal{P}\mathcal{Y}$. Thus $\mathcal{Y} = \mathcal{P}\mathcal{Y}$.
 Exactly similar is the case $\mathcal{K} = \mathcal{R}\mathcal{K}$.

(E) If $\mathcal{X}_1, \dots, \mathcal{X}_n$ are any classes of groups, we define $\mathcal{X}_1 \mathcal{X}_2 \dots \mathcal{X}_n$ to be
 the class of all groups G with a series
 $G = G_0 \triangleright G_1 \triangleright G_2 \triangleright \dots \triangleright G_n = 1$
 such that $G_{i-1}/G_i \in \mathcal{X}_i$ for each i . This multiplication of classes is not
 in general associative. We have

$(\mathcal{X}_1 \mathcal{X}_2) \mathcal{X}_3 \subseteq \mathcal{X}_1 (\mathcal{X}_2 \mathcal{X}_3) = \mathcal{X}_1 \mathcal{X}_2 \mathcal{X}_3$.

Lemma 16.51. If $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ are any three classes of groups, the either of
 the following two conditions is sufficient to ensure that $(\mathcal{X}\mathcal{Y})\mathcal{Z} = \mathcal{X}(\mathcal{Y}\mathcal{Z})$:

- (i) $\mathcal{Y} = \mathcal{A}\mathcal{Y}$ and $\mathcal{Z} = \mathcal{P}\mathcal{Z}$;
 - (ii) $\mathcal{Y} = \mathcal{A}\mathcal{Y}$ and $\mathcal{Z} = \mathcal{R}\mathcal{Z}$.
- This is clear.

With regard to the inheritance of closure properties of \mathcal{K} and \mathcal{Y} by $\mathcal{X}\mathcal{Y}$,
 we have

Lemma 16.52. (i) If \mathcal{X} and \mathcal{Y} are both closed with respect to any of

$\mathcal{D}, \mathcal{S}, \mathcal{A}, \mathcal{N}$, then so is $\mathcal{X}\mathcal{Y}$.

(ii) If \mathcal{X} and \mathcal{Y} are π -closed and $\mathcal{K} = \mathcal{R}\mathcal{K}$, then $\mathcal{X}\mathcal{Y}$ is π -closed and
 $(\mathcal{X}\mathcal{Y})G = \mathcal{X}(G \text{ mod } \mathcal{Y}G)$.

(iii) If \mathcal{X} and \mathcal{Y} are \mathcal{R} -closed and $\mathcal{Y} = \mathcal{A}\mathcal{Y}$, then $\mathcal{X}\mathcal{Y}$ is \mathcal{R} -closed and
 $G\mathcal{X}\mathcal{Y} = (G\mathcal{X})\mathcal{Y}$.

For example, every π -solvable group G is contained in one of the classes
 $\mathcal{D}\mathcal{Y}, \mathcal{S}\mathcal{Y}, \mathcal{A}\mathcal{Y}, \mathcal{N}\mathcal{Y}, \dots, \mathcal{Y}_p(\mathcal{D}\mathcal{Y}, \mathcal{S}\mathcal{Y}, \mathcal{A}\mathcal{Y}, \mathcal{N}\mathcal{Y})^k$ for some integer k . The smallest
 such k is called the p -length of G and denoted by $l_p(G)$. By 16.12 (ii),
 the π -solvable groups of p -length at most l form a π -closed class, \mathcal{Y}_p^l , where

It follows that $N \cong H = \mathcal{O}_p G$. But clearly H contains every chief p -factor of G . Hence $N = H$.

Lemma 16.62. Every A -group and every metanilpotent group is of p -length ≤ 1 for all primes p .

Here, the class of metanilpotent groups is $\mathcal{N}^2 = \mathcal{N}\mathcal{N}$.

Proof: Let G be an A -group. We may assume that $\mathcal{O}_p G = 1$ since $G/\mathcal{O}_p G$ is an A -group of the same p -length as G . Let $H = \mathcal{O}_p' G$. Then every Sylow p -subgroup S of G contains H . Since S is Abelian, $S \leq C_G(H) \leq C_G(N) = N$ by 16.61 (1). This implies that $S \leq H$, since $H = \mathcal{O}_p' G$ owing to $\mathcal{O}_p G = 1$. Hence G/H is a p -group and so $\ell_p(G) \leq 1$.

Let G be metanilpotent. Again we may assume that $\mathcal{O}_p G = 1$. Then $F = \mathcal{O}_p' G$ is a p -group. Since $G \in \mathcal{N}^2$, G/F is nilpotent and if S/F is its Sylow p -subgroup. Then $S \leq G$ and G/S is a p' -group. Thus $S = \mathcal{O}_p G$ is a Sylow p -subgroup of G and $\ell_p(G) \leq 1$.

The class \mathcal{L}_p of solvable groups which are of p -length ≤ 1 for all p then contains many of the more interesting special kinds of solvable group e.g. supersoluble groups, Z -groups, complemented groups, etc.

Theorem 16.7. Let G be a p -soluble group. Then the following conditions are equivalent.

(1) $\ell_p(G) \leq 1$.

(2) Every p -subgroup P of G is a Sylow p -subgroup of some subnormal subgroup of G i.e. $P \leq P^* \leq G$, P^* is prime to p .

(3) Every nontrivial p -subgroup of G is a Sylow p -subgroup of some normal subgroup of G .

(4) The center is G of every chief p -factor of G is a p' -group.

Proof: (1) \Rightarrow (2). For let S be a Sylow p -subgroup of G containing P . Since $\ell_p(G) \leq 1$, there exist normal subgroups L and K of G such that $L = SK$ and $S \cap K = 1$. Hence $L/K \cong S$, and, since $P \leq S$, we have KP the $KS = L$ and so KP is G . But K is a p' -group and hence is a Sylow p -subgroup of KP .

$\mathcal{O}_p = \mathcal{O}_p' \mathcal{O}_p$; if $\mathcal{O}_p = \mathcal{O}_p' \mathcal{O}_p$, then \mathcal{O}_p is also a product p -closed class, as is \mathcal{O}_p' . Indeed the classes $\mathcal{O}_p', \mathcal{O}_p, \mathcal{O}_p', \mathcal{O}_p, \mathcal{O}_p, \mathcal{O}_p'$ are closed with respect to all the primary operators except \mathcal{E} .

If G is p -soluble of p -length 1, then the series of radicals

$$1 \leq \mathcal{O}_p' G \leq \mathcal{O}_p G \leq (\mathcal{O}_p' \mathcal{O}_p) G \leq \mathcal{O}_p' G \leq \dots \leq \mathcal{O}_p' G \leq (\mathcal{O}_p' \mathcal{O}_p) G$$

is called the upper p -series of G ; and the series

$$G \geq \mathcal{O}_p' G \geq \mathcal{O}_p' \mathcal{O}_p' G \geq \mathcal{O}_p' \mathcal{O}_p' \mathcal{O}_p' G \geq \dots \geq \mathcal{O}_p' G \geq \mathcal{O}_p' \mathcal{O}_p' G = 1$$

is called the lower p -series of G . Upper and lower p -series of a p -soluble group are related rather like upper and lower central series of a nilpotent group. The k -th term from the right in (1) contains the k -th but not the $(k+1)$ -th term from the right in (2).

Note that \mathcal{O}_p is the class of all groups with a normal \mathcal{O}_p -subgroup.

(F) Lemma 16.61 Let G be a p -soluble group.

(1) If $H = \mathcal{O}_p G$ and $K = \mathcal{O}_p(H)$, then $C_G(H/K) \leq H$.

(2) The intersection of the centralizers of all chief p -factors of G is H .

Proof: (1) Let $M = \mathcal{O}_p' G$. Then H/M is a p -group and K/M is its Frattini subgroup. Clearly, we may assume that $M = 1$. Let $C = C_G(H/K)$ and suppose if possible that $C \not\leq H$. Then there is a chief factor L/H of G such that $L \leq C$. (Note that $H \leq C$ since H/K is Abelian). Since $H = \mathcal{O}_p G$, L/H is a p' -group and L also is \mathcal{O}_p -subgroup S and that $L = SH$. Since $L \leq C$, we have $[H, S] \leq K$ and so $[H, S] = 1$ by 13.2 (ii), since by hypothesis $M = 1$ and so H is a p -group. Hence L is the direct product of S and H , S char L , $S \leq G$, $S \leq M = \mathcal{O}_p' G$, contrary to $M = 1$. We conclude that $H = C$.

