ISSN: 2754-6705

Journal of Mathematical & Computer Application

SCIENTIFIC Research and Community

Research Article

Open Access

Characterising locally finite groups satisfying the strong Sylow Theorem for the prime *p* – Mathematics edition – Second edition

Dipl.-Math. Felix F. Flemisch, M.Sc., Bacc.Math. *

Mitterweg 4e, 82211 Herrsching a. Ammersee, Bavaria, Germany

Abstract

We turn a question of Otto H. Kegel [1] into a theorem [2]: *If every subgroup S of the locally finite group G contains a finite p-subgroup which is singular in S* (see below), *then G satisfies the strong Sylow Theorem for the prime p* (see below). Since the converse is also true [1; 2], this characterises the locally finite groups which satisfy the strong Sylow Theorem for the prime p. The proof is a rather old result, namely the *Charakterisierungssatz* of the author's Diplomarbeit of 1984 [2]. It is based upon the concepts of *good p-subgroups* [3] and *strongly local* – or, as we say, *very good* – *p-subgroups* [4] of Andrew Rae. To obtain the correspondence with Kegel's work we first show that a finite p-subgroup is singular in the locally finite group G if and only if it is a *p-uniqueness subgroup* of G, that is, it is contained in a unique maximal p-subgroup of G. We then proceed on the whole as in our Diplomarbeit. However, we do not publish the proof in its original form since the Diplomarbeit was written in German. Instead we translated all employed parts into English, thereby introducing a number of corrections, present the *Charakterisierungssatz* as 2.8 HAUPTSATZ, and summarise then the result as Theorem IV. Subsequently we present a few new concepts and results for Sylow theory in (locally) finite groups and a comparative outline of Kegel's previous work [8] with two open questions. We close with personal dedications, background information and deep Endnotes which introduce further new concepts. The publication has the following seven Chapters: Introduction; **1.** Good Sylow p-subgroups and p-uniqueness subgroups; **2.** Basic theorems of Sylow theory in locally finite groups and our *Charakterisierungssatz*; **3.** Novel concepts and results for Sylow theory in (locally) finite groups and comparative outline of Kegel's previous work [8]; References; About the author & Background information and annotations; Endnotes.

Mathematics Subject Classification (MSC2020)

Primary 20D20 • 20F50 • 20D15; Secondary 20E25 • 20D05 • 20E32 • 20E34

*** Corresponding author**

Felix F. Flemisch, Mitterweg 4e, 82211 Herrsching a. Ammersee, Bavaria, Germany.

e-mail: felix.flemisch@hotmail.com. ORCID: https://orcid.org/0000-0003-1612-8810.

Received: March 22, 2026; **Revised:** March 25, 2026; **Accepted:** April 2, 2026; **Published:** April 11, 2026

Four lectures on Sylow theory in locally finite groups

O.H. Kegel

Introduction

In their wording some theorems about finite groups also make sense for infinite groups; however, since there are counter examples, most of them cannot be proved. Narrowing down the class of groups considered more theorems of finite origin will become provable. In general, assuming the validity of some theorem about finite groups for some infinite group is a strong finiteness condition, and one might wonder what other finiteness conditions may be deduced from it. A good class in which to study such phenomena is the class of locally finite groups, i.e. the class of those groups in which every finite set of elements generates a finite subgroup.

The basic result on finite groups is the *Sylow Theorem* stating that for a fixed prime p the maximal p-subgroups of a finite group G are conjugate in G. This statement makes sense in arbitrary groups, but it is false in general, as we shall see even in locally finite groups. We shall refer to the discussion of the validity of the Sylow Theorem for the prime p or some variation or generalisation of it in some class of infinite groups as *Sylow Theory*. More than anyone else, Brian Hartley has contributed to Sylow Theory in locally finite groups. If not only in the locally finite group G itself, but also in every subgroup of G, the maximal p-subgroups are conjugate we shall say that G satisfies the ***strong Sylow Theorem*** for the prime p. For locally finite, locally p-soluble groups satisfying the strong Sylow Theorem for the prime p Hartley exhibited in [10] a very strong finiteness property, if $p \neq 2$. An extension of a weak form of Hartley's finiteness result is

If for the prime $p \geq 5$ the locally finite group G satisfies the strong Sylow Theorem then there is a finite series of normal subgroups N_i of G with

$$G = N_0 \supseteq \ldots \supseteq N_i \supseteq N_{i+1} \supseteq \ldots \supseteq N_k = \langle 1 \rangle$$

such that the factors N_i/N_{i+1} of this series are either a direct product of finitely many linear simple groups or locally p-soluble.

Hartley's result allows one to refine the series so that the locally p-soluble factors N_i/N_{i+1} are either p-groups or p'-groups.

[1] Kegel, Otto H.: *Four lectures on Sylow theory in locally finite groups.* In: Group Theory – Proceedings of the Singapore Group Theory Conference held at the National University of Singapore, **June 8-19, 1987**, edited by Kai Nah Cheng and Yu Kiang Leong, 3-28. ISBN 978-3-11-011366-2 (Hardcover) and ISBN 978-3-11-084839-7 (PDF) (see https://www.degruyter.com/view/title/11283). Berlin-New York: Walter de Gruyter January 1989, reprinted November 2016. **MR0981832** (MR 90c:20037 by **Brian Hartley** [March 1990]); **Zbl 0659.20024** (by **Bernhard Amberg**). https://www.degruyter.com/view/book/9783110848397/10.1515/9783110848397-004.xml

Citation: Felix F. Flemisch (2026) Characterising locally finite groups satisfying the strong Sylow Theorem for the prime *p* – Mathematics edition – Second edition. Journal of Mathematical & Computer Applications. SRC/JMCA-284. DOI: doi.org/10.47363/JMCA/2026(5)239

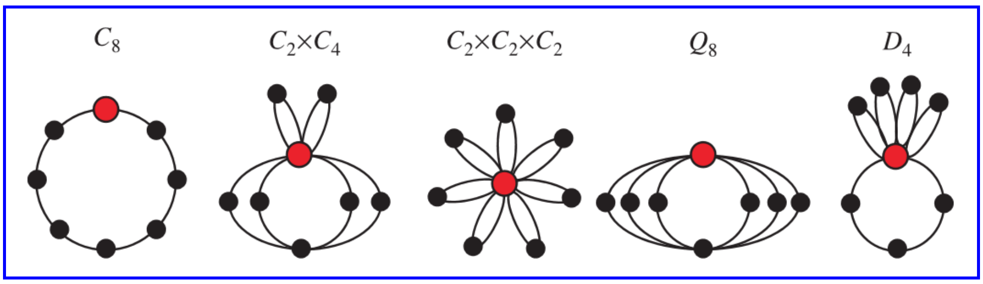
Felix Fortunatus Flemisch

Characterising locally finite groups satisfying the strong Sylow *p*-Theorem

Mathematics edition

Second edition

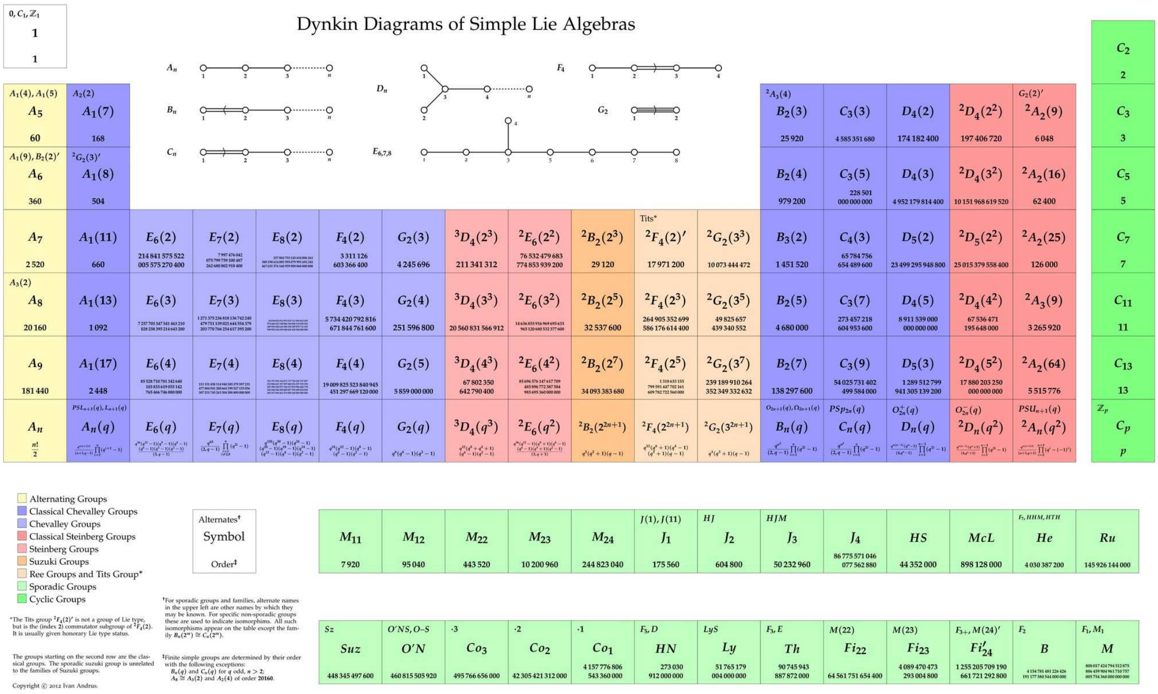
Do come in ! Hereinspaziert ! Entrez, s'il vous plaît ! Venga, per piagere !

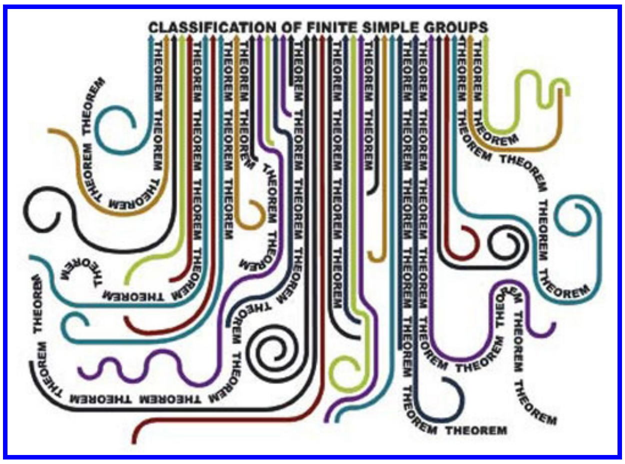


(see https://mathworld.wolfram.com/**FiniteGroup**.html)

The (2.4) Theorem of [1]

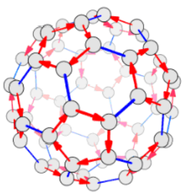
The Periodic Table Of Finite Simple Groups





Pulling together the strands of mathematics

(see https://plus.maths.org/content/os/issue41/features/elwes/index)



Alternating group $\underline{A}^5$

http://www.math.clemson.edu/~macaule/classes/m18_math4120/slides/math4120_lecture-2-03_h.pdf

L'amour immortel, éternel et infini

https://stock.adobe.com/de/images/infinity-heart-symbol-forever-vector/68328980

Citation: Felix F. Flemisch (2026) Characterising locally finite groups satisfying the strong Sylow Theorem for the prime p – Mathematics edition – Second edition. Journal of Mathematical & Computer Applications. SRC/JMCA-284. DOI: doi.org/10.47363/JMCA/2026(5)239

Dipl.-Math. **Felix F. Flemisch**, M.Sc., Bacc.Math.

Characterising locally finite groups satisfying the strong Sylow Theorem for the prime p

Manuscript on Sylow theory in locally finite groups

Research paper

Mathematics edition

Second edition

Pablo Picasso – *La Joie de vivre* – 1946

Pablo Ruiz Picasso
(25 October 1881 until 8 April 1973
{+ 19 years + 3 days = 11 April 1992
= marriage of **Helga** and **Felix** })
(see https://fr.wikipedia.org/wiki/Pablo_Picasso
and https://de.wikipedia.org/wiki/Pablo_Picasso
and https://en.wikipedia.org/wiki/Pablo_Picasso)

Picasso à côté de *La Joie de vivre*, état définitif

Bibliographic information published by the Deutsche Nationalbibliothek

The Deutsche Nationalbibliothek lists this publication in the Deutsche Nationalbibliografie; detailed bibliographic data are available on the Internet at http://dnb.dnb.de/.

DeweyDecimalClassification(DDC)-Notation:
510 Mathematics • 512 Algebra and number theory • 512.2 Groups and group theory

Imprint

One can get in touch with the author via e-mail: felix.flemisch@hotmail.com

The **Mathematics edition** extracts from the **Full edition** (see ISBN 978-3-7568-0801-4) all the *beautiful* mathematical content ☺ and omits the fair and *beautiful* personal dedications and deep information (*Richard Wagner*, *Albert Camus*, *Pablo Picasso*, *Freiburg i.Br.*, *Herrsching a.A.*, *Marcel Proust*, *Charles Baudelaire*, *Bertolt Brecht*) almost entirely ☹. It has been **Line numbered** for (Scholarly Peer) Review purposes (but the **twenty-one Footnotes** and the **eight Tables** cannot [and need not] be line numbered) ☺. The paper has vi + **19** + 4 + **11** = 40 pages. Besides **Sylow theory in locally finite groups** (**19** pages) it provides novel but classical proofs of – in our modest opinion – rather considerably historic relevance for **the very basic theorems by Lagrange and by Cauchy on finite groups** (**11** pages) ☺.

The **First edition** was produced and published on 8 September 2021 under ISBN-13: 978-3-7543-4434-7 by BoD – Books on Demand, Norderstedt, Germany (see https://buchshop.bod.de/characterising-locally-finite-groups-satisfying-the-strong-sylow-theorem-for-the-prime-p-felix-f-flemisch-9783754344347).

Cover picture: **Pablo Picasso** – La Joie de vivre
Boat varnish (Ripolin) on Asbestos cement, 1946
Collection location: **Le musée Picasso d'Antibes**

(see https://www.antibesjuanlespins.com/en/must-see-must-do/culture-and-heritage/museums/picasso-museum-2031894 and https://www.antibesjuanlespins.com/a-voir-a-faire/culture-et-patrimoine/les-musees/le-musee-picasso-2031894)

Citation: Felix F. Flemisch (2026) Characterising locally finite groups satisfying the strong Sylow Theorem for the prime p – Mathematics edition – Second edition. Journal of Mathematical & Computer Applications. SRC/JMCA-284. DOI: doi.org/10.47363/JMCA/2026(5)239

Characterising locally finite groups satisfying the strong Sylow Theorem for the prime *p*

Dipl.-Math. *Felix* **F. Flemisch**, M.Sc., Bacc.Math. [1]

Helga gewidmet, die mir so hilfreich zur Seite stand,
in Liebe und großer Dankbarkeit

Dédicacé à *Hélga*, qui était si précieuse à mes côtés,
avec tout mon amour et toute ma grande gratitude

Aimer un être, c'est accepter de vieillir avec lui. **Albert Camus**
(**Caligula**, Acte IV, Scène 13; p. 90. Barcelone: Éditions Gallimard,
folioplus classiques n° 233, 2019 (et 1944) [ISBN 978-2-07-044659-9].)

Abstract We turn a question of Otto H. Kegel [1] into a theorem [2]: *If every subgroup S of the locally finite group G contains a finite p-subgroup which is singular in S* (see below)*, then G satisfies the strong Sylow Theorem for the prime p* (see below). Since the converse is also true [1; 2], this characterises the locally finite groups which satisfy the strong Sylow Theorem for the prime p. The proof is a rather old result, namely the *Charakterisierungssatz* of the author's Diplomarbeit of 1984 [2]. It is based upon the concepts of *good p-subgroups* [3] and *strongly local* – or, as we say, *very good* – *p-subgroups* [4] of Andrew Rae. To obtain the correspondence with Kegel's work we first show that a finite p-subgroup is singular in the locally finite group G if and only if it is a *p-uniqueness subgroup* of G, that is, it is contained in a unique maximal p-subgroup of G. We then proceed on the whole as in our Diplomarbeit. However, we do not publish the proof in its original form since the Diplomarbeit was written in German. Instead we translated all employed parts into English, thereby introducing a number of corrections, present the *Charakterisierungssatz* as 2.8 HAUPTSATZ, and summarise then the result as Theorem IV. Subsequently we present a few new concepts and results for Sylow theory in (locally) finite groups and a comparative outline of Kegel's previous work [8] with two open questions. We close with personal dedications, background information and deep Endnotes which introduce further new concepts. The publication has the following seven Chapters: Introduction; **1.** Good Sylow p-subgroups and p-uniqueness subgroups; **2.** Basic theorems of Sylow theory in locally finite groups and our *Charakterisierungssatz*; **3.** Novel concepts and results for Sylow theory in (locally) finite groups and comparative outline of Kegel's previous work [8]; References; About the author & Background information and annotations; Endnotes.

Mathematics Subject Classification (MSC2020)
Primary 20D20 • 20F50 • 20D15; Secondary 20E25 • 20D05 • 20E32 • 20E34

Keywords conjugacy of maximal (i.e. Sylow) p-subgroups; singular (Sylow) p-subgroup; locally finite group satisfying the (strong) Sylow Theorem for the prime p (or the [strong] Sylow p-Theorem) ($G \in \underline{\mathrm{Syl}}$-$p$); finite simple group; split sequence of finite p-perfect subgroups; (very) good Sylow p-subgroup; p-uniqueness subgroup; singular Sylow p-subgroup; minimal p-unique subgroup; (proper) Sylow p-intersection; the (numerical) Sylow p-invariants 1) p-uniqueness and 2) Sylow p-intersection; Sylow-separated sequence of p-subgroups; maximum conditions (for $\underline{\mathrm{I}}_p G$ and for the Sylow-separated sequences of p-subgroups); locally finite group G satisfying the minimum condition for p-subgroups ($G \in \underline{\mathrm{Min}}$-$p$); p-blank of G in H.

[1] Address: **Felix F. Flemisch**, Mitterweg 4e, 82211 Herrsching a. Ammersee, Bavaria, Germany.
e-mail: felix.flemisch@hotmail.com. ORCID iD: 0000-0003-1612-8810.

Introduction

In his four workshop lectures on Sylow theory in locally finite groups at the famed Singapore Group Theory Conference of June 1987 [1], **Otto H. Kegel** stated that he could not settle the question "*If every subgroup S of the locally finite group G contains a finite p-subgroup which is singular in S, does G then satisfy the strong Sylow Theorem for the prime p?*". Recall that the group G of arbitrary cardinality is defined to be ***locally finite*** if every finite subset of G is contained in a finite subgroup of G. The finite p-subgroup P of the locally finite group G is said to be ***singular in G*** if for every finite subgroup F of G containing P there is just a unique Sylow p-subgroup of F containing P. It is well-known that a ***p-group*** for the prime p is a group of arbitrary cardinality each of whose elements has order a finite power of p. Then a p-group is finite if and only if its order is a finite power of p. The locally finite group G is defined to satisfy the ***Sylow Theorem for the prime p*** if the maximal p-subgroups of G are all conjugate in G and G satisfies the ***strong Sylow Theorem for the prime p*** if every subgroup of G satisfies the Sylow Theorem for the prime p. **Kegel**'s lectures present the basics of Sylow theory of locally finite groups, give an overview of the prodigious work by **Brian Hartley** and **Andrew Rae** on Sylow theory in locally p-soluble groups, and reveal in great detail the normal structure for groups satisfying the strong Sylow Theorem for the prime p in the general case (for $p \geq 5$). Chapters 2 and 4 of [9] give a good overview as well but without appreciating **Hartley**'s, **Rae**'s and **Kegel**'s fundamental papers properly and avoiding their *beautiful* details.

In this publication we turn **Kegel**'s question into a theorem: "*If every subgroup S of the locally finite group G contains a finite p-subgroup which is singular in S, then G satisfies the strong Sylow Theorem for the prime p.*" Since the converse is also true [1; 2], this characterises the locally finite groups which satisfy the strong Sylow Theorem for the prime p. The proof of our ***Charakterisierungssatz*** is not presented in its original form since it was written in German as the main result of the author's Diplomarbeit during 1978 - 1984 [2]: "*Lokal endliche Gruppen mit Sylow p-Satz oder mit min-p. I: Grundbegriffe, ein Charakterisierungssatz und lokale Prinzipien*". We decided against a presentation (for historical reasons) as an amalgam of English and German and translated all employed parts into English, thereby introducing a large number of corrections and embellishments, in particular **Proposition III**.

The central discovery that enabled in those days the proof was the relationship of p-subgroups which are singular to the *good p-subgroups* [3] and the *strongly local p-subgroups* [4] of **Andrew Rae**. Let G be any locally finite group and let P be a p-subgroup of G. A ***local system for G*** is a family Σ of finite subgroups such that every element of G lies in a Σ-group and for every two Σ-groups there exists another Σ-group which contains both, for example, the local system of all finite subgroups of G. The group P is said to ***reduce into a local system*** Σ ***for G*** if for every Σ-group U we have that $P \cap U$ is a Sylow p-subgroup of U, and then P is a maximal p-subgroup of G (see below), P is said to be ***good*** if there exists a local system for G into which P reduces, and P is said to be *strongly local* or, as we prefer to say, ***very good*** if given any local system Σ for G there exists a subsystem of Σ into which P reduces. A very good p-subgroup is – of course – good and, as we show below, any singular p-subgroup of the locally finite group G is contained in a unique maximal p-subgroup of G which is very good and the existence of a singular p-subgroup enforces the conjugacy of the good Sylow p-subgroups in countable locally finite groups.

The initial goal of our Diplomarbeit was to state more precisely and prove what **Kegel** ***deduces*** **"by inspection"** as **(2.4) Theorem**, namely that *if the p-subgroup P is singular in the finite simple group S which belongs to a rank-unbounded family of finite simple groups, then the rank of S is bounded* ***"in terms of P"***, and ***combines*** this with the classification of finite non-abelian simple groups, namely that *there are* ***seven*** ***"known"*** *rank-unbounded families of finite simple groups* [2], ***ten*** *families with fixed rank* [2], *and* ***26*** *sporadic finite simple groups* [5], to ***prove*** **(2.7) Theorem** [3] "*For the locally finite simple group G the following are equivalent: (i) Every countable simple subgroup of G contains a singular p-subgroup; (ii) G satisfies the strong Sylow Theorem for the prime p; (iii) G is linear.*" ***and*** the key **(3.4) Theorem** "*If* $\{F_i\}_{i \in \mathbb{N}}$ *is a smooth simple straight split sequence of finite p-perfect subgroups of the group G, then the countably infinite group* $U = \langle F_i\, ;\, i \in \mathbb{N} \rangle$ *has* $2^{\aleph_0}$ *maximal p-subgroups.*", and to ***deduce*** first "crucial configurations" and then at last his final **(4.4) Theorem** which reveals in great detail the normal structure of locally finite groups satisfying the strong Sylow Theorem for the prime $p \geq 5$. While these ***combinations*** are not all too difficult to achieve (which the author did as well and **in more detail** *in unpublished work* and **even extended them to the case** $\boldsymbol{p = 3}$), during **"the inspection"** to make the hazy **(2.4) Theorem** more specific and prove it, the author ran over many years from one dead-end street to the next cul-de-sac road to the next dead-end street and so on, having never sufficient knowledge about the "known" finite simple groups, and he succeeded only, but at least, in determining a bounding function **"in terms of** $\boldsymbol{P}$**"** for the Alternating Groups. But the ambitious goal, to determine in all finite simple groups (except possibly the exceptional and the sporadic ones) all p-subgroups, which are (minimal) singular, was not achieved. So the author was quite happy to be able to proudly present in his Diplomarbeit alternatively the above-mentioned ***Charakterisierungssatz***. The other alternative to state, but not prove, the **(2.4) "Theorem"** and submit (as a doctoral thesis) the unpublished work, which included a very nice concept for a generalised p-length as well, was not recognised since it seemed very inappropriate to submit an unproved theorem as the key argument.

L'amour immortel, éternel et infini de l'auteur pour *Hélga* et les mathématiques
(© see https://stock.adobe.com/de/images/infinity-heart-symbol-forever-vector/68328980)

[2] These are the **alternating groups** Alt_n and the **classical families** $A = PSL$, $B = P\Omega_{odd}$, $C = PSp$, $D = P\Omega^{+}_{even}$, ${}^2A = PSU$, ${}^2D = P\Omega^{-}_{even}$. The **exceptional families** E_6, E_7, E_8, F_4, G_2, ${}^2B_2$, ${}^3D_4$, ${}^2E_6$, ${}^2F_4$, ${}^2G_2$ have a fixed rank label and the **Tits group** ${}^2F_4(2)'$ belongs to ${}^2F_4$ or is regarded as the 27th sporadic group.

[3] **Brian Hartley** in his Mathematical Review of [1] stated the following: "If the simple locally finite group G satisfies the strong Sylow Theorem for the (even one) prime p, then G is linear. This depends on the classification of finite simple groups and **an assertion about singular** $\boldsymbol{p}$**-subgroups of classical groups**. Another proof of this result has since been given by the reviewer (not yet published)." However, due to the very tragic death of **Prof. Hartley** in 1994 (see [2]), aged 55, this certainly ***very interesting*** proof was never prepared for publication.

1. Good Sylow *p*-subgroups and *p*-uniqueness subgroups

We call a maximal p-subgroup of a locally finite group G a ***Sylow p-subgroup*** of G and denote the set of all Sylow p-subgroups of G by **$\underline{\mathrm{Syl}}_p G$**. If a p-subgroup of a locally finite group G reduces into a local system for G, it is a maximal p-subgroup:

1.1 [2] **Let G be a locally finite group and P be a p-subgroup of G. If there exists a local system Σ for G into which P reduces, then P is a Sylow p-subgroup of G.**

PROOF: Let $S \in \underline{\mathrm{Syl}}_p G$ with $P \subseteq S$. Suppose, $P \neq S$. Then there exists an element $x \in S \setminus P$. Let $U \in \Sigma$ with $x \in U$. It follows that $< P \cap U, x >$ is a p-subgroup of U with $P \cap U \subsetneq\ < P \cap U, x > \subseteq S$. This contradicts the prerequisite $P \cap U \in \underline{\mathrm{Syl}}_p U$. ❑

The local system Σ for the locally finite group G is said to be ***nested*** (in German ***geschachtelt***) if there exists a sequence $\{ U_n \mid n \in \mathbb{N} \}$ of finite subgroups of G such that $U_n \subseteq U_{n+1}$ for all $n \in \mathbb{N}$ and $\Sigma = \{ U_n \mid n \in \mathbb{N} \}$. If G are a countable locally finite group and $\{ x_n \mid n \in \mathbb{N} \}$ an enumeration of G, let $U_n := < x_1, x_2, \dots, x_n >$ $(n \in \mathbb{N})$. Then $\{ U_n \mid n \in \mathbb{N} \}$ is a nested local system for G. If the locally finite group G has a nested local system, then G is countable. We can identify all the good Sylow p-subgroups of countable locally finite groups by means of nested local systems for them:

1.2 [2] **Let G be a countable locally finite group. Then we have: a) If Σ is a local system for G, then Σ contains a local subsystem Σ_1 which is nested. b) Let $\Sigma = \{ U_n \mid n \in \mathbb{N} \}$ be a nested local system for G. Then there exist with respect to (w.r.t.) Σ good Sylow p-subgroups of G. In particular G contains at least one good Sylow p-subgroup.**

PROOF: **a)** Let Σ be a local system for G and $\{ x_n \mid n \in \mathbb{N} \}$ an enumeration of G. We define $U_x \in \Sigma$ with $x \in U_x$ $(x \in G)$ and $< U_x, U_y > \subseteq U_{xy}$ $(x, y \in G)$ as follows: let $U_{x_1} \in \Sigma$ with $x_1 \in U_{x_1}$; if $U_{x_1 x_2 x_3 \dots x_n} \in \Sigma$ are already defined with $x_1, x_2, x_3, \dots, x_n \in U_{x_1 x_2 x_3 \dots x_n}$ $(n \in \mathbb{N})$, let $U_{x_{n+1}} \in \Sigma$ with $x_{n+1} \in U_{x_{n+1}}$ and $U_{x_1 x_2 x_3 \dots x_n x_{n+1}} \in \Sigma$ with $< U_{x_1 x_2 x_3 \dots x_n}, U_{x_{n+1}} > \subseteq U_{x_1 x_2 x_3 \dots x_n x_{n+1}}$ $(n \in \mathbb{N})$. Then the countable subset $\Sigma_1 := \{ U_{x_1 x_2 x_3 \dots x_n} \mid n \in \mathbb{N} \}$ of Σ is a nested local system for G. **b)** Let $P_1 \in \underline{\mathrm{Syl}}_p U_1$. If $P_1 \subseteq P_2 \subseteq \dots \subseteq P_n$ are already finite p-subgroups of G with $P_i \in \underline{\mathrm{Syl}}_p U_i$ $(1 \leq i \leq n)$, let $P_{n+1} \in \underline{\mathrm{Syl}}_p U_{n+1}$ with $P_n \subseteq P_{n+1}$ $(n \in \mathbb{N})$. Define $S := \bigcup \{ P_n \mid n \in \mathbb{N} \}$. Then S is a p-subgroup of G, which reduces into Σ, and so is good with $S \in \underline{\mathrm{Syl}}_p G$ by **1.1**. [4] ❑

We can now introduce the ***p*-uniqueness subgroups** and present the very close relationship between them and the **good Sylow *p*-subgroups**.

In [2] we call *p-dominant* a p-subgroup of the locally finite group G if it is finite and is contained in a unique Sylow p-subgroup S of G, and call then S ***singular*** (in German ***einzigartig*** or ***einmalig*** or ***singulär***, in a double sense). Although "dominant" in German is "dominant" in English we now find it smarter to define such a p-subgroup of G as a ***p-uniqueness subgroup*** (in German, a bit unwieldy, ***p-Einzigartigkeits-untergruppe*** or ***p-Einmaligkeitsunterguppe***) of G ***for S*** or ***w.r.t. S***. [5]

[4] see **Kegel**'s **(1.1) Lemma** of [1] for another argument for **1.2.b)** which is very similar to **1.1**

[5] There is no danger of confounding our p-uniqueness subgroups with the p-uniqueness subgroups which play a major rôle in the classification of the finite simple groups (see **Gorenstein, Daniel; Lyons, Richard; Solomon, Ronald:** *The Classification of the Finite Simple Groups, Part 1*; p. 82. Providence: AMS, Mathematical Surveys and Monographs Volume **40.1**, 2000 [and 1994] [ISBN 978-0-8218-0960-0 resp. ISBN 978-1-4704-1267-8]. https://www.ams.org/books/surv/040.1/).

Proposition I. Let G be a locally finite group and let p be a prime. Let P be a finite p-subgroup of G. The following are equivalent:
1) P is a p-uniqueness subgroup of G.
2) P is singular in G.
3) Whenever P_1 and P_2 are finite p-subgroups of G with $P \subseteq P_1 \cap P_2$ then $< P_1, P_2 >$ is a p-group.

PROOF: **1) ⇒ 2)**: Suppose P is not singular in G. Then we have a finite subgroup F of G such that P is contained in at least two Sylow p-subgroups P_1 and P_2 of F. Let S_i be a Sylow p-subgroup of G which contains P_i (i = 1, 2). If $S_1 = S_2$ then $< P_1, P_2 > \subseteq < S_1, S_2 > \cap F$ is a p-group which contradicts $P_1 \in \underline{\mathrm{Syl}}_p F$ and $P_2 \in \underline{\mathrm{Syl}}_p F$. Thus $S_1 \neq S_2$. Therefore P is not a p-uniqueness subgroup of G.
2) ⇒ 3): Let $P \subseteq P_1 \cap P_2$ where P_1 and P_2 are finite p-subgroups of G and suppose that $F := < P_1, P_2 >$ is not a p-group. Then $P \subseteq F$ and since $< P_1, P_2 >$ is not a p-group there are two distinct Sylow p-subgroups Q_1 and Q_2 of F containing P_1 and P_2, respectively. But then $P \subseteq Q_1 \cap Q_2$ and so P is not singular in G.
3) ⇒ 1): Suppose that **3)** holds and that P is not a p-uniqueness subgroup of G. Then there are distinct Sylow p-subgroups Q_1 and Q_2 of G such that $P \subseteq Q_1 \cap Q_2$. Let $x \in Q_1 \setminus Q_2$ and $y \in Q_2 \setminus Q_1$. It follows that $P_1 := < P, x >$ and $P_2 := < P, y >$ are finite p-groups and that $< P_1, P_2 >$ is not a p-group contradicting **3)**. ❑

Kegel discovered insight gaining equivalent conditions for the conjugacy of good Sylow p-subgroups in countable locally finite groups. We expandedly restate and reprove his **(1.2) Theorem** in our terminology thereby adding another property:

(1.2) Theorem [1]. For the countable locally finite group G and the prime p the following properties are equivalent: 1) There exists a nested local system $\{G_i \mid i \in \mathbb{N}\}$ for G and an index i_0 such that for every pair $j \geq i \geq i_0$ of indices every Sylow p-subgroup P_i of G_i is contained in a unique Sylow p-subgroup P_j of G_j.
2) There exists a finite p-subgroup P_0 of G which is singular in G.
3) There exists a p-uniqueness subgroup P_0 of G.
4) Any two good Sylow p-subgroups of G are conjugate in G.