(2) If N is the intersection of the centralizers of the chief p -factors of G , then every chief p -factor of N is a central factor of N . Let $M_i = \mathcal{O}_p N$, $M_i = \mathcal{O}_p(N)$, $M_i = \mathcal{O}_p(N)$. By (1), if $M_i \leq N$, then M_i is a p' -element of N which does not centralize N_i/K_i . But N_i/K_i is an elementary p -group. Hence there must be some chief p -factor L/M of N with $K_i \in M \leq L \leq M_i$ which is not centralized by M_i . This is a contradiction. Hence $N = \mathcal{O}_p N$. Since $N \leq G$

(ii) \Rightarrow (iii). Suppose T is a pronormal p -subgroup of G . By hypothesis, $|P^{G^*}|, |P|$ is prime to p . But by 6.65, the subnormal closure $P^{G^*} \triangleleft P$ in G coincides with its normal closure $K = \langle P^G \rangle$. Hence T is a Sylow p -subgroup of the normal subgroup K of G .

(iii) \Rightarrow (i). Here we need the Lemma 16.71. Let G be a p -soluble group and suppose that $\mathcal{L}_p(G/K) \triangleleft \mathcal{L} = \mathcal{L}_p(G)$ for all normal subgroups $K \neq 1$ of G . Then G is monolithic; its unique minimal normal subgroup M is an elementary Abelian p -group and $G = MS$ where S is a subgroup complementary to M in G , $M \cap S = 1$.

Proof: Suppose G had two different minimal normal subgroups M and M_1 . By hypothesis, G/M and G/M_1 are of p -length at most $t-1$. But $M \cap M_1 = 1$ and the class $\mathcal{L}_p(G^{G^*})$ is κ -closed. Hence $\mathcal{L}_p(G) \leq t-1$, a contradiction. Hence G is monolithic, and

Since G is p -soluble, M is either an elementary Abelian p -group or else a p' -group. The second case is excluded since it would make $\mathcal{L}_p(G/M) = \mathcal{L}_p(G)$. Let $L = \mathcal{O}_p'(G \text{ and } M)$. Then $L > M$ since otherwise we should again have $\mathcal{L}_p(G/M) = \mathcal{L}_p(G)$. Hence $L = MT$, where T is an S -subgroup of L . Let $S = N_G(T)$. Then $MS = G$ by the intersection of T in L . Hence $M \cap S \triangleleft MS = G$, which since M is Abelian. If $M \leq S$, then L would be the direct product of M and T and we should have T char L , $T \triangleleft G$, $T \neq 1$, contrary to the monolithic character of G . Hence $M \neq S$ and so $M \cap S = 1$, by the minimality of M as a normal subgroup of G . This proves 16.71.

Now let G satisfy (iii) of 16.7. Let $K \triangleleft G$ and choose K to be maximal subject to $\mathcal{L}_p(G/K) = \mathcal{L}_p(G)$. Then G/K satisfies the hypothesis of 16.71. By 6.68, every pronormal p -subgroup of G/K has the form KP/K where P is some pronormal p -subgroup of G . By hypothesis, there is a normal subgroup L of G such that P is a Sylow p -subgroup of L . Then KP/K is a Sylow p -subgroup of LK/K and $LK/K \triangleleft G$. Thus T also satisfies (iii). Hence we may assume without loss of generality


that $G = T = MS$, $M \cap S = 1$, where M is a minimal normal subgroup of G and $|M| = p^m$. Suppose if possible that p divides $|S|$ and let P_0 be a Sylow p -subgroup of S . Since M is a p -group, we have $P_0 \in \{M, P\} \triangleleft M$. Now $C = C_G(M) \triangleleft MS = G$ and, since G is monolithic, it follows that $C = 1$. But $P_0 \neq 1$, by hypothesis. Hence $P_0 \neq 1$. By 6.69, $H = PP_0$ is a pronormal p -subgroup of G . Since $P_0 \neq 1$, we have $M \leq \langle H \rangle$. Since $H \cap M = P_0 \triangleleft M$, it follows that H is not a Sylow p -subgroup of $\langle H \rangle$. This contradicts (ii). We conclude that S is a p' -group and so $\mathcal{L}_p(G) = \mathcal{L}$. (i) \Rightarrow (v). For if $\mathcal{L}_p(G) = \mathcal{L}$, there are normal subgroups L and K of G such that $L = KS$, $K \cap S = 1$ where S is a Sylow p -subgroup of G . Then every chief p -factor of G is incident with one of the form E/D where $K \leq D \leq E \leq L$. By 9.3 (v), $C_G(E/D)$ contains L . But G/L is a p' -group and so $A_G(E/D)$ is a p' -group.

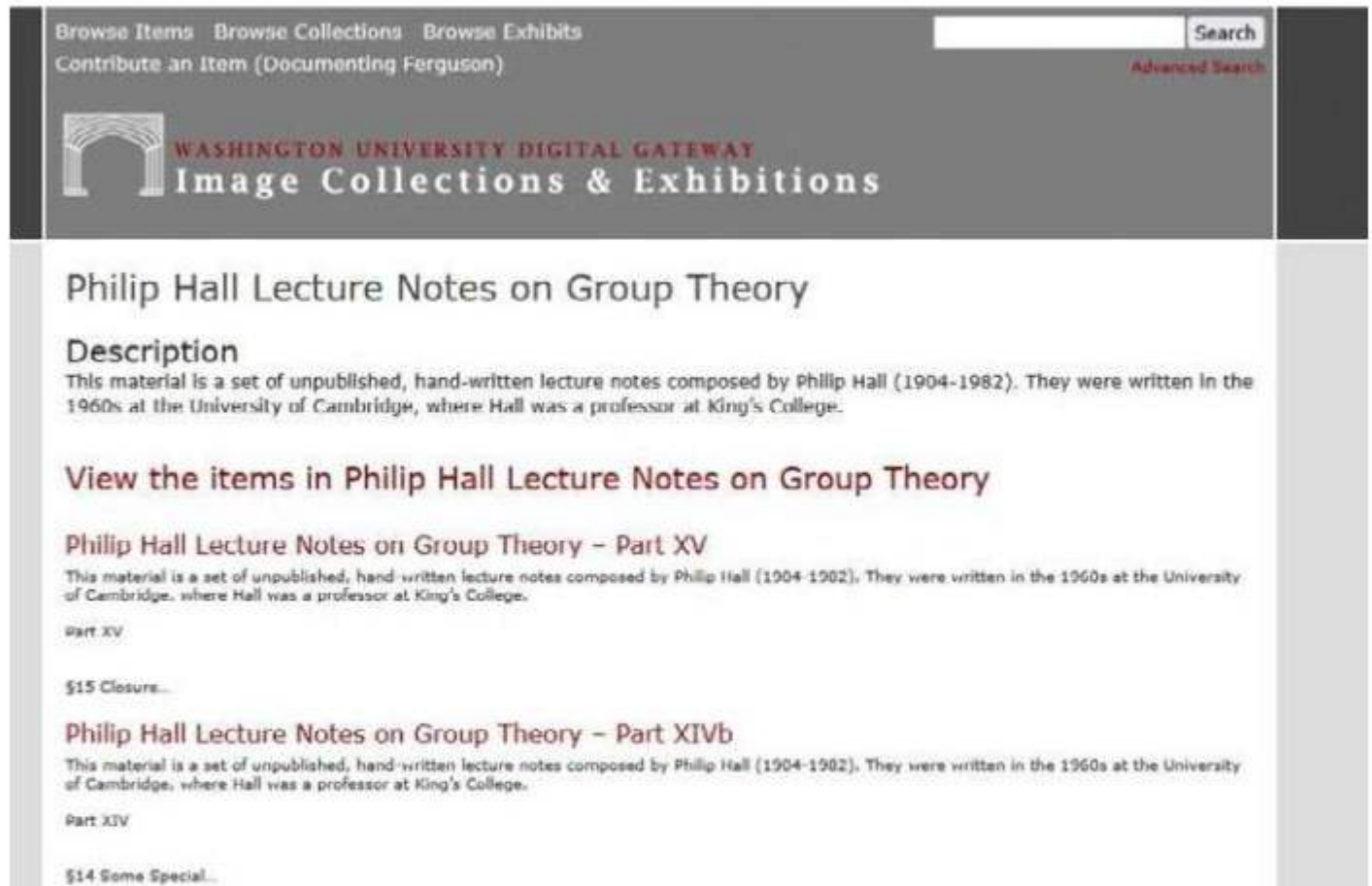
(v) \Rightarrow (i). Here, as in the proof of (ii) \Rightarrow (i), we replace G by a quotient group of smaller order having the same p -length as G . Then finally this quotient group still by G , we have $G = MS$, $M \cap S = 1$, where M is the unique minimal normal subgroup of G and so $C_G(M) = 1$. It follows that $S \triangleleft A_G(M)$. Hence S is a p' -group and $\mathcal{L}_p(G) = \mathcal{L}$.