PROOF: **1) ⇒ 2)**: Choose $P_{i_0} \in \underline{\mathrm{Syl}}_p G_{i_0}$ and put $P_0 := P_{i_0}$. Let F be any finite subgroup of G containing P_0. For every index j such that $F \subseteq G_j$, the unique Sylow p-subgroup of G_j containing P_0 must contain a Sylow p-subgroup of F, and no other Sylow p-subgroup of F can contain P_0. Clearly **2) ⇒ 1)**. From **Proposition I** follow **2) ⇒ 3)** and **3) ⇒ 2)**. To show **4) ⇒ 1)** assume that for any nested local system $\{ G_i \mid i \in \mathbb{N} \}$ for G and any index i_0 for infinitely many pairs $j \geq i \geq i_0$ of indices some (hence any by conjugacy) Sylow p-subgroup of G_i is contained in at least two Sylow p-subgroups of G_j. We then can construct, similar to **2.2** or **2.7** below, $2^{\aleph_0}$ maximal p-subgroups of G which are good by **1.2** and cannot all be conjugate in G. Thus **4)** entails **1)**, and hence **2)**. It remains to show **3) ⇒ 4)**:[6] let P and Q be good Sylow p-subgroups of G obtained as two unions of Sylow p-subgroups of nested local systems $\{G_i \mid i \in \mathbb{N}\}$ and $\{H_i \mid i \in \mathbb{N}\}$ for G (see **1.2**) and let S_0 be the unique Sylow p-subgroup of G containing P_0; we show that P is conjugate to S_0 and S_0 is conjugate to Q, and therefore P is conjugate to Q; if P and S_0 are not conjugate then one of them must have property (✪) of **2.1**

[6] **Kegel**'s argument for **2) ⇒ 4)** on page 6 f. of [1] is really not fully convincing.

(see below) which means in particular that it is not singular; so P has property (✪); now P reduces into $\{G_i \mid i \in \mathbb{N}\}$, that is, $P \cap G_i \in \underline{\mathrm{Syl}}_p G_i$ for all $i \in \mathbb{N}$; there exists an index i_0 such that $P_0 \subseteq G_{i_0}$; then $P_0 \subseteq P_{i_0}$ for some unique $P_{i_0} \in \underline{\mathrm{Syl}}_p G_{i_0}$; now let – thanks to Ludvig M. Sylow – x be an element of G_{i_0} such that $P_{i_0}{}^x = P \cap G_{i_0}$; then $P_{i_0}{}^x$ is a finite p-subgroup of P which is contained in just only one Sylow p-subgroup of G thereby contradicting property (✪) of P; for exactly the same reasons S_0 is conjugate to Q; therefore P must be conjugate to Q. ❑

Let S be a Sylow p-subgroup of the locally finite group G. A finite subgroup F of G is called ***S*-dominant** if S reduces into every subgroup U of G which contains F, that is, $S \cap U \in \underline{\mathrm{Syl}}_p U$ for all subgroups U of G such that $F \subseteq U$.

1.3 [2] **Let G be a locally finite group, p be a prime, $S \in \underline{\mathrm{Syl}}_p G$ and F be a finite subgroup of G. The following are equivalent: 1) F is S-dominant. 2) For each finite subgroup U of G with $F \subseteq U$ we have $S \cap U \in \underline{\mathrm{Syl}}_p U$.**

PROOF: **1) ⇒ 2)** is clear. **2) ⇒ 1)**: Since F is finite, there exists a local system Σ for G such that for each Σ-group U we have $F \subseteq U$. Let V be a subgroup of G with $F \subseteq V$. Then $\Sigma_1 := \{ V \cap U \mid U \in \Sigma \}$ is a local system for V into which $S \cap V$ reduces. Therefore from **1.1** follows $S \cap V \in \underline{\mathrm{Syl}}_p V$. ❑

1.4 [2] **Let G be a locally finite group and $S \in \underline{\mathrm{Syl}}_p G$. Then are equivalent: 1) S is very good. 2) There exists an S-dominant subgroup of G.**

PROOF: **1) ⇒ 2)**: Suppose, no S-dominant subgroup of G exists. Then according to **1.3** there exists to every finite subgroup F of G at least one finite subgroup U_F of G with $F \subseteq U_F$ and $S \cap U_F \notin \underline{\mathrm{Syl}}_p U_F$. Let $\Sigma := \{ U_F \mid F \text{ finite subgroup of } G \}$. Then Σ is a local system for G that possesses no local subsystem into which S reduces. **2) ⇒ 1)**: Let F be an S-dominant subgroup of G and Σ a local system for G. Let $\Sigma_1 := \{ U \mid U \in \Sigma, F \subseteq U \}$. Then Σ_1 is, because of the S-dominance of F, a local subsystem of Σ into which S reduces. ❑

1.5 [2] **Let G be a locally finite group and let p be a prime. a) If F is a p-uniqueness subgroup of G and S is the singular Sylow p-subgroup of G with $F \subseteq S$, then F is an S-dominant subgroup of G. b) Every singular Sylow p-subgroup of G is very good.**

PROOF: **a)** Let U be a subgroup of G with $F \subseteq U$. Let $P \in \underline{\mathrm{Syl}}_p U$ and $T \in \underline{\mathrm{Syl}}_p G$ with $S \cap U \subseteq P \subseteq T$. From $F \subseteq S$ and the p-uniqueness of F follows $T = S$. Therefore $S \cap U = P$. **b)** follows from **a)** and **1.4**. ❑

We spend a separate proposition for the important insight **1.5.b)**:

Proposition II. Let p be a prime and P be a p-uniqueness subgroup of the locally finite group G or, equivalently by Proposition I, let P be a singular p-subgroup of G. Then the singular Sylow p-subgroup S of G containing P is very good. ❑

We can now summarise the relationships between good Sylow p-subgroups and p-uniqueness subgroups with their containing singular Sylow p-subgroups as follows:

• singular Sylow p-subgroups are very good; • p-uniqueness subgroups are singular, and conversely; • in countable locally finite groups good Sylow p-subgroups are identified by nested local systems; • in countable locally finite groups the existence of a p-uniqueness subgroup compels the conjugacy of all good Sylow p-subgroups.

We end the discussion of good Sylow p-subgroups by pointing out that there exist **1)** countable locally finite groups with Sylow p-subgroups which are **not good** [7] and **2)** locally finite groups of cardinality $2^{\aleph_0}$ which contain **no good** Sylow p-subgroups **at all**.

First, we let G be a finite group with $|\underline{\mathrm{Syl}}_p G| \geq 2$, e.g. the symmetric group $\underline{S}^{2p}$ of degree $2p$ for the prime p for which we know surely that $|\underline{\mathrm{Syl}}_p \underline{S}^{2p}| \geq 2p - 2 \geq 2$. In ***the*** $\mathbb{N}$***-fold cartesian power*** $G^{[\mathbb{N}]} := \prod \{ G_i \mid G_i := G \text{ for all } i \in \mathbb{N} \} := \{ (x_1, x_2, \dots) \mid x_i \in G_i \text{ for all } i \in \mathbb{N} \}$ of G, which satisfies the Sylow p-Theorem [8], there exist ***the*** $\mathbb{N}$***-fold direct power*** $G^{(\mathbb{N})} := \prod^0 \{ (x_i)_{i \in \mathbb{N}} \in G^{[\mathbb{N}]} \mid x_i = 1 \text{ for almost all } i \in \mathbb{N} \}$, which does **not** satisfy the Sylow p-Theorem [9], and esp. ***the diagonal*** $D := \{ (x_i)_{i \in \mathbb{N}} \in G^{[\mathbb{N}]} \mid (\exists\, x \in G)(\forall\, i \in \mathbb{N})\ x_i = x \} \cong G$ via the isomorphism $\delta\colon D \to G$, $((x_i)_{i \in \mathbb{N}})^{\delta} := x$, from D onto G with $D \cap G^{(\mathbb{N})} = \langle 1 \rangle$. Since $G^{(\mathbb{N})}$ is a normal subgroup of $G^{[\mathbb{N}]}$, we have $\langle G^{(\mathbb{N})}, D \rangle = D \bullet G^{(\mathbb{N})}$; this is a **countable** subgroup of $G^{[\mathbb{N}]}$. The Sylow p-subgroups of $G^{[\mathbb{N}]}$ resp. of $G^{(\mathbb{N})}$ are cartesian resp. direct products of the Sylow p-subgroups of the G_is ($i \in \mathbb{N}$), namely $\prod \{ S^{\pi_i} \mid i \in \mathbb{N} \}$ resp. $\prod^0 \{ S^{\pi_i} \mid i \in \mathbb{N} \}$ for $S \in \underline{\mathrm{Syl}}_p G$, where $\pi_i\colon G^{[\mathbb{N}]} \to G_i$ is the coördinate projection $\pi_i((x_k)_{k \in \mathbb{N}}) := x_i$ on the factor G_i ($i \in \mathbb{N}$). Any $P \in \underline{\mathrm{Syl}}_p D$ normalises exactly one Sylow p-subgroup S of $G^{[\mathbb{N}]}$ resp. exactly one Sylow p-subgroup S^0 of $G^{(\mathbb{N})}$, namely $S = \prod \{ P^{\pi_i} \mid i \in \mathbb{N} \}$ resp. $S^0 = \prod^0 \{ P^{\pi_i} \mid i \in \mathbb{N} \}$. Therefore every Sylow p-subgroup of D is a p-uniqueness subgroup of $D \bullet G^{(\mathbb{N})}$. Thus by **(1.2) Theorem** all good Sylow p-subgroups of $D \bullet G^{(\mathbb{N})}$ are conjugate in $D \bullet G^{(\mathbb{N})}$, what nevertheless could be proved directly, however. The countable group $D \bullet G^{(\mathbb{N})}$ has by **1.2** indeed **good** Sylow p-subgroups and we are able to designate some distinguished of them explicitly: let $U_i := D \bullet G_1 \times D \bullet G_2 \times \dots \times D \bullet G_i$ ($i \in \mathbb{N}$); then $\Sigma := \{ U_i \mid i \in \mathbb{N} \}$ is a nested local system for $D \bullet G^{(\mathbb{N})}$; if $P \in \underline{\mathrm{Syl}}_p D$ and $P_i \in \underline{\mathrm{Syl}}_p G_i = \underline{\mathrm{Syl}}_p G$ ($i \in \mathbb{N}$), then $\boldsymbol{P^0} := P \bullet P_1 \times P \bullet P_2 \times P \bullet P_3 \times \dots \cap G^{(\mathbb{N})}$ is a p-subgroup of $D \bullet G^{(\mathbb{N})}$ which reduces into Σ and therefore is by **1.1** and **1.2** a good Sylow p-subgroup of the countable group $D \bullet G^{(\mathbb{N})}$.

The group $D \bullet G^{(\mathbb{N})}$ has indeed also (many) Sylow p-subgroups, which are **not good**: since $|\underline{\mathrm{Syl}}_p G| \geq 2$ we can construct by the method of **2.2** or that of **2.7** an infinitely ($\aleph_0$) high tree of finite p-subgroups of $D \bullet G^{(\mathbb{N})}$ with $\langle 1 \rangle$ as the root which branches properly at each location with proper inclusions and where two immediate successors of each point do not generate a p-group; this tree has $2^{\aleph_0}$ branches which constitute $2^{\aleph_0}$ many ascending unions of finite p-subgroups and thus $2^{\aleph_0}$ many p-subgroups P_ι where any two of them do not generate a p-group; choosing for each P_ι a Sylow p-subgroup S_ι of $D \bullet G^{(\mathbb{N})}$ containing P_ι now delivers $2^{\aleph_0}$ Sylow p-subgroups of $D \bullet G^{(\mathbb{N})}$ ($1 \leq \iota \leq 2^{\aleph_0}$) on the treetop; since by Kegel's (1.2) Theorem the good

[7] (see the note on page 5 of [1]: "It may be worthwhile to point out that a countable infinite locally finite group G may have maximal p-subgroups" which are not good.)

[8] For $S, T \in \underline{\mathrm{Syl}}_p G^{[\mathbb{N}]}$ there are $S_i, T_i \in \underline{\mathrm{Syl}}_p G_i = \underline{\mathrm{Syl}}_p G$ ($i \in \mathbb{N}$) such that S resp. T is the cartesian product of the S_is resp. the T_is. If $x_i \in G_i = G$ with $S_i^{x_i} = T_i$ ($i \in \mathbb{N}$) and $x := (x_i)_{i \in \mathbb{N}}$, then $S^x = T$.

[9] Let $S, T \in \underline{\mathrm{Syl}}_p G^{(\mathbb{N})}$. If there is an $x \in G^{(\mathbb{N})}$ with $S^x = T$, then $S^{x\pi_i} = T^{\pi_i}$ for almost all $i \in \mathbb{N}$. Thus for $P, Q \in \underline{\mathrm{Syl}}_p G$ with $P \neq Q$, the groups $P^{(\mathbb{N})}$ and $Q^{(\mathbb{N})}$ are not in $G^{(\mathbb{N})}$ – but in $G^{[\mathbb{N}]}$ – conjugate Sylow p-subgroups of $G^{(\mathbb{N})}$. Alternatively follows from $|G^{(\mathbb{N})}| = \aleph_0$ and $|\underline{\mathrm{Syl}}_p G^{(\mathbb{N})}| = 2^{\aleph_0}$ – we have **2.1** and **2.2** because of $|\underline{\mathrm{Syl}}_p G| \geq 2$ (see below) – that not all Sylow p-subgroups of $G^{(\mathbb{N})}$ can be conjugate.
In uncountable locally finite groups the Sylow p-Theorem is not inherited by normal subgroups.

Sylow p-subgroups of the countable group $D \bullet G^{(\mathbb{N})}$ are conjugate, *at most* $\aleph_0$ of these $2^{\aleph_0}$ Sylow p-subgroups can be good; there then remain (with or without the continuum hypothesis) *at least* $2^{\aleph_0} - \aleph_0$ many Sylow p-subgroups in the treetop which are not good and too many to be conjugate in $D \bullet G^{(\mathbb{N})}$.

Second, let p and q be primes with $q \equiv 1 \pmod{p}$ and $A := < a, b \mid a^p = b^q = (ab)^p = 1 >$. Then $|A| = pq$ and A has q many Sylow p-subgroups and a normal Sylow q-subgroup, hence is metabelian. If $(p,q) = (2,3)$ then $A = \underline{S}^3$ is the symmetric group of degree 3. The group A contains the elements a and $a' := ab$ of order p which are not p-consonant, that is, they do not generate a p-group. The $\mathbb{N}$-fold cartesian power $A^{[\mathbb{N}]}$ of A is locally finite and metabelian of exponent pq. **László G. Kovács**, **Bernhard H. Neumann** and **Hugo de Vries** constructed, based on the elements a and a' (and exemplarily for $(p,q) = (2,3)$), an $\mathbb{N}$-fold interdirect power H of A, that is, $A^{(\mathbb{N})} \subseteq H \subseteq A^{[\mathbb{N}]}$, with the following properties [6, Theorem 3.7]: H is metabelian of exponent q and order $2^{\aleph_0}$ with a countable Sylow p-subgroup and a Sylow p-subgroup of order $2^{\aleph_0}$ (hence without Sylow Theorem for the prime p). They also constructed, using again a and a', an $\mathbb{N}$-fold interdirect power H of A with the following amazing properties [6, Theorem 4.4]: H has order $2^{\aleph_0}$, each Sylow p-subgroup of H is countable, H has a countable normal (hence unique) Sylow q-subgroup, which has no complement in H, and each Sylow p-subgroup has a complement in H, which is normal in H and contains elements of order p. **No Sylow p-subgroup of H can be good**: suppose a Sylow p-subgroup S of H reduces into a local system Σ for H; we then choose a Σ-group U that contains an element x of order p of a complement of S, and a $P \in \underline{\mathrm{Syl}}_p U$ containing x; since $S \cap U \in \underline{\mathrm{Syl}}_p U$ there is a $y \in U$ with $P^y = S \cap U$; then $< x >^y \subseteq S$ whereas $< x >^y$ belongs to the normal complement of S, a contradiction. The arithmetic p-structure and the local structure of these examples get on very poorly with one another.

2. Basic theorems of Sylow theory in locally finite groups and our *Charakterisierungssatz*

In this Chapter we first present – with quite considerably improved proofs – the basics of Sylow theory in locally finite groups (**2.1** to **2.5 Theorem**) and subsequently prepare and carry out the proof of our ***Charakterisierungssatz*** (**Proposition III** to **2.8 HAUPTSATZ**) which allows us to prove very easily our main theorem (**Theorem IV**).

2.1 [2] **Any locally finite group G which does not satisfy the Sylow Theorem for the prime p contains a Sylow p-subgroup S with the following property:**
(✪) "Every finite subgroup of S lies in at least two Sylow p-subgroups of G." [10] [11]

PROOF: Let S and T be two Sylow p-subgroups of G which are not conjugate (in G). If T is not singular, that is, T does have property (✪), the result is immediate, so suppose that T is singular and let Y be a p-uniqueness subgroup for T. We show that then S has property (✪), that is, S is not singular. To this end let X be an arbitrary finite subgroup of S. Then $< X, Y >$ is a finite group. By the Sylow p-Theorem for finite groups there exists an $x \in G$ such that X and Y^x lie in the same Sylow p-subgroup of $< X, Y >$. Then $< X, Y^x >$ is a p-group. From the assumption on Y it thus follows that $< X, Y^x > \subseteq T^x$. Hence X lies in at least the two Sylow p-subgroups S and T^x of G. ❑

[10] The property (✪) means that ***S* is not singular** (see the same property (∗) on page 8 of [1]). This property was for the first time discovered by **Ali O. Asar** [7].

[11] see **Proposition III** below for a generalisation

We now prepare an alternative proof of the basic theorem of Sylow theory known as the "Asar-Hartley Theorem" (see [9, 2.3.11. Theorem] for the basically original proof).

2.2 [2] **Let G be a locally finite group and P be a p-subgroup of G for the prime p. a) P shall have the following property: (⦿) "To every finite subgroup F of P there exists an $x = x(F) \in G$ with $F^x \subseteq P$ such that $< P, P^x >$ is not a p-group." Then there are $2^{\aleph_0}$ infinite ascending chains $X_{i_1} < X_{i_1 i_2} < \ldots < X_{i_1 i_2 \ldots i_n} < \ldots$ of finite p-subgroups of G with indices $i_k \in \{0,1\}$ ($k \in \mathbb{N}$) such that for all $n \in \mathbb{N}$ and each choice of indices i_k ($1 \le k \le n$) the group $< X_{i_1 i_2 \ldots i_n 0}, X_{i_1 i_2 \ldots i_n 1} >$ is not a p-group. b) Let $P \in \underline{\underline{\mathrm{Syl}}}_p G$ with the property (✪) from 2.1. Then P has property (⦿) from a).** [12] **c) Let $P \in \underline{\underline{\mathrm{Syl}}}_p G$ with the property (✪) from 2.1 and X be a finite subgroup of P. Then there are $2^{\aleph_0}$ many infinite ascending chains $X < X_{i_1} < X_{i_1 i_2} < \ldots < X_{i_1 i_2 \ldots i_n} < \ldots$ with the properties from a).** [13]

PROOF: **a)** Let X be a finite subgroup of P and y be an element of G with "**1)** $< P, P^y >$ is not a p-group" and "**2)** $X^y \subseteq P$". Because of the first property there exists a finite subgroup X_0 of P with $X \subseteq X_0$ such that $< X_0, X_0^y >$ is not a p-group, and because of the second property we have $< X, X^y > \subseteq P$, hence $X_0 \neq X \neq X_0^y$. If we substitute in the last two sentences X by X_0, we get two finite p-subgroups X_{00} and X_{01} of G with $X_0 < X_{00}$ and $X_0 < X_{01}$ such that $< X_{00}, X_{01} >$ is not a p-group. Since P^y has the property (⦿), too, we can analogously substitute X by $X_1 := X_0^y$ and thereby get two finite p-subgroups X_{10} and X_{11} of G with $X_1 < X_{10}$ and $X_1 < X_{11}$ such that $< X_{10}, X_{11} >$ is not a p-group. We thereby have altogether constructed four ascending chains $X < X_0 < X_{00}$, $X < X_0 < X_{01}$, $X_1 < X_{10}$ and $X_1 < X_{11}$ of finite p-subgroups of G such that the three groups $< X_0, X_1 >$, $< X_{00}, X_{01} >$ and $< X_{10}, X_{11} >$ are not p-groups. Now let $n \in \mathbb{N}$ with $n \ge 2$ and let already be constructed 2^n ascending chains $X_{i_1} < X_{i_1 i_2} < \ldots < X_{i_1 i_2 \ldots i_n}$ of finite p-subgroups of G with indices $i_k \in \{0,1\}$ ($1 \le k \le n$) such that for each $m \in \mathbb{N}$ with $m \le n-1$ and each choice of indices i_k ($1 \le k \le m$) the group $< X_{i_1 i_2 \ldots i_m 0}, X_{i_1 i_2 \ldots i_m 1} >$ is not a p-group. Whilst repeating the construction of the first two sentences successively with the 2^n groups $X_{i_1 i_2 \ldots i_n}$ in place of X, we get, because each conjugate of P possesses the property (⦿), in each case two p-subgroups $X_{i_1 i_2 \ldots i_n 0}$ and $X_{i_1 i_2 \ldots i_n 1}$ of G with $X_{i_1 i_2 \ldots i_n} < X_{i_1 i_2 \ldots i_n 0} \cap X_{i_1 i_2 \ldots i_n 1}$ such that $< X_{i_1 i_2 \ldots i_n 0}, X_{i_1 i_2 \ldots i_n 1} >$ is not a p-group. Therewith we now have constructed 2^{n+1} ascending chains $X_{i_1} < X_{i_1 i_2} < \ldots < X_{i_1 i_2 \ldots i_n} < X_{i_1 i_2 \ldots i_n i_{n+1}}$ with the requested properties. Therefore we can recursively w.r.t. inclusion construct a tree of height $\aleph_0$ of finite p-subgroups of G, which branches properly at each location with proper inclusions, hence must contain $2^{\aleph_0}$ infinite branches. Also any two immediate successors of an arbitrary point do not generate a p-group. These branches are just the required chains. **b)** Let F be a finite subgroup of P and R be a Sylow p-subgroup of G with $F \subseteq R \neq P$. Then there is an element x in R with $x \notin P$ and the group $< F, x >$ is a finite p-group. Let $Y := < F, x > \cap P$. Then we have $Y \neq < F, x >$. It is well known that as a finite p-group, $< F, x >$, satisfies the **normaliser condition**. [14] Therefore Y

[12] Our proofs of **a)** and of **b)** with reference to **a)** are much clearer and detailed than the original proof by **Ali O. Asar** [7] which may be considered rather cumbersome.

[13] In the proof of his **(1.3) Theorem** [1] **Kegel** sagely combines **2.1** with **2.2.c)**.

[14] **Normaliser condition** of finite p-groups:
If U is a p-subgroup of the finite group G and p divides $[G:U]$, then $U \subsetneq \underline{\mathrm{N}}_G U$.

is a proper subgroup of $\underline{N}_{<F,\,x>}Y$. Let y be an element in $<F, x>$, but not in Y, which normalises Y. Then $y \notin P$. Since y is a p-Element and P is by assumption a Sylow p-subgroup of G, it follows that $y \notin \underline{N}_G P$ and that $<P, P^y>$ is not a p-group. This just is the property (⦿) from **a)** for P to be proved. [15] **c)** We combine the proofs of **a)** and **b)**. So let $R \in \underline{\mathrm{Syl}}_p G$ with $\boldsymbol{X} \subseteq R \neq P$. Let $x \in R \setminus P$ and $T := P \cap <\boldsymbol{X}, x>$. As a finite p-group, $<\boldsymbol{X}, x>$ satisfies the **normaliser condition**. So let $t \in <\boldsymbol{X}, x> \setminus T$ with $t \in \underline{N}_{<X,\,x>}T$. Then $<P, P^t>$ is not a p-group and therefore there exists a finite subgroup X_0 of P with $\boldsymbol{X} \subseteq X_0$ such that with $X_1 := X_0{}^t$ the group $<X_0, X_1>$ is not a p-group. We have $X_0 \neq \boldsymbol{X} \neq X_1$ since $<\boldsymbol{X}, \boldsymbol{X}^t> \subseteq T$ is a p-group. **Not only $\boldsymbol{X} \subseteq \boldsymbol{X_0}$ pertains but also $\boldsymbol{X} \subseteq \boldsymbol{X_1}$ because of $\boldsymbol{t \in \underline{N}_{<X,\,x>}T}$.** We can repeat this construction whilst replacing $\boldsymbol{X}$ by X_0 and also by its conjugate X_1. Thereby we construct X_{00}, X_{01}, X_{10}, X_{11} and four ascending chains $\boldsymbol{X} < X_0 < X_{00}$, $\boldsymbol{X} < X_0 < X_{01}$, $\boldsymbol{X} < X_1 < X_{10}$ and $\boldsymbol{X} < X_1 < X_{11}$ of finite p-subgroups of G. We subsequently repeat this construction with each of $X_{i_1 i_2}$ and whilst doing this infinitely often we construct $2^{\aleph_0}$ many chains $\boldsymbol{X} < X_{i_1} < X_{i_1 i_2} < \ldots < X_{i_1 i_2 \ldots i_n} < \ldots$ of finite p-subgroups of G with the properties from **a)**. Therefore we can, **starting from an arbitrary subgroup $\boldsymbol{X}$ of P as a "minimal point" or a "root"**, recursively w.r.t. inclusion construct a tree of height $\aleph_0$ of finite p-subgroups of G, which branches properly at each location with proper inclusions, hence must contain $2^{\aleph_0}$ infinite branches. Also any two immediate successors of an arbitrary point do not generate a p-group. These branches are just the required chains. ❑

2.2 enables us to prove very easily the "Asar-Hartley Theorem" which characterises primally locally finite groups satisfying the strong Sylow p-Theorem for the prime p by a cardinality result without the need to endeavour the **continuum hypothesis**:

2.3 Theorem (Ali O. Asar [7]; Brian Hartley [10; 11])**: Let G be a locally finite group and p be a prime. For every countable subgroup H of G shall apply $|\underline{\mathrm{Syl}}_p H| < 2^{\aleph_0}$. Then G satisfies the strong Sylow Theorem for the prime p.** [16] [17]

PROOF: Suppose G does not satisfy the strong Sylow Theorem for the prime p. Then there is a subgroup U of G which does not satisfy the Sylow Theorem for the prime p. Thus according to **2.1** and **2.2** there are $2^{\aleph_0}$ many infinite ascending chains $X_{i_1} < X_{i_1 i_2} < \ldots < X_{i_1 i_2 \ldots i_n} < \ldots$ of finite p-subgroups of U with the properties from **2.2.a)**. Let $\underline{M}$ be the set of all p-subgroups of U which are an ascending union of one of these chains. Then follows $|\underline{M}| = 2^{\aleph_0}$ and any two $\underline{M}$-groups cannot generate a p-group. Now let $H_n := <X_{i_1 i_2 \ldots i_n} \mid i_k \in \{0,1\},\ 1 \leq k \leq n>$ $(n \in \mathbb{N})$ and $H := \bigcup \{ H_n \mid n \in \mathbb{N} \}$. Then H is a countable subgroup of U and hence of G. Since H contains every $\underline{M}$-group it follows herefrom that $|\underline{\mathrm{Syl}}_p H| = 2^{\aleph_0}$. This contradicts the assumption on G. ❑

2.3 Theorem has an immediate **first corollary** for countable locally finite groups:

[15] In his **Lemma 1** [7] **Ali O. Asar** considers **very unwieldy** instead of R a p-subgroup Y of G such that $Y \subsetneq U\,(= P)$, chooses $y \in Y \setminus U$, defines $F^* := U \cap <F, y>$ with $F \subseteq U \cap Y$, finds $F \subseteq F^*$ and $\underline{N}_{<F,\,y>}F^* \supsetneq F^*$, and finally concludes $\underline{N}_G F^* \subsetneq \underline{N}_G U$, since U is the unique maximal p-subgroup of $\underline{N}_G U$ and $\underline{N}_{<F,\,y>}F^* \subsetneq U$.

[16] see the pages 8 and 9 of [1] for a rather different argument close to the original argument of **Ali O. Asar**

[17] The result for **countable** locally finite groups was obtained independently by **Brian Hartley** using a quite different method which allowed him to generalise it from the prime p to a set of primes π when the finite groups of a nested local system have each a nilpotent Hall π-subgroup (see [10]). However, **Brian Hartley** has extended his proof in [11] to **uncountable** locally finite groups by another *beautiful* method.

2.4 Theorem: Let G be a countable locally finite group. Then are equivalent:
1) For every (countable) subgroup H of G we have $|\underline{\mathrm{Syl}}_p H| < 2^{\aleph_0}$.
2) G satisfies the strong Sylow Theorem for the prime p.
3) G satisfies the Sylow Theorem for the prime p.
4) $|\underline{\mathrm{Syl}}_p G| < 2^{\aleph_0}$.
5) Every (countable) subset of G lies in a subgroup U of G with $|\underline{\mathrm{Syl}}_p U| < 2^{\aleph_0}$. ❑

The **second corollary** of **2.3 Theorem** would as a conjugacy assertion be only very difficultly to be proved but is as a cardinality statement trivial:

2.5 Theorem: The locally finite group G satisfies the strong Sylow Theorem for the prime p if and only if every countable subgroup of G satisfies the strong Sylow Theorem for the prime p. The class $\underline{\mathrm{Syl}}$-p of all locally finite groups satisfying the strong Sylow Theorem for the prime p is therefore *countably recognisable* in the spirit of Reinhold Baer. [18] ❑

We now can prove our **key discovery** if the Sylow Theorem for the prime p is not valid in a countable locally finite group which shows a surprising symmetry between non-conjugated Sylow p-subgroups:

Proposition III. Let G be a countable locally finite group and let p be a prime. If two Sylow p-subgroups of G are not conjugate, then neither is singular. [19]

PROOF: Let S and T be Sylow p-subgroups of G which are not conjugate. We saw in **2.1** that one of S or T is not singular. Without loss of generality (**w.l.o.g.**) we may suppose that S is not singular. To prove the result we must show that T is not singular either. Suppose **that T is singular** and let P_0 be a p-uniqueness subgroup for T. It follows by **(1.2) Theorem** that the good Sylow p-subgroups of G are conjugate. However, since T is singular it is very good by **Proposition II** and hence good. Since S and T are not conjugate, S is not good. Clearly, $S \neq T$.

Let T be good w.r.t. the nested local system $\{G_n \mid n \in \mathbb{N}\}$ for G and let $\boldsymbol{F}$ be an arbitrary finite subgroup of T. We show that $\boldsymbol{F}$ cannot be a p-uniqueness subgroup for T and so **T is not singular** since $\boldsymbol{F}$ is chosen arbitrarily, a contradiction. There exists an $m = m(\boldsymbol{F}) \in \mathbb{N}$ with $\boldsymbol{F} \subseteq G_m$. After the renumeration $\{ n \mapsto n + m - 1 \mid n \in \mathbb{N} \}$ we may assume $\boldsymbol{F} \subseteq G_1$. Then $\boldsymbol{F} \subseteq T \cap G_1 \in \underline{\mathrm{Syl}}_p G_1$. If $T \cap G_n$ would be the unique Sylow p-subgroup of G_n for all $n \in \mathbb{N}$ then T would be the unique Sylow p-subgroup of G and we obtain the contradiction that $S = T$. Hence there exists an $n \in \mathbb{N}$ such that G_n has a Sylow p-subgroup R with $R \neq T \cap G_n$. Renumbering again if needed we may assume that $R \in \underline{\mathrm{Syl}}_p G_1$ with $R \neq T \cap G_1$. Choose $y \in R \setminus (T \cap G_1)$. Then $y \notin T$. By Ludvig M. Sylow's Sylow p-Theorem for finite groups there exists an $x \in G_1$ such that $(T \cap G_1)^x = R$ and so $\boldsymbol{F}^x \subseteq R$ since $\boldsymbol{F} \subseteq T \cap G_1$. From $< \boldsymbol{F}^x, y > \subseteq R$ it follows that $< \boldsymbol{F}^x, y >$ is a finite p-group. Let $Y := < \boldsymbol{F}^x, y > \cap T$. Then $Y \neq < \boldsymbol{F}^x, y >$ since $y \notin T$. But Y satisfies, as is well known, the **normaliser condition** and so we can choose $z \in \underline{\mathrm{N}}_{< \boldsymbol{F}^x, y >} Y \setminus Y$. Then $z \notin T$ since otherwise $z \in T \cap < \boldsymbol{F}^x, y > = Y$.

[18] **Baer, Reinhold**: *Abzählbar erkennbare gruppentheoretische Eigenschaften.* Math. Z. **79**, Issue **1** (December 1962), 344-363. **MR0148740** (MR 26-6 #6246 by **Walter Ledermann** [December 1963]); **Zbl 0105.25901** (by **Joachim Neubüser**). https://link.springer.com/journal/209/volumes-and-issues/79-1

[19] We could prove **Proposition III** **explicitly** by using **2.1** and **2.2** and not **Proposition II** but prefer to give the following elegant proof which depends on **Proposition II** (and the **normaliser condition**).

However, z is a p-element outside of T and $T \in \underline{\underline{\mathrm{Syl}}}_p G$, and so $z \notin \underline{\underline{\mathrm{N}}}_G T$. Therefore $< T, T^z >$ is not a p-group. In particular, $T \neq T^z$ and $\boldsymbol{F} \subseteq T \cap T^z$. Hence $\boldsymbol{F}$ is not a p-uniqueness subgroup for T. **Q.E.D.** (Quod Erat Demonstrandum)

If a countable locally finite group contains ***a*** singular Sylow p-subgroup then **all good** Sylow p-subgroups are conjugate by **(1.2) Theorem**. If ***every*** countable subgroup of a (countable) locally finite group contains a singular Sylow p-subgroup then **really all** Sylow p-subgroups are conjugate. This is spelled out by the following theorem:

2.6 Theorem [2]: Let G be a locally finite group and let p be a prime. Suppose that every countable subgroup of G contains a singular Sylow p-subgroup. Then G satisfies the strong Sylow Theorem for the prime p.