Remark: The equivalence of 16.7 (i) and (ii) is proved in a rather more general form (not noticed by the hypothesis that G is p -soluble) by Wielandt.


(G) Lemma 16.8 (i) The class of all groups G with $\mathcal{L}_p G = 1$ in \mathcal{D} , \mathcal{E} , κ - κ - and κ -closed, but not p -, α - or s -closed

(B) Dually, the class of all groups G with $G = G^{G^*}$ is the groups which are generated by their κ -elements is \mathcal{D} -, \mathcal{E} -, p -, and α -closed, but not κ -, κ - or s -closed.

The URL http://omeka.wustl.edu/omeka/exhibits/show/philip_hall/ was available only until some time in 2024 before the Department of Mathematics of Washington University in St. Louis (WashU) , Missouri (see <https://math.wustl.edu/>), had to migrate the digital content (so-called Omeka migration of all items of <http://omeka.wustl.edu/omeka/>) of “WUSTL Digital Gateway Image Collections & Exhibitions”. The new URL is <https://wayback.archive-it.org/4726/20150221174119/http://omeka.wustl.edu/omeka/collections/show/71> which, however, is not easily accessible for everybody. We show the beginning of that Web page:



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Philip Hall Lecture Notes on Group Theory

Description

This material is a set of unpublished, hand-written lecture notes composed by Philip Hall (1904-1982). They were written in the 1960s at the University of Cambridge, where Hall was a professor at King's College.

View the items in Philip Hall Lecture Notes on Group Theory

Philip Hall Lecture Notes on Group Theory – Part XV

This material is a set of unpublished, hand-written lecture notes composed by Philip Hall (1904-1982). They were written in the 1960s at the University of Cambridge, where Hall was a professor at King's College.

Part XV

§15 Closure...

Philip Hall Lecture Notes on Group Theory – Part XIVb

This material is a set of unpublished, hand-written lecture notes composed by Philip Hall (1904-1982). They were written in the 1960s at the University of Cambridge, where Hall was a professor at King's College.

Part XIV

§14 Some Special...

Clicking on **Part XV** of the Lecture Notes, that is <https://wayback.archive-it.org/4726/20150221230212/http://omeka.wustl.edu/omeka/items/show/10799>, one gets the Web page

Philip Hall Lecture Notes on Group Theory – Part XV

Title

Philip Hall Lecture Notes on Group Theory – Part XV

Subject

Hall, Philip, 1904-1982

Description

This material is a set of unpublished, hand-written lecture notes composed by Philip Hall (1904-1982). They were written in the 1960s at the University of Cambridge, where Hall was a professor at King's College.

Part XV

§15 Closure Properties of Classes of Groups. Groups of p -length 1 (pp. 150-160)

Creator

Hall, Philip, 1904-1982

Source

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Date

1951-1967

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Hall, Philip, 1904-1982, "Philip Hall Lecture Notes on Group Theory – Part XV," *WUSTL Digital Gateway Image Collections & Exhibitions*, accessed February 21, 2015, <http://omeka.wustl.edu/omeka/items/show/10799>.

which allows one to download from <https://wayback.archive-it.org/4726/20150221230238/http://omeka.wustl.edu/omeka/archive/files/0aab7374c705b2d24168adb0cafc6042.pdf> §15 of the Lecture Notes, that is Pages 150 to 160. Similarly one gets the Web pages for **Part XIVb**, **Part XIVa** and **Part XIII** to **Part I** of the Lecture Notes, that is §14b and §14a (Pages 117 to 130) and §13 to §1 (Pages 1 to 116). All these Web pages have the same structure: Title, Subject, Description, Creator, Source, Date, Rights, Files, Collection and Citation, and differ only in Description, Files and Citation.



29.04.25, 05:35

Philip Hall (1904 - 1982) - Biography - MacTutor History of Mathematics



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Philip Hall

Quick Info

Born
11 April 1904
[Hamstead, London, England](#)

Died
30 December 1982
Cambridge, England

Summary
Philip Hall was the main impetus behind the British school of group theory and the growth of group theory to be one of the major mathematical topics of the 20th Century was largely due to him.



[View three larger pictures](#)

Biography

Philip Hall's father was George Hall and his mother was Mary Laura Sayers. They were not married and George left without making any provision for Laura or their newly born son Philip. Laura called herself Mrs Hall and she brought Philip up in her parents home in Hampstead, London, until he was seven years old. At the time Philip was born Laura was employed as dressmaker as was her twin sister Lois, and also her two elder sisters, who were also twins, Ada and Ethel. In 1909, while living in his grandfather Joseph Sayers' house, Philip entered New End Primary School.

In 1910 Laura, along with her three sisters, bought a house in Well Walk which they ran as a boarding house. Philip attended Sunday School and was baptised in 1911. He excelled at his Primary School and in 1915 he won a scholarship to Christ's Hospital West Horsham. This was a boarding school for children of ability whose parents were not able to afford the normal boarding school fees. Hall entered Christ's Hospital in May 1915 at age eleven. Although these were the years of World War I, Hall was too young to be affected by the war, other than serving in the Officer's Training Corps.

It was at Christ's Hospital that Hall came to love mathematics. He was fortunate to have teachers who were both accomplished mathematicians and also able to transmit their enjoyment of the subject. It was not only mathematics in which he excelled, winning the Gold Medal in his final year, but also in English winning a medal for the best English essay. Hall was House Captain in 1921-22, his final year at Christ's Hospital, and his popularity says much about his character [4]:-

He was kind and helpful to the younger boys, who respected and admired him; this was remarkable in those days for he was neither an extroverted leader nor any kind of athlete. He played rugby for his House as a rather clumsy forward, but gave up cricket, being content for a number of years to be official scorer for the school First Eleven.

His fellow school pupils recollected that at school he was [4]:-

... likeable and cheerful, with a sense of humour, gentle and reserved.

Hall went up to King's College Cambridge in October 1922 having won an Open Foundation Scholarship in December 1921. He wrote in his first letter home (see for example [4]):-

I am getting on beautifully and enjoying it very much; there are such opportunities of learning that it is about all you can do to make the most of them. ... I have made the acquaintance of Mr Littlewood and Mr Pollard, the two most progressive of the mathematicians here, so I am going on fairly well for a start.

The 'Mr Littlewood' that Hall referred to is [Dudley Littlewood](#). However, he fails to mention in this letter one other extremely promising mathematician in his year at Cambridge, namely [William Hodge](#). Among his teachers at Cambridge were [Hobson](#), the Sadleirian professor, and [Baker](#), the Lowndean professor of Astronomy and Geometry. [Richmond](#) was also on the staff when Hall arrived in Cambridge, but he retired in 1923.

Hall's interest in [group theory](#) came from [Burnside's](#) book which he was encouraged to read by Arthur Berry, the Assistant Tutor in Mathematics at King's College. Hall later wrote:-

I began with Berry's encouragement to study the works of [William Burnside](#), especially his magnificent treatise on the "Theory of Groups" and some of his later papers.

Hall offered parts of that book for examination in the Tripos and gave a proof that no group of order p^n , $n > 1$, can be simple. He graduated with a B.A. in 1925 and was elected to an Open Senior Foundation Scholarship which supported him for a further year at King's College. It was a year in which Hall wondered about his future, unsure whether to try for an academic career or not. He sat the Civil Service Examination in June 1926 which, if he had been successful, would have given him a fast route to the Administrative Grades. Fortunately for mathematics, and particularly group theory, he was not successful. He did spend some time on learning languages during this year; he spent the summer of 1925 in Italy learning Italian and studied German in London in March 1926.

In October 1926 Hall submitted an essay *The Isomorphisms of Abelian Groups* as his attempt to gain a Fellowship. It shows many signs of having been written hurriedly, even to the extent that it ends in the middle of a proof! It is fairly clear that Hall only made the decision to try for an academic career after much thought and late on in terms of writing up his dissertation. Despite its deficiencies, it shows that already Hall was way ahead of his time in his approach to group theory and certainly nobody at Cambridge could have been in a position to properly evaluate the work. It considers subgroups of $PGL(2, C)$ and, among other results, proves that a group of order p^n , $n > 1$, in which every characteristic abelian subgroup has order p , is the central product of non-abelian groups of order p^3 . [John Thompson](#) writes in [4] that the dissertation:-

... suffers from unwise use of the word 'obvious', a trait common to the young, but not always confined there. It is a trait which Hall did not retain.