PROOF: According to **2.5 Theorem** we can assume that G is countable, and according to **2.4 Theorem** it suffices to show that G satisfies the Sylow Theorem for the prime p. However, this is now immediate since G has a singular Sylow p-subgroup S by assumption. Let T be an arbitrary Sylow p-subgroup of G. If S and T are not conjugate, then by **Proposition III** neither is singular. With this contradiction S and T are conjugate and the result follows. ❑

We finally supplement **2.6 Theorem** with an example for a countable locally finite group H without the (strong) Sylow Theorem for the prime p but with a (countable) subgroup U without singular Sylow p-subgroups. Let $H := D \bullet G^{(\mathbb{N})}$ be the group of **Page 7** and let $U := H$. Let $\boldsymbol{F}$ be a finite subgroup **of $\boldsymbol{P^0}$**. We show that $\boldsymbol{F}$ cannot be a p-uniqueness subgroup of U. Since $\boldsymbol{F}$ is finite, there is an $\mathrm{m} = \mathrm{m}(\boldsymbol{F}) \in \mathbb{N}$ with $\boldsymbol{F} \subseteq U_\mathrm{m}$. Because of $| \underline{\underline{\mathrm{Syl}}}_p G | \geq 2$ there is a $Q_{\mathrm{m}+1} \in \underline{\underline{\mathrm{Syl}}}_p G_{\mathrm{m}+1}$ with $Q_{\mathrm{m}+1} \neq P_{\mathrm{m}+1}$. Then $Q^0 := P \bullet P_1 \times P \bullet P_2 \times \ldots \times P \bullet P_\mathrm{m} \times P \bullet Q_{\mathrm{m}+1} \times P \bullet P_{\mathrm{m}+2} \times \ldots \cap G^{(\mathbb{N})}$ contains the group $\boldsymbol{F}$ and we have $Q^0 \neq \boldsymbol{P^0}$. Now let $R \in \underline{\underline{\mathrm{Syl}}}_p H$ be **good**. A finite subgroup **$\boldsymbol{F}$ of $\boldsymbol{R}$** cannot be a p-uniqueness subgroup of U either. We choose $\mathrm{k} = \mathrm{k}(\boldsymbol{F}) \in \mathbb{N}$ with $\boldsymbol{F} \subseteq U_\mathrm{k}$ and let $S^0 := P \bullet P_1 \times P \bullet P_2 \times \ldots \times P \bullet P_\mathrm{k} \times P \bullet Q_{\mathrm{k}+1} \times P \bullet P_{\mathrm{k}+2} \times \ldots \cap G^{(\mathbb{N})}$ with $Q_{\mathrm{k}+1} \in \underline{\underline{\mathrm{Syl}}}_p G_{\mathrm{k}+1}$ and $Q_{\mathrm{k}+1} \neq P_{\mathrm{k}+1}$. Since all good Sylow p-subgroups of H are conjugate in H, there exists an $x \in H$ with $\boldsymbol{P^0}^x = S^0$. We now can conclude **as in the proof of Proposition III**. Let $y \in S^0 \setminus \boldsymbol{P^0}$ and $z \in \underline{\underline{\mathrm{N}}}_{< \boldsymbol{F}^x, y >} Y \setminus Y$ for $Y := < \boldsymbol{F}^x, y > \cap \boldsymbol{P^0}$. Then $< \boldsymbol{P^0}, \boldsymbol{P^0}^z >$ cannot be a p-group, that is, $\boldsymbol{P^0}^z \neq \boldsymbol{P^0}$. There also exists an element $w \in H$ with $R^w = \boldsymbol{P^0}$. Then $\boldsymbol{F}^w$ lies in $\boldsymbol{P^0}$ and $\boldsymbol{F}^{wz}$ lies in $\boldsymbol{P^0}^z$. Hence $\boldsymbol{F}$ lies in the both different even good Sylow p-subgroups $\boldsymbol{P^0}^{w^{-1}}$ and $\boldsymbol{P^0}^{z^{-1}w^{-1}}$ of H. Therewith R is not singular. If $R \in \underline{\underline{\mathrm{Syl}}}_p H$ should be **not good**, it can on no account be singular since according to **Proposition II** singular Sylow p-subgroups are even very good. ❑

The **good** Sylow p-subgroup $\boldsymbol{P^0}$ of $H := D \bullet G^{(\mathbb{N})}$ provides an example of a Sylow p-subgroup which is **even very good but not singular**: let Σ be a local system for H; by **1.2.a)** exists a nested local subsystem $\Sigma_1 = \{ V_\mathrm{n} \mid \mathrm{n} \in \mathbb{N} \}$ of Σ and by **1.2.b)** exists a Sylow p-subgroup Q of H which is good w.r.t. Σ_1; by **(1.2) Theorem** exists an $x \in H$ with $\boldsymbol{P^0}^x = Q$; then $\boldsymbol{P^0}^x \cap V_\mathrm{n} = Q \cap V_\mathrm{n} \in \underline{\underline{\mathrm{Syl}}}_p V_\mathrm{n}$ and so $| \boldsymbol{P^0} \cap V_\mathrm{n} | = | Q \cap V_\mathrm{n} |$ whence $\boldsymbol{P^0} \cap V_\mathrm{n}$ has the size of a Sylow p-subgroup of V_n ($\mathrm{n} \in \mathbb{N}$); thus $\boldsymbol{P^0}$ reduces into Σ_1. ❑

The following result may be well-known but we can present a novel and shorter proof:

2.7 [2] **Let G be a locally finite group. To any finite p-subgroup P of G shall pertain two finite p-subgroups P_1 and P_2 of G with $P \subseteq P_1 \cap P_2$ such that $< P_1, P_2 >$ is not a p-group. Then there will exist a countable subgroup H of G with $| \underline{\underline{\mathrm{Syl}}}_p H | = 2^{\aleph_0}$.**

PROOF: We construct recursively an infinite ascending chain $F_0 < F_1 < \ldots < F_\mathrm{n} < \ldots$ of finite subgroups of G and for every $\mathrm{n} \in \mathbb{N}_0$ a set Σ_n of p-subgroups of F_n such that for every $\mathrm{n} \in \mathbb{N}_0$ we have: (i) $| \Sigma_\mathrm{n} | = 2^\mathrm{n}$. (ii) Every two Σ_n-groups do not generate a p-group. (iii) For $\mathrm{n} \geq 1$ every $\Sigma_{\mathrm{n}-1}$-group lies in at least two Σ_n-groups.

Let $F_0 := \langle 1 \rangle$ and $\Sigma_0 := \{ \langle 1 \rangle \}$. Let $n \in \mathbb{N}$ and let $F_0 < F_1 < \ldots < F_{n-1}$ and $\{ \Sigma_i \mid i < n \}$ already be constructed. We put $\Sigma_n := \{ P_1, P_2 \mid P_1$ and P_2 are finite p-subgroups of G such that $\langle P_1, P_2 \rangle$ is not a p-group and there exists exactly one Σ_{n-1}-Gruppe P with $P \subseteq P_1 \cap P_2 \}$. From the properties (i)-(iii) of Σ_{n-1} and from the prerequisite on G then follow (i)-(iii) for Σ_n. Let F_n be the span of all Σ_n-groups. Hereafter F_n is a finite subgroup of G with $F_{n-1} < F_n$. Let $H := \bigcup \{ F_i \mid i \in \mathbb{N}_0 \}$. Then H is a countable subgroup of G. Let $\underline{M}$ be the set of all p-subgroups of G which are an ascending union of a chain $S_0 < S_1 < \ldots < S_n < \ldots$ of finite p-subgroups $S_i \in \Sigma_i$ $(i \in \mathbb{N}_0)$. According to (i) and (iii) we have $|\underline{M}| = 2^{\aleph_0}$ and according to (ii) any two $\underline{M}$-groups cannot generate a p-group. H contains every $\underline{M}$-group, so that from the properties of $\underline{M}$ (and the countability of H) $|\underline{\mathrm{Syl}}_p H| = 2^{\aleph_0}$ follows. We have constructed an infinitely high ($\aleph_0$) tree of finite p-subgroups of G which branches properly at each location with proper inclusions and in which any two immediate successors of an arbitrary point do not generate a p-group. ❑

2.8 HAUPTSATZ [2]**: Let *G* be a locally finite group and let *p* be a prime. The following are equivalent: 1) *G* satisfies the strong Sylow Theorem for the prime *p*.**
2) In every subgroup *U* of *G* every Sylow *p*-subgroup of *U* is singular.
3) Every countable subgroup *H* of *G* contains a *p*-uniqueness subgroup of *H*.
4) Every countable subgroup *H* of *G* contains a singular Sylow *p*-subgroup of *H*.
5) Every countable subgroup of *G* satisfies the Sylow Theorem for the prime *p*.
6) If *H* is a countable subgroup of *G*, then $|\underline{\mathrm{Syl}}_p H| < 2^{\aleph_0}$.

PROOF: **2)** ⇒ **3)** and **3)** ⇒ **4)** are clear. **4)** ⇒ **5)** is valid by **2.6 Theorem**, **5)** ⇒ **6)** is valid by **2.4 Theorem**, and **6)** ⇒ **1)** is valid by **2.3 Theorem**. It remains to show **1)** ⇒ **2)**:[20] let $U \subseteq G$; then U satisfies the strong Sylow Theorem for the prime p; by **2.5 Theorem** and **2.4 Theorem** we have that $|\underline{\mathrm{Syl}}_p H| < 2^{\aleph_0}$ for any countable subgroup H of U; by **2.7** there is a finite p-subgroup P of U such that for all finite p-subgroups P_1 and P_2 of U with $P \subseteq P_1 \cap P_2$ the group $\langle P_1, P_2 \rangle$ is a p-group; by **Proposition I** it follows that P is a p-uniqueness subgroup of U; let $S \in \underline{\mathrm{Syl}}_p U$ with $P \subseteq S$; let $T \in \underline{\mathrm{Syl}}_p U$ and $x = x(T) \in U$ with $S = T^{x^{-1}}$; then P^x is a p-uniqueness subgroup of U with $P^x \subseteq T$; hence T is singular by means of P^x. ❑

Having proved the **HAUPTSATZ** which is our ***Charakterisierungssatz*** we can be very proud of, we are now ready to prove the announced main theorem characterising the locally finite groups which satisfy the strong Sylow Theorem for the prime p:

Theorem IV. Let *G* be a locally finite group and let *p* be a prime. The following are equivalent: 1) *G* satisfies the strong Sylow Theorem for the prime *p*.
2) Every subgroup *S* of *G* contains a finite *p*-subgroup which is singular in *S*.

PROOF: **1)** ⇒ **2)** follows directly from **2.8 HAUPTSATZ** and **Proposition I**, and **2)** ⇒ **1)** follows directly from **Proposition I** and **2.8 HAUPTSATZ**. ❑

[20] In his **(1.5) Theorem** [1] "*If the locally finite group G satisfies the strong Sylow Theorem for the prime p there exists a finite p-subgroup P which is singular in G*." **Kegel** constructs, by contradiction, **directly** a countable subgroup of *G* which has $2^{\aleph_0}$ maximal p-subgroups and therefore contradicts **2.4 Theorem**.

3. Novel concepts and results for Sylow theory in (locally) finite groups and comparative outline of Kegel's previous work [8]

We continue with some further thoughts, some early results, and some questions that could be quite useful for future researchers in Sylow theory of (locally) finite groups. The status quo of Sylow theory of locally finite groups is *beautifully* summarised in [1] and [9], in particular the contributions by **Prof. Brian Hartley** (see **Footnote 17**, [2], [10] and [11]) who has contributed very eye-opening to simple locally finite groups as well [21].

In any locally finite group G for all subgroups U of G the set ***$\underline{\mathrm{Unique}}_pU$*** of finite p-subgroups which are p-uniqueness subgroups of U is non-empty if G satisfies the strong Sylow Theorem for the prime p, that is, if G belongs to the class $\underline{\mathrm{Syl}}$-p of locally finite groups satisfying the strong Sylow Theorem for the prime p, and should this set be non-empty for a countable U then all the good Sylow p-subgroups of U are conjugate. Let U be finite. Then we have already $\underline{\mathrm{Unique}}_pU \neq \emptyset$ because of $\underline{\mathrm{Syl}}_pU \subseteq \underline{\mathrm{Unique}}_pU$. The Sylow p-subgroups of U are the **maximal** members of $\underline{\mathrm{Unique}}_pU$, w.r.t. inclusion and order. It is a very considerable **challenge** to determine the **minimal** members of $\underline{\mathrm{Unique}}_pU$, with respect to either inclusion or order, in case U and $\underline{\mathrm{Syl}}_pU$ are sufficiently "known", in particular if U will be a "known" finite simple group or a p-soluble group. Note that whenever $P \subsetneq Q \subsetneq R$ are p-subgroups of U where Q is a minimal p-uniqueness subgroup, or is minimal singular in U, then P is contained in at least two, in fact $p+1$, Sylow p-subgroups of U and R will be another p-uniqueness subgroup of U. It is much to be hoped that some progress be made to this **challenge** in the future. For example, the question of whether resp. when *minimal* p-uniqueness subgroups are conjugate, quite similar to the *maximal* ones, is of some interest, or, whether minimal w.r.t. inclusion implies minimal w.r.t. order, the converse being clearly obvious. We would then also come to better know the p-uniqueness subgroups of locally finite groups, in particular the simple and the locally p-soluble ones, and, many thanks to **Kegel**'s **(4.4) Theorem**, of locally finite groups in general which belong to the lovely class $\underline{\mathrm{Syl}}$-p. A good starting point would be to study minimal p-uniqueness subgroups of the finite Symmetric and Alternating Groups where a Sylow 2-subgroup of an Alternating Group is just a next to maximal 2-uniqueness subgroup of the corresponding Symmetric Group so that we have to scrutinise only the Sylow p-subgroups of the Symmetric Groups.

Let G be a locally finite group, $S \in \underline{\mathrm{Syl}}_pG$ and $F \subseteq G$. We call F ***minimal p-unique w.r.t. S***, if F is a w.r.t. **order** minimal p-uniqueness subgroup of G with $F \subseteq S$, that is, F is p-unique with $F \subseteq S$ and each (finite) subgroup P of S with $|P| < |F|$ lies in at least two Sylow p-subgroups of G. If there exists an $S \in \underline{\mathrm{Syl}}_pG$, such that F is w.r.t. S minimal p-unique, then F is called ***minimal p-unique*** (in G). Obviously G is p-closed if and only if $< 1 >$ is minimal p-unique (in G).

3.1 [2] **Let G be a locally finite group satisfying the strong Sylow Theorem for the prime p. Then we have: a) Each Sylow p-subgroup of G contains at least one (w.r.t. order) minimal p-unique subgroup of G. b) Each two (w.r.t. order) minimal p-unique subgroups of G have the same order.**

[21] The results of **Kegel**'s lectures [1] with his fine discovery of split sequences of finite p-perfect subgroups depend heavily upon the hazy **(2.4) Theorem** about the "known" finite simple groups (see **Page 3**). **Brian Hartley** made the remark quoted in **Footnote 3** in 1990 and wrote 1994 an eye-opening paper on simple locally finite groups which, however, did not refer to **Kegel**'s work and not even included it in its list of 56 references (see **Hartley, Brian:** *Simple locally finite groups.* In: Finite and Locally Finite Groups, Proceedings of the NATO Advanced Study Institute on Finite and Locally Finite Groups, Istanbul, Turkey, August 14-27, 1994, 1-44. NATO ASI Series C: Mathematical and Physical Sciences – Vol. **471**. Dordrecht: Springer Science + Business Media, B.V., 1995. ISBN 978-94-010-4145-4 [Softcover reprint] and ISBN 978-94-011-0329-9 [eBook]. **MR1362804**; **Zbl 0855.20030** [by **Gernot Stroth**]. https://www.springer.com/de/book/9780792336693). The paper could appear only posthumously and was completed and prepared for publication by **Richard E. Phillips** (3 December 1936 until 9 November 1999). It might be worthwhile to inspect **Hartley**'s estate … .

PROOF: **a)** Let $S \in \underline{\underline{\mathrm{Syl}}}_p G$ and $\underline{\underline{\mathrm{U}}}(G,S) := \{ F \mid F$ is a p-uniqueness subgroup of G with $F \subseteq S \}$. According to **2.8 HAUPTSATZ** we have $\underline{\underline{\mathrm{U}}}(G,S) \neq \emptyset$ and of course each $\underline{\underline{\mathrm{U}}}(G,S)$-group has finite order. Thus $\underline{\underline{\mathrm{U}}}(G,S)$ contains a (w.r.t. S) minimal p-unique subgroup due to the well ordering of $\mathbb{N}$. **b)** Let F_1 and F_2 be two minimal p-unique subgroups of G. For symmetry reasons it then suffices to show $|F_1| \leq |F_2|$. Let $S_1, S_2 \in \underline{\underline{\mathrm{Syl}}}_p G$ with $F_1 \subseteq S_1$ and $F_2 \subseteq S_2$. Since $G \in \underline{\underline{\mathrm{Syl}}}\text{-}p$ there exists an $x \in G$ with $S_1 = S_2^{\,x}$. Then $F_2^{\,x}$ is a p-uniqueness subgroup of G with $F_2^{\,x} \subseteq S_1$. It follows that $|F_1| \leq |F_2^{\,x}| = |F_2|$ since F_1 is minimal p-unique w.r.t. S_1. ❑

Let G be a locally finite group satisfying the strong Sylow Theorem for the prime p and $S \in \underline{\underline{\mathrm{Syl}}}_p G$. According to **3.1.a)** S contains a (w.r.t. S) minimal p-unique subgroup F. We define $\mathrm{a}_p = \mathrm{a}_p(G) \in \mathbb{N}_0$ by $|F| =: p^{\mathrm{a}_p}$, that is, we let $\mathbf{a}_p$ be the composition length of F. According to **3.1.b)** this definition is independent of the special choice of the Sylow p-subgroup S of G. Whereby consequently a_p is a (numerical) Sylow p-invariant of G. We call a_p the ***p-uniqueness*** of G. This Sylow p-invariant enqueues into the list – even is in the vanguard … – of other Sylow p-invariants which play a major rôle in (locally) finite group theory, e.g. the order p^{b_p} of a Sylow p-subgroup, its nilpotency class $\mathbf{c}_p$, its solubility length $\mathbf{d}_p$, its exponent p^{e_p}, the composition length $\mathbf{i}_p - 1$ of a proper w.r.t. order maximal Sylow p-intersection (see below) and further. The real peculiarity of a_p is that it is not determined by a Sylow p-subgroup as abstract p-group alone but depends on its embedding into the whole group and the respective relationships to the other Sylow p-subgroups. Then (w.r.t. inclusion or order maximal) intersections of two or several Sylow p-subgroups are of interest and deserve further study.

Closely related to p-uniqueness subgroups are the so-called Sylow p-intersections. If G is any locally finite group we define $\underline{\underline{\mathbf{I}}}_p \boldsymbol{G} := \{ S \cap T \mid S, T \in \underline{\underline{\mathrm{Syl}}}_p G$ with $S \neq T \}$. The elements of $\underline{\underline{\mathrm{Syl}}}_p G \cup \underline{\underline{\mathrm{I}}}_p G$ resp. of $\underline{\underline{\mathrm{I}}}_p G$ are called ***Sylow p-intersections*** resp. ***proper Sylow p-intersections*** (in German ***(echte) Sylow p-Durchschnitte***) of G. We then consider two sets of subgroups of G: $\underline{\underline{\mathbf{I}}}_p^{\mathrm{e}} \boldsymbol{G} := \{ Q \subseteq G \mid Q$ is finite and is a with respect to **inclusion** maximal $\underline{\underline{\mathrm{I}}}_p G$-group $\}$, $\underline{\underline{\mathbf{I}}}_p^{\mathrm{m}} \boldsymbol{G} := \{ Q \subseteq G \mid Q$ is finite and is a with respect to **order** maximal $\underline{\underline{\mathrm{I}}}_p G$-group $\}$. Then we have $\underline{\underline{\mathrm{I}}}_p^{\mathrm{m}} G \subseteq \underline{\underline{\mathrm{I}}}_p^{\mathrm{e}} G$ and $\underline{\underline{\mathrm{I}}}_p G = \emptyset$ if and only if G is p-closed. We now define $\mathrm{i}_p = \mathrm{i}_p(G) \in \mathbb{N}_0$ as follows: if $\underline{\underline{\mathrm{I}}}_p^{\mathrm{m}} G = \emptyset$ let $\mathbf{i}_p := 0$ and if $\underline{\underline{\mathrm{I}}}_p^{\mathrm{m}} G \neq \emptyset$ let $p^{\mathrm{i}_p - 1} := |Q|$ for $Q \in \underline{\underline{\mathrm{I}}}_p^{\mathrm{m}} G$, that is, we let $\mathbf{i}_p - 1$ be the composition length of an arbitrary $\underline{\underline{\mathrm{I}}}_p^{\mathrm{m}} G$-group. i_p is a (numerical) Sylow p-invariant of G: if $Q_{\mathrm{i}} \in \underline{\underline{\mathrm{I}}}_p^{\mathrm{m}} G$ (i = 1, 2) then there is a finite subgroup F of G containing Q_1 and Q_2; let $P_{\mathrm{i}} \in \underline{\underline{\mathrm{Syl}}}_p F$ with $Q_{\mathrm{i}} \subseteq P_{\mathrm{i}}$ (i = 1, 2) and $x \in F$ with $P_2 = P_1^{\,x}$; then $Q_1^{\,x} \in \underline{\underline{\mathrm{I}}}_p G$ and so $|Q_1| = |Q_1^{\,x}| \leq |Q_2|$; by symmetry $|Q_2| \leq |Q_1|$. We call i_p the ***Sylow p-intersection*** of G. Apparently $\mathrm{i}_p(G) = 1$ if and only if $S = T$ or $S \cap T = \langle 1 \rangle$ for all $S, T \in \underline{\underline{\mathrm{Syl}}}_p G$. In particular, if $\mathrm{i}_p(G) = 1$ then each Sylow p-subgroup of G is a ***TI-set***, that is, either $S \cap S^x = \langle 1 \rangle$ or $S = S^x$ for all $x \in G$. The observation that $\underline{\underline{\mathrm{I}}}_p^{\mathrm{e}} G \neq \emptyset$ implies the existence of **localised** p-uniqueness subgroups of G delivers the relationship between Sylow p-intersections and p-uniqueness subgroups: let $Q \in \underline{\underline{\mathrm{I}}}_p^{\mathrm{e}} G$; then there are $S, T \in \underline{\underline{\mathrm{Syl}}}_p G$ such that $S \neq T$ and $Q = S \cap T$; if $x \in S \setminus Q$ then $\langle Q, x \rangle$ is a p-uniqueness subgroup of G **with respect to *S***, and if $y \in T \setminus Q$ then $\langle Q, y \rangle$ is a p-uniqueness subgroup of G **with respect to *T***. Therefore, if $\underline{\underline{\mathrm{I}}}_p^{\mathrm{e}} G \neq \emptyset$ then $\mathrm{a}_p(G)$ is defined and if $\underline{\underline{\mathrm{I}}}_p^{\mathrm{m}} G \neq \emptyset$ then $\mathrm{i}_p(G) \leq \mathrm{a}_p(G) + 1$. Suppose the locally finite group G contains a **finite** Sylow p-subgroup S and let P be any finite p-subgroup of G; since $\langle S, P \rangle$ is finite, S contains a conjugate of P; thus $|S|$ is an upper bound for the orders of the finite p-subgroups of G; hence $T \in \underline{\underline{\mathrm{Syl}}}_p G$ has a maximal finite subgroup M; if $x \in T$ then $\langle M, x \rangle$ is finite and so $\langle M, x \rangle = M$; hence $T = M$ is finite; thus S and T

are Sylow p-subgroups of the finite group $< S, T >$ and so are conjugate; so G satisfies the strong Sylow Theorem for the prime p and all Sylow p-subgroups of G have order $|S| =: p^{\mathrm{b}_p(G)}$ and we have $\mathrm{i}_p(G) - 1 \le \mathrm{a}_p(G) \le \mathrm{b}_p(G)$. **Hence in a countable locally finite group all Sylow *p*-subgroups have the same order.** We summarise our first findings about Sylow p-intersections of G: **G contains a finite Sylow p-subgroup $\Rightarrow \underline{\mathbf{I}}_p{}^{\mathbf{m}}G \neq \varnothing$ and G satisfies the strong Sylow Theorem for the prime $p \Rightarrow \underline{\mathbf{I}}_p{}^{\mathbf{e}}G \neq \varnothing$ and $\mathbf{i}_p(G) - 1 \le \mathbf{a}_p(G) \le \mathbf{b}_p(G) \Rightarrow G$ contains through $\underline{\mathbf{I}}_p{}^{\mathbf{e}}G$ localised p-uniqueness subgroups.**

Otto H. Kegel not only had discovered during the eighties ("während der 80er Jahre"; see https://en.wikipedia.org/wiki/1980s) conditions for locally finite groups to satisfy the strong Sylow Theorem for the prime p (and presented them **in June 1987** in four lectures [1]) **but already had presented** such conditions, **which are related to Sylow *p*-intersections**, in lectures **during December 11, 1973** [8]. Sequences of subgroups of a group G are either finite or countably infinite. The sequence ω_1 is ***larger*** than the sequence ω if and only if ω is an initial segment of ω_1. **Kegel** calls a sequence $\omega = \{ S_i \mid i \in \mathbb{N} \}$ of p-subgroups of a (locally finite) group G ***Sylow-separated*** if there exists a sequence $\pi = \{ P_i \mid i \in \mathbb{N} \}$ of maximal p-subgroups of G (separating ω) so that if $S_i, S_{i+1} \in \omega$ one has $S_i \subseteq P_i \cap S_{i+1}$ and $S_{i+1} \not\subset P_i$ ($i \in \mathbb{N}$). Clearly, there is an infinite Sylow-separated sequence of p-subgroups in the group G if and only if the set of Sylow-separated sequences of p-subgroups of G does not satisfy the maximum condition in the partial order defined above. This condition is countably recognisable (**Lemma 1** of [8]). Similar to **2.1** and **2.2**, **Kegel** ***shows*** that if the locally finite group G has a subgroup S with two maximal p-subgroups which are not conjugate then there exists an infinite Sylow-separated sequence of finite p-subgroups of G (**Lemma 2** of [8]) and ***deduces*** from this, rather similar to **2.3 Theorem**, that if every Sylow-separated sequence of p-subgroups of G is finite, then G satisfies the strong Sylow Theorem for the prime p (**Proposition 3** of [8]). Vice versa to his **question** of 1987 (see **Page 2**), he ***asks*** the **question**, if the converse holds, that is, "*if the strong Sylow Theorem for the prime p forces the maximum condition for Sylow-separated sequences of p-subgroups*" [8, § 3]. The set $\underline{\mathrm{I}}_pG$ is partially ordered by inclusion and every ascending sequence of elements of $\underline{\mathrm{I}}_pG$ is a Sylow-separated sequence of p-subgroups of G. Now the maximum condition on Sylow-separated sequences of p-subgroups of G entails, of course, that $\underline{\mathrm{I}}_pG$ satisfies the maximum condition, while the converse is not true. **For the proofs of 2.2 and Proposition III the normaliser condition of finite *p*-groups is crucial.** With a quite similar argument, **Kegel** ***shows*** that if $\underline{\mathrm{I}}_pG$ satisfies the maximum condition and $\underline{\mathrm{N}}_S(S \cap T) \neq S \cap T \neq \underline{\mathrm{N}}_T(S \cap T)$ for all $S \cap T \in \underline{\mathrm{I}}_pG$, then G satisfies the Sylow Theorem for the prime p (**Proposition 4** of [8]). He then ***shows*** that if $\underline{\mathrm{I}}_pG$ satisfies the maximum condition and all p-subgroups of G satisfy the **normaliser condition** and G has a maximal p-subgroup which is **countable**, then G satisfies the strong Sylow Theorem for the prime p (**Proposition 5** of [8]). Here the premise of existence of a countable maximal p-subgroup is not dispensable since the maximum condition on proper Sylow p-intersections is not countably recognisable. **Kegel** then ***shows***, spending considerable effort, that if in the locally finite group G every $\underline{\mathrm{I}}_pG$-group satisfies the minimum condition for subgroups then G satisfies the strong Sylow Theorem for the prime p if and only if G satisfies the maximum condition for Sylow-separated sequences of p-subgroups (**Theorem 8** of [8]), ***deduces*** quite easily for G **satisfying the minimum condition for *p*-subgroups**, that is,

$G \in \underline{\mathrm{Min}}\text{-}p$, two equivalent conditions for satisfying the strong Sylow Theorem for the prime p, namely the maximum conditions for Sylow-separated sequences of p-subgroups and for $\underline{\underline{\mathrm{I}}}_p G$, and then ***asks*** the final **question** "*if the maximum condition for $\underline{\underline{\mathrm{I}}}_p G$ implies the strong Sylow Theorem for the prime p*". It is much to be hoped that our ***Charakterisierungssatz* 2.8** and our ***p*-uniqueness subgroups** will help to answer this question and the **question** "*if G belonging to* $\underline{\mathrm{Syl}}\text{-}p$ *entails the maximum condition for Sylow-separated sequences of p-subgroups of G*", in a similar fashion as they answered **Kegel**'s **question** of **Page 2** which was **the actual reason and motivation** for this publication.

Acknowledgements The author wishes to thank heartfeltly his most fabulous wife *Helga*. Without her tenderest and unconditional support over so many years this publication would never have been born. Most importantly, he is forever and ever grateful to **Prof. Brian Hartley** and to his teacher **Prof. Otto H. Kegel** (see [1], p. 25) for their *beautiful* papers which provide incredible insights and give marvellous pleasure in reading and understanding. We prepared this **Line numbered edition** of the manuscript in order to lively encourage **(Scholarly Peer) Review** which would lead to further acknowledgements.

Declarations On behalf of all authors, the corresponding author (as the only author) states firmly that there is no conflict of (competing) interest(s). **Data availability statement**: this publication consists "only" of the **Manuscript** with **thirty-two Figures** (as parts of the Manuscript), **twenty-one Footnotes**, **six Tables**, and the **four Endnotes**; all data is immediately accessible through the original PDF-file. We do not have associated data. This research **did not receive**, and did not need, any specific grant from funding agencies in the public, commercial, or not-for-profit sectors.

The discoverer and explorer of Sylow theory

Ludvig M. Sylow
(12 December 1832 until 7 September 1918)
(see https://en.wikipedia.org/wiki/Peter_Ludwig_Mejdell_Sylow
and https://mathshistory.st-andrews.ac.uk/Biographies/Sylow/
and https://en.wikipedia.org/wiki/Sylow_theorems)

Monsieur L. Sylow (Frederiksiiald en Norvege)**:** *Théorèmes sur les groupes de substitutions.*
Mathematische Annalen **5**, Issue **4** (December 1872), 584-594
(see https://link.springer.com/journal/208/volumes-and-issues/5-4).
MR1509796 (no review: see https://mathscinet.ams.org/mathscinet-getitem?mr=1509796);
Zbl JFM 04.0056.02 (by **Eugen O. Netto**:
see https://www.zbmath.org/?q=an%3A04.0056.02
and https://www.zbmath.org/pdf/02718820.pdf).

References

MR AMS Mathematical Reviews ® ( ) considered always together with MathSciNet ® ( ) and with MR Lookup ( ) (see https://www.ams.org/mr-database, https://mathscinet.ams.org/mathscinet, https://mathscinet.ams.org/mrlookup)

Zbl Zentralblatt MATH (zbMATH Open ® THE FIRST RESOURCE FOR MATHEMATICS) (see https://www.zbmath.org/)

[1] Kegel, Otto H.: *Four lectures on Sylow theory in locally finite groups.* In: Group Theory – Proceedings of the Singapore Group Theory Conference held at the National University of Singapore, **June 8-19, 1987**, edited by Kai Nah Cheng and Yu Kiang Leong, 3-28. ISBN 978-3-11-011366-2 (Hardcover) and ISBN 978-3-11-084839-7 (PDF) (see https://www.degruyter.com/view/title/11283). Berlin-New York: Walter de Gruyter January 1989, reprinted November 2016. **MR0981832** (MR 90c:20037 by **Brian Hartley** [March 1990]); **Zbl 0659.20024** (by **Bernhard Amberg**). https://www.degruyter.com/view/book/9783110848397/10.1515/9783110848397-004.xml

[2] Flemisch, Felix F.: *Lokal endliche Gruppen mit Sylow p-Satz oder mit min-p. I: Grundbegriffe, ein Charakterisierungssatz und lokale Prinzipien.* Diplomarbeit an der Mathematischen Fakultät der Albert-Ludwigs-Universität zu Freiburg im Breisgau, Bundesland Baden-Württemberg, Deutschland.

Submitted **on October 8, 1984**, **exactly ten years** before the tragic death of **Brian Hartley** – whose contributions to **locally finite group theory** the author had studied in great detail and with the deepest admiration – **while mountain hiking**. **Prof. Hartley** was a keen hill walker, and it was while descending *Helvellyn* (see https://en.wikipedia.org/wiki/Helvellyn) in the English Lake District, nearby his homeland, **on October 8, 1994**, that he collapsed with a heart attack and died ☹ (see https://en.wikipedia.org/wiki/**Brian_Hartley** and the references cited there, in particular MacTutor).