Despite having written the work hurriedly, his quality shone through and Hall was elected to a Fellowship at King's College in March 1927. By that time he was already working as a research assistant to [Karl Pearson](#) in University College, London. He took up this post in January 1927 and his first published papers are on the theory of correlation. However, he found his main task of computing tables for the Incomplete Beta Function less than inspiring.

Hall wrote to [Burnside](#) in the summer of 1927 and, in 1942 he described this:-

I asked [Burnside's](#) advice on topics of group theory which would be worth investigation and received a postcard in reply containing valuable suggestions as to worth-while problems. ... shortly afterwards [Burnside](#) died. I never met him, but he has been the greatest influence on my ways of thinking.

Returning to Cambridge in September 1927 to take up the Fellowship at King's he made an important discovery in group theory, generalising the Sylow theorems for finite [soluble groups](#) to prove what are now called Hall's theorems. These fundamentally important results were published in a note on soluble groups in the *Journal of the London Mathematical Society* in 1928.

There is then a rather surprising gap in Hall's publication record. Here are his own words, written fifty years later:-

My Fellowship at King's had been renewed in 1930 but, sometime in 1931 it was intimated to me that a second renewal would be unlikely unless I showed signs of mathematical life; before then I had only produced one short note in 1928, so there was some justification in their warning and I obviously had to make a bit of an effort.

Hall certainly made 'a bit of an effort' for in 1932 he wrote what is perhaps his most famous paper *A contribution to the theory of groups of prime power order*. It is a beautiful paper which is one of the fundamental sources of modern group theory. In it, in addition to its main aims of developing the theory of regular p -groups, Hall introduces the commutator calculus, commutator collection, and the connection between p -groups and [Lie rings](#). Not only did he get his Fellowship renewed but in 1933 he was appointed as a Lecturer at Cambridge.

In June 1939 Hall gave a series of lectures at a small meeting at the Mathematical Institute in Göttingen. Four of Hall's lectures were published as separate papers in [Crelle's Journal](#). These papers are *Verbal and marginal subgroups*, *The classification of prime-power groups*, *On groups of automorphisms*, and *The construction of soluble groups* all of which appear in volume 182 published in 1940. In *The classification of prime-power groups* Hall introduces an equivalence relation called isoclinism to aid the classification of prime power groups. This important concept continues to play a major role. We should note that Hall was criticised for going to Germany at this difficult time but defended his actions saying:-

... the German mathematicians ... [are] as little responsible for the present situation (and probably enjoy it as little) as you or I do.

During World War II he made an important contribution with his work at the Code and Cypher School at Bletchley Park where he began work in September 1941. In particular he worked on Italian ciphers, then on Japanese ciphers learning about 1500 Japanese characters to help him in this task. During these war years he lived with his mother in Little Gaddesden where she had moved with her elder sister Ada at the beginning of the war in 1939 to be away from London. This meant that he had to travel about 20 miles to Bletchley Park each day and he made the 40 mile round trip partly by train and partly by motorcycle.

Hall returned to King's College Cambridge in July 1945. In 1946 he wrote letters to the authorities supporting [Hasse's](#) reinstatement, and also wrote encouraging letters to [Hasse](#) who had shown great kindness to Hall in 1939. Hall was promoted to Reader at Cambridge in 1949, then in 1953, after [Mordell](#) retired from the Sadleirian Chair, Hall was appointed to succeed him. In 1955 he was one of the main speakers at the Edinburgh Mathematical Colloquium in St Andrews where he gave five lectures on *Symmetric Functions in the Theory of Groups*. His picture at this colloquium is one of those in the set above.

Before giving his lectures he wrote to [Edge](#) saying:-

The subject I have in mind is symmetric functions, in relation to various branches of the theory of groups. I think I can find something to say on that which will not be too trite.

In particular he spoke about partitions and their connection to [representation theory](#):-

... whenever in mathematics you meet with partitions, you have only to turn over the stone or lift up the bark, and you will, almost infallibly, find symmetric functions underneath. More precisely, if we have a class of mathematical objects which in a natural and significant way can be placed in one-to-one correspondence with the partitions, we must expect the internal structure of these objects and their relations to one another to involve sooner or later ... the algebra of symmetric functions.

In 1956 Hall published, jointly with [Graham Higman](#), *On the p -length of p -soluble groups and reduction theorems for Burnside's problem*. This is a paper of major importance as was seen by [Baker](#) when he wrote a review saying that he could:-

... do no more than just indicate the wealth of material contained in this investigation.

The paper has indeed proved highly influential and much of the rapid development of group theory in the 1960s was built on this foundation. In August 1957 Hall gave a series of lectures at the Canadian Mathematical Congress Summer Seminar in Edmonton, Canada, on [nilpotent groups](#) which have had great influence ever since.

His major contribution to infinite groups is seen in highly significant papers of 1952, 1959 and 1961. The ideas of these papers continue to be one of the main areas of group theory research. For example *The Frattini subgroups of finitely generated groups* is the important paper on infinite groups which he published in 1961. In it Hall considers many different classes of groups and investigates whether the [Frattini](#) subgroup of groups in these classes needs to be nilpotent.

In *On non-strictly simple groups* published in 1963 Hall established the existence of simple groups which were the infinite union of a chain of subgroups, each normal in the next. The paper, like so many of Hall's papers, introduces important ideas which are widely applicable. Karl Gruenberg explains some further features of this paper:-

Besides containing a discussion of the possible order types of abelian series in simple groups, the paper also presents an extremely informative survey of the inter-relations that are known or conjectured to exist between the various classes of generalized soluble groups. This discussion is kept concise by the use of an elegant calculus of closure operations on group properties.

Hall received many honours for his work. He was elected to the [Royal Society](#) in 1942, then he was awarded its [Sylvester Medal](#) in 1961:-

... in recognition of his distinguished researches in algebra.

Hall was a great supporter of the [London Mathematical Society](#), and he was awarded its Senior Berwick Prize (1958) and the [De Morgan Medal](#) and Larmor Prize in 1965. He was elected President of the [London Mathematical Society](#) in 1955 and served the Society in this capacity until 1957. He delivered his Presidential address on 21 of November 1957 on *Some word-problems*. In this talk Hall spoke about word problems in general and specifically mentioned word problems for groups, semigroups, and cancellative semigroups. He introduced the idea of a normal form which he used in the solution of the word problem for [Lie rings](#) and also for nilpotent groups. He ended his talk with these words:-

Problems such as these still seem to present a formidable challenge to the ingenuity of algebraists. In spite of, or perhaps because of, their relatively concrete and particular character, they appear, to me at least, to offer an amiable alternative to the ever popular pursuit of abstractions.

The collected works of Philip Hall [2] were published in 1988. A McIver in a review wrote:-

This beautiful book consists of almost fifty years of publications by one of the greatest mathematicians of this century. ... Hall's elegant works (both in content and exposition) are allowed to speak for themselves ... however, the tremendous impact which his research has had on algebra is discussed... The reader glimpses a little of his character: his universal kindness and his invigorating enthusiasm both for mathematics and the world in general. ... Altogether we are presented with an all-around picture of a most remarkable mathematician.

We have made various comments about Hall's character in this article, but we should end by making a few more. He had a deep love of poetry which he recited beautifully in English, Italian or Japanese. He also loved music, art, flowers, and country walks. However, he was a rather shy man who avoided large gatherings and was only really happy in company when he was with one or two friends. When [Olga Tausky-Todd](#) accused him of being the worst recluse in Cambridge, Hall replied "No, [Turing](#) is worse"! He had an incredibly broad knowledge, not only of mathematics but, it seemed, on almost any subject [1]:-

Hall's range of knowledge was extraordinary, embracing anything from agriculture to poetry, ... combined with his complete integrity, high intellectual standards and sound judgement ...

Although a man of a few words, his comments were always significant. His modesty was clear when you spoke to him or heard him lecture as I [EFR] had the good fortune to do on several occasions. Roseblade, one of Hall's research students, writes in [4]:-

His students loved him and he them. Writing so lucidly and elegantly himself, he must have found painful much of what they first wrote; but whenever he had strong criticism to make of their work, he always found a way to soften the blow and never failed to suggest effective improvements. Nor did he abandon them when they had completed their dissertations: he wrote them helpful and stimulating letters, often very long and always by hand. ... He was a wonderful person; gentle, amused, kind, and the soul of integrity.

[Other Mathematicians born in England](#)
[A Poster of Philip Hall](#)

References [\(hide\)](#)

1. Obituary in *The Times*
See [THIS LINK](#)
2. P Hall, *The collected works of Philip Hall* (New York, 1988).
3. J L Brenner, Philip Hall - A famous mathematician, *Pi Mu Epsilon Journal* 9 (1990), 110-111.
4. J A Green, J E Roseblade and J G Thompson, Philip Hall, *Bull. London Math. Soc.* 16 (6) (1984), 603-626.

Additional Resources [\(hide\)](#)

Other pages about Philip Hall:

1. [Times obituary](#)
2. [Multiple entries in *The Mathematical Gazetteer of the British Isles*.](#)

Other websites about Philip Hall:

1. [Dictionary of National Biography](#)
2. [Royal Society obituary](#)
3. [Mathematical Genealogy Project](#)
4. [MathSciNet Author profile](#)
5. [zbMATH entry](#)

Honours [\(hide\)](#)

Honours awarded to Philip Hall

1. [Lecturer at the EMS](#)
2. [Fellow of the Royal Society 1942](#)
3. [BMC morning speaker 1949](#)
4. [LMS President 1955 - 1957](#)
5. [LMS Berwick Prize winner 1958](#)
6. [Royal Society Sylvester Medal 1961](#)
7. [LMS De Morgan Medal 1965](#)

Cross-references [\(hide\)](#)

1. [History Topics: A history of the Burnside problem](#)
2. [Other: 1958 ICM - Edinburgh](#)
3. [Other: Cambridge Colleges](#)
4. [Other: Cambridge Individuals](#)
5. [Other: Cambridge professorships](#)
6. [Other: Colloquium photo 1955](#)
7. [Other: Earliest Known Uses of Some of the Words of Mathematics \(M\)](#)
8. [Other: Edinburgh Mathematical Society Lecturers 1883-2024](#)
9. [Other: London Learned Societies](#)
10. [Other: London Schools](#)
11. [Other: London individuals H-M](#)

Written by J J O'Connor and E F Robertson
Last Update October 2003

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If you have comments, or spot errors, we are always pleased to [hear from you](#).

<http://mathshistory.st-andrews.ac.uk/Biographies/Hall/>

The screenshot shows a web browser displaying the Pantheon website for Philip Hall. The page features a teal background with a silhouette of a person. The title "MATHEMATICIAN PHILIP HALL" is centered, with the dates "1904 - 1982" below it. The navigation bar includes "Visualizations", "Rankings", "Portals", "PANTHEON", "About", "Data", "Games", and "Webbook". Below the main content, there are tabs for "MEMORABILITY METRICS", "AMONG MATHEMATICIANS", "CONTEMPORARIES", "IN UNITED KINGDOM", and "AMONG MATHEMATICIANS IN UNITED KINGDOM". A small portrait of Philip Hall is shown on the left, and a text block on the right provides a brief biography: "Philip Hall FRS (11 April 1904 – 20 December 1982), was an English mathematician. His major work was on group theory, notably on finite groups and solvable groups. [Read more on Wikipedia](#)". Below this, it states: "Since 2007, the English Wikipedia page of Philip Hall has received more than 70,785 page views. His biography is available in 16 different languages on Wikipedia (up from 12 in 2019). Philip Hall is the 840th most popular mathematician (down from 845th in 2019), the 4,796th most popular biography from United Kingdom (up from 4,853rd in 2019) and the 95th most popular British Mathematician."

WIKIPEDIA – Classification of finite simple groups

https://en.wikipedia.org/wiki/Classification_of_finite_simple_groups
26 languages

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The Free Encyclopedia

Classification of finite simple groups

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From Wikipedia, the free encyclopedia

In mathematics, the **classification of finite simple groups** (popularly called the **enormous theorem**^[17]) is a result of group theory stating that every finite simple group is either cyclic, or alternating, or belongs to a broad infinite class called the groups of Lie type, or else it is one of twenty-six exceptions, called *sporadic* (the Tits group is sometimes regarded as a sporadic group because it is not strictly a group of Lie type,^[2] in which case there would be 27 sporadic groups). The proof consists of tens of thousands of pages in several hundred journal articles written by about 100 authors, published mostly between 1965 and 2004.

Simple groups can be seen as the basic building blocks of all finite groups, reminiscent of the way the prime numbers are the basic building blocks of the natural numbers. The Jordan-Hölder theorem is a more precise way of stating this fact about finite groups. However, a significant difference from integer factorization is that such "building blocks" do not necessarily determine a unique group, since there might be many non-isomorphic groups with the same composition series, or, put in another way, the extension problem does not have a unique solution.

Daniel Coorenstein (1923-1992), Richard Lyons, and Ronald Solomon are gradually publishing a simplified and revised version of the proof.

Statement of the classification theorem [edit]

Main article: List of finite simple groups

Theorem—Every finite simple group is isomorphic to one of the following groups:

- a member of one of four infinite classes of such, namely:
 - the cyclic groups of prime order,
 - the alternating groups of degree at least 5,
 - the groups of Lie type,
 - the derived subgroup of the groups of Lie Type, such as the Tits group^[18] 1)
- or one of 26 groups called the "sporadic groups"

The classification theorem has applications in many branches of mathematics, as questions about the structure of finite groups (and their action on other mathematical objects) can sometimes be reduced to questions about finite simple groups. Thanks to the classification theorem, such questions can sometimes be answered by checking each family of simple groups and each sporadic group.

Daniel Coorenstein announced in 1963 that the finite simple groups had all been classified, but this was premature as he had been misinformed about the proof of the classification of *quasithin* groups. The completed proof of the classification was announced by Aschbacher (2004) after Aschbacher and Smith published a 1221-page proof for the missing quasithin case.

Overview of the proof of the classification theorem [edit]

Coorenstein (1962, 1963) wrote two volumes outlining the low rank and odd characteristic part of the proof, and Michael Aschbacher, Richard Lyons, and Stephen D. Smith et al. (2011) wrote a 3rd volume covering the remaining characteristic 2 case. The proof can be broken up into several major pieces as follows:

Groups of small 2-rank

Groups of component type

Groups of characteristic 2 type

Existence and uniqueness of the simple groups

Group theory

Basic notions [show]

Finite groups [hide]

Cyclic group Z_n · Symmetric group S_n · Alternating group A_n · Dihedral group D_n · Quaternion group Q · Cauchy's theorem · Lagrange's theorem · Sylow theorems · Hall's theorem · p -group · Elementary abelian group · Frobenius group · Burnside's theorem

Classification of finite simple groups

cyclic · alternating · Lie type · sporadic

Discrete groups - Lattices [show]

Topological and Lie groups [show]

Algebraic groups [show]

The classification of the finite simple groups

Look at https://en.wikipedia.org/wiki/Classification_of_finite_simple_groups for the full article and see also [61] and [61.1] to [61.14]. We introduce the *family \mathcal{T} of types of known finite simple groups* by using some assumed well-known symbols (see **Page 5**): $\mathcal{T} := \{ \text{abelian}_p, \underline{A}^n, A = \text{PSL}_n, B = \text{P}\Omega_{\text{odd } n}, C = \text{PSp}_n, D = \text{P}\Omega_{\text{even } n}^+, {}^2A = \text{PSU}_n, {}^2D = \text{P}\Omega_{\text{even } n}^-, E_6, E_7, E_8, F_4, G_2, {}^2B_2, {}^3D_4, {}^2E_6, {}^2F_4, {}^2G_2, \text{sporadic } \star \}$. These are 19 families and 7 have unbounded rank.