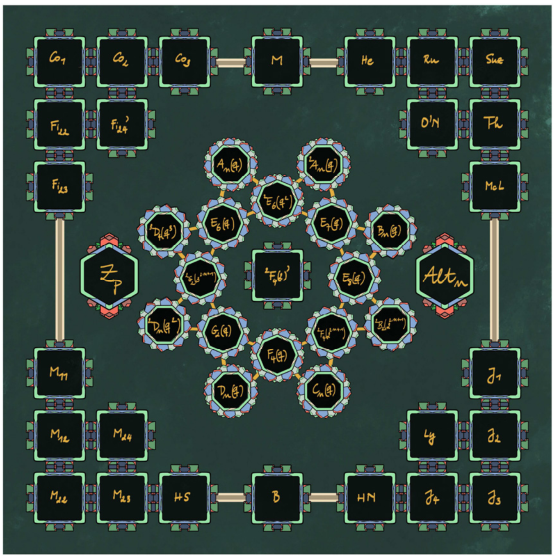
[3] Rae, Andrew: *Local Systems and Sylow Structure in Locally Finite Groups. I.* Proc. Camb. Phil. Soc. **72**, Issue **2** (September 1972), 141-160. **MR0302766** (MR 46-2 #1909 by **Rüdiger Göbel** [August 1973]); **Zbl 0247.20024** (Autorreferat). https://www.cambridge.org/core/journals/mathematical-proceedings-of-the-cambridge-philosophical-society/issue/15C07559DD5656BF16C0EEE87507AD31

[4] Rae, Andrew: *Local Systems and Sylow Structure in Locally Finite Groups. II.* Proc. Camb. Phil. Soc. **75**, Issue **1** (January 1974), 1-22. **MR0360836** (MR 50-6 #13283 by **Bertram A. F. Wehrfritz** [December 1975]); **Zbl 0275.20060** (Autorreferat). https://www.cambridge.org/core/journals/mathematical-proceedings-of-the-cambridge-philosophical-society/issue/484A09783837877807F3E856353F83CD

[5] WIKIPEDIA: *Classification of finite simple groups.* This page was last edited on 23 January 2024, at 07:37 (UTC). WIKIPEDIA The Free Encyclopedia https://en.wikipedia.org/wiki/Classification_of_finite_simple_groups

Solomon, Ronald M.: *A brief history of the classification of the finite simple groups.* Bull. Amer. Math. Soc., New Ser. **38**, No. **3** (July 2001), 315-352. **MR1824893** (MR 2002k:20002 by **Gernot Stroth** [November 2002]); **Zbl 0983.20001** (by **Ulrich Dempwolff**). https://www.ams.org/journals/bull/2001-38-03/S0273-0979-01-00909-0/S0273-0979-01-00909-0.pdf

Aschbacher, Michael G.: *The Status of the Classification of the Finite Simple Groups.* Notices of the AMS, Volume **51**, Number **7** (August 2004), 736-740. **MR2072045** (in the online edition no review but MSC and References; in the print edition of 2004 no proper entry but entries in Author index with MSC and in Subject index under 20D05 each without reference of type 2004x:20abc); **Zbl 1113.20302** (no review but two lines Summary and MSC). https://www.ams.org/notices/200407/fea-aschbacher.pdf



The classification of the finite simple groups into **13 sporadic groups** above **18** ***beautiful*** **infinite families** around another “sporadic” group and **further 13 sporadic groups below**
© https://commons.wikimedia.org/wiki/File:Classification_of_the_finite_simple_groups.jpg

Elwes, Richard: *An enormous theorem: the classification of finite simple groups.* Plus Magazine, Issue **41** (December 2006). https://plus.maths.org/content/os/issue41/features/elwes/index

Steingart, Alma: *A group theory of group theory: Collaborative mathematics and the 'uninvention' of a 1000-page proof.* Social Studies of Science, Volume **42**, Issue **2** (April 2012 [February 23, 2012]), 185-213. https://journals.sagepub.com/toc/sssb/42/2

Guralnick, Robert M.: *Commentary on "A brief history of the classification of the finite simple groups" by Ronald Solomon.* Bull. Amer. Math. Soc., New Ser. **55**, No. **4** (October 2018), 451-452. **MR3854073** (inadequate Summary only [eMRS 1F January 2019]); **Zbl 1395.20009** (no review but MSC and References). https://www.ams.org/journals/bull/2018-55-04/S0273-0979-2018-01638-8/S0273-0979-2018-01638-8.pdf

Solomon, Ronald M.: *Afterword to the article "A brief history of the classification of the finite simple groups".* Bull. Amer. Math. Soc., New Ser. **55**, No. **4** (October 2018), 453-457. **MR3854074** (insufficient Summary only [eMRS 1F January 2019]); **Zbl 1395.20012** (no review but MSC and References). https://www.ams.org/journals/bull/2018-55-04/S0273-0979-2018-01639-X/S0273-0979-2018-01639-X.pdf

Solomon, Ronald M.: *The Classification of Finite Simple Groups: A Progress Report.* Notices of the AMS, Volume **65**, Number **6** (June/July 2018), 646-651. **MR3792856** (by **Jürgen Müller** [eMRS 1F June 2019]); **Zbl 1398.20001** (no review but MSC). https://www.ams.org/journals/notices/201806/rnoti-p646.pdf

Wolffe, Julia: *Michael Aschbacher and the sociology of mathematical proof w.r.t. the classification theorem for finite simple groups.* Julia Wolffe's Notes (28 January 2020). https://juliawolffenotes.home.blog/2020/01/28/michael-aschbacher-and-the-sociology-of-mathematical-proof-w-r-t-the-classification-theorem-for-finite-simple-groups/

[6] Kovács, László G.; Neumann, Bernhard H.; de Vries, Hugo: *Some Sylow subgroups.* Proc. Royal Soc. London, Series **A**, Volume **260**, Issue **1302** (7 March 1961), 304-316. **MR0167531** (MR 29-5 #4803 by **Dudley E. Littlewood** [May 1965]); **Zbl 0102.26205** (by **Wolfgang P. Kappe**). https://royalsocietypublishing.org/toc/rspa1938/1961/260/1302

[7] Asar, Ali O.: *A Conjugacy Theorem for Locally Finite Groups.* J. London Math. Soc. **(2) 6**, Issue **2** (February 1973), 358-360. **MR0311775** (MR 47-1 #337 by **James E. Roseblade** [January 1974]); **Zbl 0253.20042** (Aus der Arbeit). https://londmathsoc.onlinelibrary.wiley.com/toc/14697750/1973/s2-6/2

[8] Kegel, Otto H.: *Chain conditions and Sylow's theorem in locally finite groups.* In: Symposia Matematica, Volume **XVII**, Convegno sui Gruppi Infiniti, Istituto Nazionale di Alta Matematica (INδAM) "Francesco Severi", Roma, **11-14 Dicembre 1973**, 251-260. ISBN 978-0-12612-217-6. London-New York: Academic Press 1976. **MR0414714** (MR 54-2 #2809 by **Rüdiger Göbel** [August 1977]); **Zbl 0328.20029** (Autorreferat).

[9] Dixon, Martyn R.: *Sylow Theory, Formations and Fitting Classes in Locally Finite Groups.* Series in Algebra, Volume **2**. Singapore: World Scientific Publishing Co. Pte. Ltd. 1994. ISBN 978-981-02-1795-2 (Hardcover) and ISBN 978-981-4518-13-0 (eBook). https://www.worldscientific.com/worldscibooks/10.1142/2386

[10] Hartley, Brian: *Sylow subgroups of locally finite groups.* Proc. London Math. Soc. **(3) 23**, Issue **1** (September 1971), 159-192. **MR0304488** (MR 46-3 #3623 by **Bertram A. F. Wehrfritz** [September 1973]); **Zbl 0221.20040** (Autorreferat). https://londmathsoc.onlinelibrary.wiley.com/toc/1460244x/1971/s3-23/1

[11] Hartley, Brian: *Sylow theory in locally finite groups.* Compositio Mathematica, Tome **25**, Fascicule **3** (1972), 263-280. **MR0316557** (MR 47 #5104 [1974]); **Zbl 0248.20036** (Autorreferat). http://www.numdam.org/issues/CM_1972__25_3/

[12] Flemisch, Felix F.: *The Strong Sylow Theorem for the Prime p in Simple Locally Finite Groups.* Ischia Group Theory (IGT 2024), April 8-13, 2024, Journal of Mathematical & Computer Applications (JMCA), Volume 4, Issue 1 (see https://www.onlinescientificsresearch.com/journals/jmca/archives/2025/4/1). Preprint at https://buchshop.bod.de/the-strong-sylow-theorem-for-the-prime-p-in-simple-locally-finite-groups-dipl-math-felix-f-flemisch-9783758321207 and Conference Presentation at https://drive.google.com/drive/folders/1rD92YXaJ5fs6VvK1SSNfXBpMK6uDjjtr and https://drive.google.com/drive/folders/1WqjFnFJzragVU3A4YqP7hc5_B2FrKXGZ

Note The MR number in brackets refers to the print edition of Mathematical Reviews®, which was printed very regrettably only until 2012 ☹, and includes the reviewer and the month of publication. Since 2013 reference is to the online edition of MathSciNet® and the **electronic Mathematical Reviews® (eMR) Sections** (see http://www.ams.org/publications/mrsections/mrsections).

About the author

Felix **F. Flemisch** proudly received his first-ever degree **Baccalaureus der Mathematik** (**Bacc.Math.**) in July 1974 with the alas unpublished **bachelor's thesis** "Über einfache Punkte affiner Varietäten" from the most venerable Albert-Ludwigs-Universität at *beautiful* Freiburg im Breisgau, Baden-Württemberg, Germany, under the rather thorough supervision of **Akadem. Rat Dr. Herbert Götz**, and his degree **Master of Science** (**M.Sc.**) from the Faculty of Science of the highly esteemed University of London, Bedford College, United Kingdom, in August 1975 under the supervision of greatly adored **Prof. Paul M. Cohn**. From October 1975 until July 1976 he was employed as ***a recognised Teaching Assistant with two graduations***. Subsequently he rather enthusiastically continued his postgraduate mathematical studies in Freiburg i.Br. – with some decent interruptions as ***a teacher*** and as ***a tutor*** – and received his degree **Diplom-Mathematiker** (**Dipl.-Math.**) in 1985 under the easy supervision of **Prof. Otto H. Kegel**. The publication at hand publishes the most essential and partly well corrected portions of his German **Diplomarbeit** [2] of 1984 and also a "sprinkling" of new considerations and results which try to propose coming directions of research for Sylow theory in (locally) finite groups. From February 1981 until April 1985 he was affiliated to the **Institut für Medizinische Biometrie und Statistik (IMBI)** at Freiburg im Breisgau as ***a Wissenschaftlicher Mitarbeiter***. Since May 1985 he was based in native Munich and working in the telecom industry first as ***a System Software Developer***, then as ***a Systems Engineer***, and finally as ***a Director for International Standardisation of telecom software and concepts***. On April 11, 1992, he blissful happily married the most fabulous woman *Helga* in *beautiful* Florence in Italy, which was a memorable marriage celebrated along with about twenty friends and uniting the venerable city Weiden in ***Upper Palatinate*** (i.d.OPf.) (*Helga*) with the huge cosmopolitan city Munich in ***Upper Bavaria*** (*Felix*). It was built really for eternity: *Helga* and *Felix* were meant to last forever.

Address: Mitterweg 4e, **82211 Herrsching a. Ammersee**, Bavaria, Germany

ORCID iD: 0000-0003-1612-8810

Felix Flemisch
ORCID iD
https://orcid.org/0000-0003-1612-8810

e-mail: felix.flemisch@hotmail.com

Background information and annotations

Note The author was born in 1951 and is now for some time retired. During retirement he had pretty fortunately rediscovered his **great enthusiasm for mathematical work**. This publication "Characterising locally finite groups satisfying the strong Sylow Theorem for the prime *p*" is a matter truly dear to his heart as one can see from the vital personal dedications in the **Full edition**. The publication is the essential part of his German Diplomarbeit during 1978-1984 but he does not submit it in its original form since to keep the original form would have resulted in an amalgam of English and German. The author claims to speak and write a (British) English which is fairly good for a non-native speaker. In any case, the many standard mathematical symbols used quite abundantly – $\underline{\mathrm{Syl}}_p G$; $P_n \subseteq P_{n+1}$ ($n \in \mathbb{N}$); $x \in S \setminus P$; $S = T^{x^{-1}}$; $Q^g \cap X \supsetneq P \cap Q$; etc. – should much facilitate the understanding even if the English should be a bit unwieldy at some places.

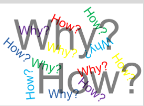
Note We are a little proud of two discoveries: **1)** **Proposition III** on **Page 11**, which found a symmetry between non-conjugated Sylow *p*-subgroups, and **2)** that the **minimal** members of the set $\underline{\mathbf{Unique}}_p \boldsymbol{U}$ from **Page 14** should play a similar important rôle as its **maximal** members, which are the Sylow *p*-subgroups, so that it becomes a major challenge to determine the **minimal** members for sufficiently "known" (locally) finite groups, in particular for all the "known" finite simple groups and the *p*-soluble groups. These should be very interesting mathematical ideas which propose exciting new directions for (timeless and eternal) Sylow theory in (locally) finite groups during the coming years where we intend to join in and shape. We are planning to revise Sylow theory starting with **a new proof** for **Cauchy**'s theorem in group theory (see https://en.wikipedia.org/wiki/Cauchy%27s_theorem_(group_theory)) based on ideas by **Galois**.[a][b][c][d] The **Endnotes** describe and provide new but classical proofs for the fundamental theorems of **Lagrange** and **Cauchy** on finite groups of – in our opinion – considerably historic relevance (see **Page 39** of the **Full edition** and **Page 24**).

Note The **Endnotes** were considerably expanded into **Part 3** of the **Second Trilogy about Sylow Theory in Locally Finite Groups** (see ISBN 978-3-7412-8392-5).

Note The appearance of the publication may be considered unusual. However, it supports reading and understanding with, exempli gratia, **bold face** or blue text or shading. The numbering of items does not correspond to the numbering of the author's Diplomarbeit but has been changed in accordance with the Chapter numbers: **Chapter.1**, **Chapter.2**, **…** . A more serious concern is that the publication could possibly be regarded as "*old-fashioned*" ("*altmodisch*", "*unzeitgemäß*"; "*ringard*", "*dépassé*") since locally finite group theory seems at present to attract no actual interest. The publication dates back to the eighties – although it was reworked considerably –, when locally finite group theory and Sylow theory and also the classification of finite simple groups were quite popular, as well as still in the nineties (and already in the seventies). The author is curious about the assessment by the group theory community since he considers it very rewarding to take up these topics (again) and not to let them fall into oblivion. With his publication he solves a famous open problem of Sylow theory within locally finite group theory and announces solutions of further significant problems. He is confident that his publication fulfils the high quality standards required since it is in accordance with the standards adhered to by all contributions of the **References** section and uses abundantly standard mathematical symbols (from set theory) which facilitates reading and understanding for the erudite mathematical reader. So our "serious" concern should indeed not be serious.

Note Perorating we express our hope that in the near future the Sylow p-subgroups of all "known" finite simple groups and of the p-soluble groups will be discovered in all details and that in particular the secret of their **minimal p-uniqueness subgroups** will be unveiled.

Note We feel obliged to include a **very personal statement** about the classification of the finite simple groups since the Sylow theory in locally finite groups depends on the existence of **only** seven rank-unbounded families (see **Page 3** and [5] with the references cited there): 26 sporadic finite simple groups deserve God-fearing further study (Why and how just and only ?



Is there a yet unknown much closer relationship to Ludvig M. Sylow, the bedrock of all ?).

• Mais le pire de tout, c'est que le peintre n'a jamais terminé. Il n'y a jamais un moment où tu peux dire: j'ai bien travaillé et demain c'est dimanche. Dès que tu t'arrêtes, c'est que tu recommences. Tu peux laisser une toile de côté en disant que tu n'y touches plus. Mais tu ne peux jamais mettre le mot FIN. **Pablo Picasso** • Doch das Schlimmste von allem ist, daß der Maler nie vollendet hat. Niemals kommt ein Augenblick, in dem du sagen kannst: ich hab' gut gearbeitet, und morgen ist Sonntag. Sobald du aufgehört hast, beginnst du schon wieder von vorn. Du kannst ein Bild wegstellen und sagen, du rührst nicht mehr daran. Doch du kannst niemals das Wort ENDE daruntersetzen. • But the worst thing of all is that the painter has never finished. There is never a moment when you can say "I've done a good day's work and tomorrow is Sunday." As soon as you stop you have to start again. You can put aside a canvas and say you won't touch it any more. But you can never write the words THE END. • Ma la cosa peggiore di tutte è che non si termina mai. Non c'è mai un momento in cui puoi dire: ho lavorato bene e domani è domenica. Non appena ti fermi, è ora di ricominciare. Si può lasciare da parte una tela dicendo che non si ha più voglia di metterci mano. Ma non si può mai scrivere la parola FINE.