History of the proof

Gorenstein's program



In 1972 Daniel Gorenstein (1979, Appendix) (see [23] and also <https://www.nasonline.org/wp-content/uploads/2024/06/gorenstein-daniel.pdf>) announced quite (self-)assertively a truly ambitious program for completing the classification of finite simple groups, consisting of the following 16 steps:

- 1. Groups of low 2-rank.** This was essentially done by Gorenstein and Harada, who classified the groups with sectional 2-rank at most 4. Most of the cases of 2-rank at most 2 had been already done by the time Gorenstein announced his program.
- 2. The semisimplicity of 2-layers.** The problem is to prove that the 2-layer of the centralizer of an involution in a simple group is semisimple.
- 3. Standard form in odd characteristic.** If a group has an involution with a 2-component that is a group of Lie type of odd characteristic, the goal is to show that it has a centralizer of involution in “standard form”, meaning that a centralizer of involution has a component that is of Lie type in odd characteristic and also has a centralizer of 2-rank 1.
- 4. Classification of groups of odd type.** The problem is to show that if a group has a centralizer of involution in “standard form” then it is a group of Lie type of odd characteristic. This was solved by Aschbacher's classical involution theorem.
- 5. Quasi-standard form**
- 6. Central involutions**
- 7. Classification of alternating groups**
- 8. Some sporadic groups**
- 9. Thin groups.** The simple thin finite groups, those with 2-local p -rank at most 1 for odd primes p , were classified by Aschbacher in 1978.
- 10. Groups with a strongly p -embedded subgroup for p odd**
- 11. The signalizer functor method for odd primes.** The main problem is to prove a signalizer functor theorem for nonsolvable signalizer functors. This was solved by McBride in 1982.
- 12. Groups of characteristic p type.** This is the problem of groups with a strongly p -embedded 2-local subgroup with p odd, which was handled by Aschbacher.
- 13. Quasithin groups.** A quasithin group is one whose 2-local subgroups have p -rank at most 2 for all odd primes p , and the problem is to classify the simple ones of characteristic 2 type. This was completed by Aschbacher and Smith in 2004.
- 14. Groups of low 2-local 3-rank.** This was essentially solved by Aschbacher's trichotomy theorem for groups G with $e(G) = 3$. The main change is that 2-local 3-rank is replaced by 2-local p -rank for odd primes.
- 15. Centralizers of 3-elements in standard form.** This was essentially done by the trichotomy theorem.
- 16. Classification of simple groups of characteristic 2 type.** This was handled by the Gilman–Griess theorem, with 3-elements replaced by p -elements for odd primes.

Timeline of the proof

Many of the items in the table below are taken from Solomon's great history paper (2001) (see [61.4]). The date given is usually the publication date of the complete proof of a result, which is sometimes several years later than the proof or the first announcement of the result, so some of the items appear in the “wrong” order.

Date Development

- 1832 Galois introduces normal subgroups and finds the simple groups A_n ($n \geq 5$) and $\text{PSL}_2(F_p)$ ($p \geq 5$).
- 1854 Cayley defines abstract groups.
- 1861 Mathieu describes the first two Mathieu groups M_{11} and M_{12} , the first sporadic simple groups, and announces the existence of M_{24} .
- 1870 Jordan lists some simple groups, the alternating and the projective special linear ones, and emphasizes the importance of the simple groups.
- 1872 Sylow proves the Sylow theorems.
- 1873 Mathieu introduces three more Mathieu groups: M_{22} , M_{23} , and M_{24} .
- 1892 Hölder proves that the order of any nonabelian finite simple group must be a product of at least four (not necessarily distinct) primes, and asks for a classification of finite simple groups.
- 1893 Cole classifies simple groups of order up to 660.
- 1896 Frobenius and Burnside begin the study of character theory of finite groups.
- 1899 Burnside classifies the simple groups such that the centralizer of every involution is a non-trivial elementary abelian 2-group.
- 1901 Frobenius proves that a Frobenius group has a Frobenius kernel, so in particular is not simple.

- 1901 Dickson defines classical groups over arbitrary finite fields, and exceptional groups of type G_2 over fields of odd characteristic.
- 1901 Dickson introduces the exceptional finite simple groups of type E_6 .
- 1904 Burnside uses character theory to prove Burnside's theorem that the order of any non-abelian finite simple group must be divisible by at least 3 distinct primes.
- 1905 Dickson introduces simple groups of type G_2 over fields of even characteristic.
- 1911 Burnside conjectures that every non-abelian finite simple group has even order.
- 1928 Philip Hall proves the existence of Hall subgroups of solvable groups.
- 1933 Philip Hall begins his study of p -groups.
- 1935 Brauer begins the study of modular characters.
- 1936 Zassenhaus classifies finite sharply 3-transitive permutation groups.
- 1938 Fitting introduces the Fitting subgroup and proves Fitting's theorem that for solvable groups the Fitting subgroup contains its centralizer.
- 1942 Brauer describes the modular characters of a group divisible by a prime to the first power.
- 1954 Brauer classifies simple groups with $GL_2(F_2)$ as the centralizer of an involution.
- 1955 The Brauer–Fowler theorem implies that the number of finite simple groups with given centralizer of involution is finite, suggesting an attack on the classification using centralizers of involutions.
- 1955 Chevalley introduces the Chevalley groups, in particular introducing the new exceptional simple groups of types F_4 , E_7 , and E_8 .
- 1956 The Philip Hall–Graham Higman theorem describes the possibilities for the minimal polynomial of an element of prime power order for a representation of a p -solvable group and bounds groundbreakingly its p -length λ_p by the Sylow p -invariants order p^{b_p} of a Sylow p -subgroup, its nilpotency class c_p , its solubility length d_p , and its exponent p^{e_p} (see Page 98 to Page 100).
- 1957 Suzuki shows that all finite simple CA groups of odd order are cyclic.
- 1958 The Brauer–Suzuki–Wall theorem characterizes the projective special linear groups of rank 1, and classifies the simple CA groups.
- 1959 Steinberg introduces the Steinberg groups, giving some new finite simple groups, of types 2D_4 and 2E_6 (the latter were independently found at about the same time by Tits).
- 1959 The Brauer–Suzuki theorem about groups with generalized quaternion Sylow 2-subgroups shows in particular that none of them are simple.
- 1960 Thompson proves that a group with a fixed-point-free automorphism of prime order is nilpotent.
- 1960 Feit, Marshall Hall, and Thompson show that all finite simple CN groups of odd order are cyclic.
- 1960 Suzuki introduces the Suzuki groups, with types 2B_2 .
- 1961 Ree introduces the Ree groups with types 2F_4 and 2G_2 .
- 1963 Feit and Thompson prove the odd order theorem.
- 1964 Tits introduces BN pairs for groups of Lie type and finds the Tits group.
- 1965 The Gorenstein–Walter theorem classifies groups with a dihedral Sylow 2-subgroup.
- 1966 Glauberman proves the Z^* theorem.
- 1966 Janko introduces the Janko group J_1 , the first new sporadic group for about a century.
- 1968 Glauberman proves the ZJ theorem.
- 1968 Higman and Sims introduce the Higman–Sims group.
- 1968 Conway introduces the Conway groups.
- 1969 Walter's theorem classifies groups with abelian Sylow 2-subgroups.
- 1969 Introduction of the Suzuki sporadic group, the Janko group J_2 , the Janko group J_3 , the McLaughlin group, and the Held group happens.
- 1969 Gorenstein introduces signalizer functors based on Thompson's ideas.
- 1970 MacWilliams shows that the 2-groups with no normal abelian subgroup of rank 3 have sectional 2-rank at most 4. (The simple groups with Sylow subgroups satisfying the latter condition were later classified by Gorenstein and Harada.)
- 1970 Bender introduced the generalized Fitting subgroup.
- 1970 The Alperin–Brauer–Gorenstein theorem classifies groups with quasi-dihedral or wreathed Sylow 2-subgroups, completing the classification of the simple groups of 2-rank at most 2.
- 1971 Fischer introduces the three Fischer groups.
- 1971 Thompson classifies quadratic pairs.
- 1971 Bender classifies group with a strongly embedded subgroup.
- 1972 Gorenstein proposes a 16-step program for classifying finite simple groups (see above); the final classification follows his outline quite closely.
- 1972 Lyons introduces the Lyons group.
- 1973 Rudvalis introduces the Rudvalis group.
- 1973 Fischer discovers the baby monster group (unpublished), which Fischer and Griess use to discover the monster group, which in turn leads Thompson to the Thompson sporadic group and Norton to the Harada–Norton group (also found in a different way by Harada).

- 1974 Thompson classifies N -groups, groups all of whose local subgroups are solvable.
- 1974 The Gorenstein–Harada theorem classifies the simple groups of sectional 2-rank at most 4, dividing the remaining finite simple groups into those of component type and those of characteristic 2 type.
- 1974 Tits shows that groups with BN pairs of rank at least 3 are groups of Lie type.
- 1974 Aschbacher classifies the groups with a proper 2-generated core.
- 1975 Gorenstein and Walter prove the L-balance theorem.
- 1976 Glauberman proves the solvable signalizer functor theorem.
- 1976 Aschbacher proves the component theorem, showing roughly that groups of odd type satisfying some conditions have a component in standard form. The groups with a component of standard form were classified in a large collection of papers by many authors.
- 1976 O’Nan introduces the O’Nan group.
- 1976 Janko introduces the Janko group J_4 , the last sporadic group to be discovered.
- 1977 Aschbacher characterizes the groups of Lie type of odd characteristic in his classical involution theorem. After this theorem, which in some sense deals with “most” of the simple groups, it was generally felt that the end of the classification was in sight.
- 1978 Timmesfeld proves the O_2 extraspecial theorem, breaking the classification of groups of $GF(2)$ -type into several smaller problems.
- 1978 Aschbacher classifies the thin finite groups, which are mostly rank 1 groups of Lie type over fields of even characteristic.
- 1981 Bombieri uses elimination theory to complete Thompson’s work on the characterization of Ree groups, one of the hardest steps of the classification.
- 1982 McBride proves the signalizer functor theorem for all finite groups.
- 1982 Griess constructs the monster group by hand.
- 1983 The Gilman–Griess theorem classifies groups of characteristic 2 type and rank at least 4 with standard components, one of the three cases of the trichotomy theorem.
- 1983 Aschbacher proves that no finite group satisfies the hypothesis of the uniqueness case, one of the three cases given by the trichotomy theorem for groups of characteristic 2 type.
- 1983 Gorenstein and Lyons prove the trichotomy theorem for groups of characteristic 2 type and rank at least 4, while Aschbacher does the case of rank 3. This divides these groups into 3 subcases: the uniqueness case, groups of $GF(2)$ type, and groups with a standard component.
- 1983 Gorenstein announces the proof of the classification is complete, somewhat prematurely as the proof of the quasithin case was incomplete.
- 1985 Conway, Curtis, Norton, Parker, Wilson, and Thackray publish the Atlas of Finite Groups with basic information about 93 finite simple groups.
- 1994 Gorenstein, Lyons, and Solomon begin publication of the revised classification.
- 2004 Aschbacher and Smith publish their work on quasithin groups (which are mostly groups of Lie type of rank at most 2 over fields of even characteristic), filling the last gap in the classification known at that time.
- 2008 Harada and Solomon fill a minor gap in the classification by describing groups with a standard component that is a cover of the Mathieu group M_{22} , a case that was accidentally omitted from the proof of the classification due to an error in the calculation of the Schur multiplier of M_{22} .
- 2012 Gonthier and collaborators announce a computer-checked version of the Feit–Thompson theorem using the Coq proof assistant.

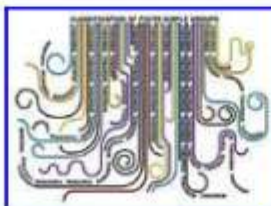
... .. 😊

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Look at https://en.wikipedia.org/wiki/Classification_of_finite_simple_groups for the full article and see also [61] and [61.1] to [61.14].



Prof. Otto H. Kegel and Prof. Philip Hall at the Mathematisches Forschungsinstitut Oberwolfach (MFO)

Prof. Otto H. Kegel has very frequently been at the famous Mathematisches Forschungsinstitut Oberwolfach (MFO) quite near Freiburg im Breisgau (see <https://www.mfo.de>) both as a guest and as a great speaker and notably as an organiser of fascinating conferences (see Page 69). The MFO (Oberwolfach Research Institute for Mathematics) is an international research centre situated in the German Black Forest. Leading experts from all over the world meet at the Institute in order to pursue their research activities, to discuss aptly recent developments in their field with others, and in particular to generate **new ideas** 😊.

The Institute is a member of the Leibniz Association and is funded jointly by the German federal government and the German state governments. It is closely related to the ancient Albert-Ludwigs-University of Freiburg i.Br. where Prof. Kegel happily occupied his chair from 1975 until 1999. Some of his visits meeting Group Theory researchers have been **beautifully** documented in the Oberwolfach Photo Collection (OPC) which we show with great pleasure:

https://opc.mfo.de/person_detail?id=2061

https://opc.mfo.de/person_detail?id=2061

Oberwolfach Photo Collection

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1 2

H. G. Tilmann O. H. Kegel (2012)	O. H. Kegel (2013)	O. H. Kegel (2010)	O. H. Kegel (2010)
V. Remmert O. H. Kegel (2010)	O. H. Kegel (2010)	H. E. Müller O. H. Kegel (2007)	O. H. Kegel D. Held (2007)
W. Kimmerle O. H. Kegel (2009)	G. Huiskes O. H. Kegel (2006)	H. Kuzuroglu B. Shumratskiy O. H. Kegel (2003)	O. H. Kegel (2002)

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2003



On the Photo:
 Eusebio de S. Toledo (left)
 Shrawvabir Patel (right)
 Bernd Ollig (middle)

Occasion: [EPC\(2003\): Research in Pairs 2003](#)
 Location: Oberwolfach
 Author: [Schmid, Renate](#) (photos provided by Schmid, Renate)
 Source: HFO
 Year: 2003
 Copyright: HFO

Photo ID: 7429

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1 2




O. H. Kegel
H. Lenz
(1982)




H. Kacber
B. Wroscheld
M. Bärner
W. Föllmer
W. Jaeger
H. Korte
T. Schneider
J. K. Moser
S. Hildebrandt
B. Büllrich
O. H. Kegel
K. P. Göttemeyer
G. Hammerlin
R. Lutz
R. Remmert
(1984)



J. Szpoc
A. Lashl
O. H. Kegel
(1984)



O. H. Kegel
(1980)



O. H. Kegel
(1978)

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1984



On the Photo:

Burton, Herman (front row, left)

Edwards, Richard (front row, 2. from left)

Barnes, Harold (front row, 2. from right)

Johnson, David (front row, right)

1984, 2001 (2. row, left)

Scott, David (2. row, 2. from left)

Schwartz, Thomas G. (2. row, 2. from right)

Thompson, John G. (2. row, right)

Edwards, Richard (3rd row, left)

Scott, David (3rd row, 2. from left)

Johnson, David (3rd row, 2. from right)

Thompson, John G. (3rd row, 4. from left)

Hammerlin, Günther (3rd row, 3. from right)

Lutz, Jürgen (3rd row, 3. from right)

Reichardt, Günther (3rd row, right)

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1978



On the Photo:

Lutz, Jürgen

Location: Berlin

Author: Jürgen, Bernd (photos provided by Jürgen, Bernd)

Source: Jürgen Lutz, Bielefeld

Year: 1978

Copyright: NFG

Photo ID: 2061

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1980



On the Photo:

Ehrhard, Arthel

Location: Ludwigsburg

Author: Arthel, Richard (photos provided by Arthel, Ehrhard)

Source: Arthel, Ehrhard

Year: 1980

Copyright: Arthel, Ehrhard

Photo ID: 23960

J Mathe & Comp Appli, 2025

Volume 4(3): 263–266

Prof. Philip Hall visited the MFO several times as well showing great interest in but not contributing to the Classification of Finite Simple Groups (CFSG). In the OPC is **beautifully** documented an overview of ten photographs of him, which we show with great pleasure together with seven of the photos.

https://opc.mfo.de/person_detail?id=1528

The screenshot shows the 'Oberwolfach Photo Collection' website. The page title is 'Details for Philip Hall'. On the left is a navigation menu with links for 'Main', 'Occasions', 'Persons', 'Links', and 'Legal Notice/Impressum'. Below the menu is a search bar with the text 'Search' and 'Extended Search'. The main content area displays a grid of ten photographs of Philip Hall. Each photo has a caption below it, such as 'P. Hall (1960)', 'P. Hall (1960)', 'P. Hall (1960)', 'P. Hall (1960)', 'P. Hall (1953)', 'P. Hall J. G. Thompson O. Tamaschka', 'P. Hall', 'P. Hall', 'J. G. Thompson P. Hall S. Huppert', and 'P. Hall'. At the bottom of the photo grid, there is a note: 'For more detailed information please click on the photos.'

https://opc.mfo.de/detail?photo_id=1531

The screenshot shows the 'Oberwolfach Photo Collection' website for a specific photo. The page title is '1960'. On the left is the same navigation menu as in the previous screenshot. The main content area features a large photograph of Philip Hall standing on a balcony. To the right of the photo is a section titled 'On the Photo:' with the following details: 'Hall, Philip', 'Location:', 'Author: Jacobs, Konrad (photos provided by Jacobs, Konrad)', 'Source: Konrad Jacobs, Erlangen', 'Year: 1960', 'Copyright: MFO', and 'Photo ID: 1531'.

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Author: Jacobs, Konrad (photos provided by Jacobs, Konrad)
Source: Konrad Jacobs, Birmingen
Year: 1960
Copyright: MFO
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Author: Jacobs, Konrad (photos provided by Jacobs, Konrad)
Source: Konrad Jacobs, Birmingen
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https://opc.mfo.de/detail?photo_id=11861

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1960



On the Photo:
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Author: Gatzert, Wilfried (photos provided by Gatzert, Wilfried)
Source: Wolfgang Gatzert
Copyright: Wolfgang Gatzert
Photo ID: 11861

https://opc.mfo.de/detail?photo_id=11857

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Location:
Author: [Gasschutz, Wilfried](#) (provided by [Gasschutz, Wilfried](#))
Source: [Wolfgang Gasschutz](#)
Copyright: [Wolfgang Gasschutz](#)
Photo ID: 11857

https://opc.mfo.de/detail?photo_id=11856

https://opc.mfo.de/detail?photo_id=11856

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On the Photo:
[Thomson, John Lewis](#) (MFO)
[Tull, John](#) (MFO)
[Tull, John](#) (MFO)
Location:
Author: [Gasschutz, Wilfried](#) (provided by [Gasschutz, Wilfried](#))
Source: [Wolfgang Gasschutz](#)
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Photo ID: 11856

https://opc.mfo.de/detail?photo_id=11862

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[Thomson, John Lewis](#) (MFO)
[Thomson, John](#) (MFO)
Location:
Author: [Gasschutz, Wilfried](#) (provided by [Gasschutz, Wilfried](#))
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Photo ID: 11862

Philip Hall Archive | London Mathematical Society

<https://www.lms.ac.uk/archive/philip-hall-archive>

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Archive of Philip Hall (1906-1982)

Philip Hall was elected to membership of the London Mathematical Society on 16 February 1928 and was Honorary Secretary from 1938 to 1941 and 1949 to 1948. Hall was elected the forty-seventh President of the Society in 1965. He received the Senior Serwick Prize in 1958 and the De Morgan Medal in 1965.

In his first paper on algebra he proved that the classical Sylow theorems extend to finite soluble groups to a new class of subgroups now known as Hall π -subgroups. Ten years later he produced a characterization of finite soluble groups based on this earlier discovery. In 1932 Hall wrote his famous 'A contribution to the theory of groups of prime-power order' which is one of the fundamental sources of modern group theory. Hall also made significant contributions to infinite group theory in the 1950s; the ideas of these papers continue to be one of the main areas of research in group theory. Hall supervised a large number of research students; many of them achieved academic distinction in algebra. J.S. Roseblade said of him: "His students loved him and he them."

When he died in 1982, an extensive collection of his correspondence, lecture notes and notebooks and Hall's Sylvester Medal were inherited by Dr Jim Roseblade, who offered the collection to the London Mathematical Society in 2007. In December 2007, Peter Harper of the former National Cataloguing Centre for the Archives of Contemporary Scientists (NCCACS) in Bath inspected the archive and reported that Cambridge University Library would be the best place to house the archive, where the collection now resides. The collection is now available for study in the library and anyone wishing to access it should first seek permission from the keeper of Archives and Modern Manuscripts.

Further details about the papers can be found at <http://www.maths.cam.ac.uk/interactions/2008archive/01/>

The London Mathematical Society, together with the Cambridge Philosophical Society and Kings College, Cambridge, contributed a total of £12,000 towards cataloguing the collection. Under the supervision of Imperial College Archivist, Anne Bartlett, and joint Project LMS Archival Consultant, the LMS Annual Library and Archive Studies Placement holders in 2014 listed a portion of the correspondence as well as sorting and arranging it by name of correspondent in chronological order.

For any inquiries regarding the collection, please email msa@lms.ac.uk

<https://www.lms.ac.uk/archive/philip-hall-archive>

"LECTURE NOTES ON GROUP THEORY", HANDWRITTEN BY PHILIP HALL (§ 1 - § 14)



<https://www.advgrouptheory.com/GTArchivum/Hall/HallLectureNotes.pdf>

Felix F. Flemisch fairly proudly received his first degree **Bacc.Math.** 🎓 in 1974 from the Albert-Ludwigs-Universität at lovely Freiburg im Breisgau, his postgraduate degree **M.Sc.** 🎓 in 1975 from the honourable University of London, UK, and finally his degree **Dipl.-Math.** 🎓 at marvellous and fabulous Freiburg i.Br. in 1985. From February 1981 until April 1985 he was quite happily affiliated to the Albert-Ludwigs-Universität Freiburg i.Br., Universitätsklinikum Freiburg, Institut für Medizinische Biometrie und Statistik (IMBI). Since May 1985 he was enthusiastically and with great joy working for the telecom industry. On April 11, 1992, he married beloved **Helga** in beautiful Florence in Tuscany in Italy ❤️. Since October 2016 he is retired and is still resp. is again loving to work on mathematics, in particular on the very beautiful Group Theory 🧠 😊.

This **Research Article** continues [15]. We begin with giving a quite profound overview of the structure of arbitrary simple groups and in particular of the simple locally finite groups and reduce their Sylow theory for the prime p to a rather famous conjecture by **Prof. Otto H. Kegel** (see [44], Theorem 2.4: “Let the p -subgroup P be a p -uniqueness subgroup in the finite simple group S which belongs to one of the seven rank-unbounded families. Then the rank of S is bounded in terms of P .”) about the **rank-unbounded ones** of the 19 known families of finite simple groups. We introduce a **new scheme** to describe the known 19 families, the **family \mathcal{T} of types**, define the **rank** of each type, and emphasise the rôle of **Kegel covers**. This part presents a unified picture of known results all of whose proofs are by reference.

Subsequently we apply **new ideas** to prove the conjecture for the **Alternating Groups**.

Thereupon we are remembering Kegel covers and \star -sequences. Next we suggest some **future research** by stating a **way 1)** and a **way 2)** how to prove and even how to optimise Kegel’s conjecture step-by-step or peu à peu which is leading to **Conjecture 1**, to **Conjecture 2** and to **Conjecture 3** thereby unifying *Sylow theory in locally finite simple groups* with *Sylow theory in locally finite and p -soluble groups* whose joint study directs very reliably the Sylow theory in (locally) finite groups. For any unexplained terminology we refer to [15].

We then continue the program begun above to optimise along the **way 1)** the theorem about the **first type** $\Xi = “\mathbf{A}^n”$ of infinite families of finite simple groups step-by-step to further types by proving it for the **second type** $\Xi = “\mathbf{A} = \text{PSL}_n”$. We start with applying **new ideas** to prove **Conjecture 2** about the **General Linear Groups** over (commutative) locally finite fields, stating that their rank is bounded in terms of their p -uniqueness, and then break down this insight to the **Special Linear Groups** and to the **Projective Special Linear (PSL) Groups** over locally finite fields. We close with a number of suggestions for **future research** ► regarding the remaining five rank-unbounded types (the “Classical Groups”) and the **way 2)**, ► regarding (locally) finite and p -soluble groups, and ► regarding Cauchy’s and Galois’ contributions to Sylow theory in finite groups. **We hope to enthuse group theorists with them and are ready to coordinate related work.**

We include the beautiful predecessor research paper [15] as **Appendix 1** for good reasons.

The **Research Article** has the following **seventeen beautiful Chapters**:

Sketch of proof for \mathbf{A}^n ; Sketch of proof for $\mathbf{A} = \text{PSL}_n$; 1 Introduction; 2 Proof of Theorem 1; 3 About Kegel covers; 4 Planning future research – Part 1; 5 Proof of Theorem 2; 6 Proof of Theorem 3; 7 Proof of Theorem 4; 8 Planning future research – Part 2; 9 The First Trilogy and The Second Trilogy and their reviews; Acknowledgements; Postscript, Luciano De Crescenzo, Felix F. Flemisch, Conflicts of Interest, Pablo Picasso’s *La Joie de vivre*; About the author in Munich, in Freiburg i.Br., in London, in Weiden i.d.OPf., and in Florence in Tuscany in Italy; 75 References; Appendix 1 – Reference [15] with MR Review and Zbl Review; Appendix 2 – Talk at IGT 2024 on Thursday, April 11, the 120th birthday of Prof. Philip Hall.



May 2025 Issue

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