Note The quotation from **Pablo Picasso** (see the **Full edition** ISBN 978-3-7568-0801-4 for the sources) reflects the author's overall experiences when writing down mathematical ideas and discoveries and then reading and improving them again and again and again ☺.

Note The **Abstract**, the **Introduction**, and **Chapters 1, 2 and 3** were submitted to the open access mathematical journal AGTA and very kindly published in June 2022 (see https://www.advgrouptheory.com/journal/index.php#vol13). The review process changed the layout substantially, e.g. AGTA did not allow footnotes, and resulted in a *less beautiful* issue. Subsequently the paper was revised into a *more beautiful* issue (see ISBN 978-3-7562-3416-5) and expanded into two editions of **Part 1** of the **First Trilogy about Sylow Theory in Locally Finite Groups** (see ISBN 978-3-7543-6087-3 and ISBN 978-3-7568-0801-4) which were supplemented by **Part 2** (see ISBN 978-3-7543-3642-7) and by **Part 3** (see ISBN 978-3-7578-6001-1). ☺ **Pages 22 and 23** contain a summary.

The **AGTA-paper** "**Characterising Locally Finite Groups Satisfying the Strong Sylow Theorem for the Prime** p" (see under https://www.advgrouptheory.com/journal/Volumes/13/Flemisch.pdf and MR4441631 and also Zbl 1496.20065) has almost as many pages as there are "known" sporadic finite simple groups. As the overwhelming majority of group theorists (including the author) believe, these 26 groups are now really all and *never* in the future further "sporadics" will appear (not counting the Tits group $^2F_4(2)$' [as some do] because it did in fact not appear *sporadically* at the stage). A central question of Sylow theory in locally finite group theory is, as pointed out by Prof. Otto H. Kegel (see [1], (2.4) Theorem), how the rank of the 7 rank-unbounded families of the finite simple groups $\{\underline{A}^n$, $A = PSL_n$, $B = P\Omega_{odd\ n}$, $C = PSp_n$, $D = P\Omega^+_{even\ n}$, $^2A = PSU_n$, $^2D = P\Omega^-_{even\ n}\}$ can be bounded, say, *someway* ("in terms of") by any given p-uniqueness subgroup P. More precisely, let us (try to) discover a function f_p of the order of P or (more challenging) of the p-uniqueness $a_p(G)$ of each of these (called classical) groups G, that bounds the rank, that is, $n \leq f_p(|P|)$ or $n \leq f_p(a_p(G))$. The author has answered that question in the affirmative for the *beautiful* Alternating Groups $\underline{A}^n$ $(n \geq 5)$ in his German Diplomarbeit [2] and is currently preparing to publish his answer in a fine **follow-up research paper**: $n \leq f_p(|P|) := (p+2) \bullet |P| \bullet 2^{|P|-1} - 1$. We could optimise this answer if only we would very happily come to know $a_p(\underline{A}^n)$, that is, the minimal p-unique subgroups of the Alternating Groups. In a **third research paper** we shall answer the question (but not the optimisation) for the *beautiful* PSL Groups $A = PSL_n$ thereby completing for the time being our *beautiful* (First) Trilogy about Sylow Theory in Locally Finite Groups which provides quite a number of rather tough suggestions to stimulate and encourage future research. All of these should become *beautiful* challenging open problems for the community of (locally finite) group theory researchers and we began (as a pioneer) to solve them ☺.

A good overview of the 19 families of "known" finite simple groups is given by the figure "The Periodic Table Of Finite Simple Groups" of Iván Andrus (see **Page ii**) and by the figure at https://upload.wikimedia.org/wikipedia/commons/a/a9/Classification_of_the_finite_simple_groups.jpg which shows the 19 families as 13 sporadic groups above 18 infinite families around another "sporadic" group (the Tits group $^2F_4(2)$') and further 13 sporadic groups below (see **Page 18**).

Our **follow-up research paper** "About the Strong Sylow Theorem for the Prime *p* in Simple Locally Finite Groups" gives a very profound overview of all simple locally finite groups, including the fundamental Kegel covers and Kegel sequences, introduces an innovative overview of the allegedly known finite simple groups, and provides a rather detailed plan how to prove Kegel's conjecture step-by-step for all of them together with some quite interesting and challenging conjectures to guide future research thereby unifying *Sylow theory in locally finite simple groups* with *Sylow theory in locally finite and p-soluble groups*. The detailed proof for the Alternating Groups offers a large number of rather new ideas based on **1)** *the number* j_k *of conjugacy classes of subgroups of index* p^{b-k} *in* P $(0 \le k \le b)$, if P is a finite p-group with $|P| =: p^b$ acting on $\underline{A}^n$ $(n \in \mathbb{N})$, on **2)** the lower bound $|\underline{\mathrm{Syl}}_p \underline{S}^n| \ge n - 2$, if $p \le n \in \mathbb{N}$, and on **3)** analysing what it really means that P has at least $k + p + 1$ many P-isomorphic P-orbits on $\{1, 2, \dots, n\}$ when not being contained in $k \in \mathbb{N}$ Sylow p-subgroups of $\underline{S}^n$.

A periodic linear group is locally finite and satisfies the strong Sylow Theorem for *every* prime p. Our **third research paper** "The Strong Sylow Theorem for the Prime p in Projective Special Linear Locally Finite Groups" proves Kegel's conjecture for the PSL Groups $A = \mathrm{PSL}_n$ by first proving it for all the **General Linear Groups** over locally finite fields (which is the peak work), then breaking down the result to the **Special Linear Groups** over locally finite fields, and finally breaking down to the **Projective Special Linear (PSL) Groups**:

Theorem 3 *Let* $n \in \mathbb{N}$ *and let* p *be a prime. Let* $\mathcal{F}$ *be a locally finite field and* P *be a minimal* p*-unique subgroup of* $\mathrm{PSL}(n, \mathcal{F})$.

a) *If* $\mathcal{F}$ *has characteristic* p *and* $a_p = a_p(\mathrm{PSL}(n, \mathcal{F}))$ *then* $n \le f_p(|P|) := (p + 2) \bullet p^{a_p} - 1$.

b) *If* $\mathcal{F}$ *has characteristic* $\neq p$ *and* $a_p = a_p(\mathrm{PSL}(n, \mathcal{F}))$ *then* $n \le f_p(|P|) := (p + 2) \bullet p^{2\, a_p} - 1$.

Note that the bounds for the PSL Groups are very similar to the bound for the Alternating Groups **but much better** thereby asking how we could improve the bound for the Alternating Groups by new ideas or could even optimise it.

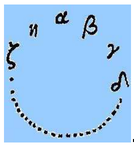
Our proofs for the types $\underline{A}^n$ and $A = \mathrm{PSL}_n$ are characterised by the fact that *we need not at all know their Sylow p-subgroups*. There is no doubt at all that we can extend those fine proofs immediately and fairly straightforwardly to the types $B = \mathrm{P\Omega}_{\text{odd } n}$, $C = \mathrm{PSp}_n$, $D = \mathrm{P\Omega}^{+}_{\text{even } n}$, ${}^2A = \mathrm{PSU}_n$, ${}^2D = \mathrm{P\Omega}^{-}_{\text{even } n}$ by considering very thoroughly the respective bilinear forms defining these groups of Lie type, resp. the underlying vector spaces they are acting upon as isometries, and their resulting Sylow p-subgroups (*without knowing them at all*). They can be considered proved which we shall confirm in a follow-up paper that is Part 1 of our "Second Trilogy about Sylow Theory in Locally Finite Groups". A bit more challenging will be breaking down to the "twisted" Chevalley groups defined through a Dynkin diagram automorphism followed by a field automorphism which we shall also do in the same follow-up paper. Thus the proofs for the further five types will eventually also be based on the basic theorem for the **General Linear Groups**.

Note The follow-up paper resp. the third paper were published as Part 2 resp. Part 3 of the "First Trilogy about Sylow Theory in Locally Finite Groups" whose Part 1 is the Full edition. Subsequently Part 2 and Part 3 were merged and considerably enhanced to [12] on the occasion of Prof. Otto H. Kegel's 90th birthday and dedicated to him thankworthily at IGT 2024. [12] is therefore a continuation of Part 1, that is, the AGTA-Paper, which is included in [12] for good reasons as an Appendix. The pioneering Endnotes were not included in the First Trilogy but are in a comprehensively expanded version Part 3 of the Second Trilogy. For a summary of the Endnotes see Page 24 (Endnotes a and b), Page 39 of the Full edition, and in greater detail at Page 22, Page 23, Page 24 and Page 25 of [12]. ☺ ♡

Endnotes

a

We can describe consequences of the absence of group elements of prime order p in spite of their availability in overgroups thereby providing a unified and heretofore undiscovered approach to the theorems of **Lagrange** and **Cauchy** and their implications for p-groups. This bears considerable historic relevance since it uses only ideas from a well-known paper of **Augustin-Louis Cauchy** presented in 1812 and published in 1815 (see **Page 34**). While it is acknowledged that **Cauchy** had *published* his fundamental group theorem in 1845/1846 (and had based it on **double cosets** of the finite permutation group and a **Sylow** p-subgroup of its symmetric overgroup …), one could henceforth argue that he had *presented* his theorem *in a concealed way* already a good thirty years earlier. **Évariste Galois** knew both **Cauchy**'s paper of 1815 and – based on his own considerations – **Cauchy**'s group theorem (and even **Sylow**'s existence theorem …). **Cauchy**'s and **Galois**' ideas are particularly lucid in the embryonic case of permutation groups of prime degree $p\ (\geq 5)$ when **Sylow** p-subgroups of the symmetric overgroup exist quite obviously. If $G \subseteq H$ with H being finite, then the unified method of proof consists **in arranging** the elements of H **in a rectangle** with $|G|$ columns and $[H:G]$ rows **resp.** the (right) cosets of G in H **in a rectangle** with p **resp.** $|H|_p$ columns and $[H:G]/p$ **resp.** $[H:G]/|H|_p$ rows to obtain information about $[H:G]$. **Cauchy**'s theorem of 1812/1815 is a consequence of $[H:<x>] \geq |G|$ if x is an element of H of order p with $x \notin G$ which we call a ***p-blank of G in H***.

Cauchy depicts 1815 a p-**cycle** for some prime p as a regular p-gon: .

b

We can present the *classical* proof of **Lagrange's theorem** due to **Cauchy** and *classical* proofs of the **three group theorems by Cauchy** of which he *presented* the first in 1812 and *published* it in 1815 but the second and the third had been *concealed* in 1812/1815 and were *published* not until 1845/1846. Comparing **Cauchy**'s classical proofs we find that they are like **two sides of a coin** where "Lagrange" is representing the case $p^0 = 1$ and "Cauchy" is representing the case $p^1 = p$ thereby offering **a unified approach** to both theorems. Consequently, "Cauchy" is not only a partial converse of "Lagrange" but is also a "swapping" of p for 1:

$$p^0 = 1 \quad \circlearrowleft \quad p = p^1$$

c

We state with proof **Lagrange's theorem** and with complete proofs the **three group theorems by Cauchy**. His second theorem has two corollaries and his third theorem has one corollary all three of which we prove as well.

> **Lagrange's theorem.** If G is a subgroup of the finite group H, then
> **a)** $|G|$ and $[H:G]$ both divide $|H|$, and $|H| = |G| \cdot [H:G]$;
> **b)** the prime p divides $|H|$ if and only if p divides either $|G|$ or $[H:G]$ or both;
> **c)** if F is a subgroup of G then $[H:F] = [H:G] \cdot [G:F]$

PROOF: **a)** (**Cauchy**; see **Page 34**) The proof is based on our arranging the elements of H in the form of a rectangle of $M := |G|$ columns and $R := [H:G]$ rows :

complete right transversal for G in H	the **first row** consists of ***all*** elements z_k of G $(1 \leq k \leq M)$ acting on H in the **following rows** via multiplication from the left by their inverses				correspondence	set $_H$**Orbi(*G*)** $:= G \backslash H$ of ***all*** orbits of H under G acting by left translation
$t_1 := 1 =: z_1$	z_2	z_3	…	z_M	$\leftrightarrow$	$G = {}_1\mathrm{Orb}(G)$
t_2	$z_2\ t_2$	$z_3\ t_2$	…	$z_M\ t_2$	$\leftrightarrow$	$Gt_2 = {}_{t_2}\mathrm{Orb}(G)$
t_3	$z_2\ t_3$	$z_3\ t_3$	…	$z_M\ t_3$	$\leftrightarrow$	$Gt_3 = {}_{t_3}\mathrm{Orb}(G)$
…	…	…	…	…	…	…
t_R	$z_2\ t_R$	$z_3\ t_R$	…	$z_M\ t_R$	$\leftrightarrow$	$Gt_R = {}_{t_R}\mathrm{Orb}(G)$

The first row is composed of the M elements of G; note that $\{1, z_2, \ldots, z_M\}$ and $\{1, z_2^{-1}, \ldots, z_M^{-1}\}$ are identical sets (and its not that they are multisets) thanks to $z \mapsto z^{-1}$ $(z \in G)$ being a bijection; t_2 is an element of H not found among those of the first row, and the second row is composed of the M elements of G multiplied from the right with the element t_2; t_3 is an element of H not found among those of the first two rows, and the third row is composed of the M elements of G multiplied from the right with the element t_3; etc. … . New rows are added in this manner **until every element of H has been accounted for**. The resulting rectangle is a multiset (or a family), that is, there could be one or more *repetitions* of the same element. It thus remains for us to prove that the M·R elements obtained are *all distinct*. First, ***the elements in any one row are distinct***. For the assumption $z_k\, t_i = z_l\, t_i$ gives $z_k = z_l$ after multiplication with t_i^{-1} (or by cancellation of t_i), and therefore $k = l$ since the first row is composed of the *distinct* M many elements of G. Second, ***the elements from two different rows are distinct***. For if, say, $z_k\, t_i = z_l\, t_j$, $i < j$, we should have $z_l^{-1}(z_k\, t_i) = z_l^{-1} z_l\, t_j$, that is, $t_j = y\, t_i$, where $y := z_l^{-1} z_k$, and therefore ***y belongs to** G*, since z_l^{-1} and z_k do that; but then t_j would occur in a previous row (namely the i th), ***contrary to* hypothesis**. It follows that the arranged M × R rectangle contains *all* the elements of H, *once each*. Therefore M·R = $|H|$ as claimed.

b) is an immediate consequence of **a)** (and the friendliness of the integers).

c) $\{t_i\}\{s_j\} = \{t_i\, s_j\}$ is a transversal for F in H since $H = \bigcup\{G\,t_i\} = \bigcup\{\bigcup\{F s_j\}\, t_i\} = \bigcup\{F s_j\, t_i\}$ and $(F s_j\, t_i = F s_l\, t_k \Rightarrow G\,t_i \supseteq F s_j\, t_i \subseteq G\,t_k \Rightarrow t_i = t_k \Rightarrow F s_j\, t_i = F s_l\, t_i \Rightarrow F s_j = F s_l \Rightarrow s_j = s_l)$. ❑

If $\{1 =: t_1, t_2, t_3, \ldots, t_R\}$ is a right transversal for G in H, then $G\backslash H = \{G, Gt_2, Gt_3, \ldots, Gt_R\} =:$ $_H$**Orbi(*G*)** is the set of all orbits of H under G acting half-transitively (:= having all orbits of the same size) via left translation. Therefore H is decomposed into R right cosets of G each of size M. Following **Évariste Galois**, we can record this basic ***observation*** symbolically as $H = G + Gt_2 + Gt_3 + \ldots + Gt_R$. It now follows in particular that $|H| = \text{R}\cdot\text{M} = |G|\cdot[H{:}G]$ since each orbit (= right coset) has size M. Altogether we have supplemented the ***classical* Cauchy**-style proof of **Lagrange**'s theorem by a ***modern*** proof. ❑

Cauchy's published *first* group theorem. Let $G \subseteq H := \underline{S}^n$ for n $\in \mathbb{N}$.
a) If $[H{:}G] < p \le$ n for the prime p then G contains all elements of H of order p.
b) If $p \le$ n for the prime p and G contains all p-cycles of H, equivalently, G contains all elements of H of order p, then G contains all 3-cycles of H.
c) If G contains all 3-cycles of H then G contains $\underline{A}^n$.
d) If $[H{:}G] < p \le$ n for the prime p then $[H{:}G] \le 2$.
e) If $[H{:}G] \le 2$ then G is normal in H and either $G = \underline{A}^n$ or $G = \underline{S}^n$.

Let G be a subgroup of some group H, that is, let H be any overgroup of G, and let p be a prime. We say that ***G is p-blank in H***, if **no element** (equivalently, no subgroup) of H of order p **at all** is contained in G. Obviously, if G contains (own) elements of order p, then G cannot be p-blank. Clearly, G is p-blank in every overgroup which is a p'-group (so that it is itself a p'-group). An element of H of order p which is not contained in G is called a ***p-blank of G in H***.

Part **a)** of **Cauchy**'s published *first* group theorem is a consequence of **the influence of a single p-blank on** $[H{:}G]$, namely that if x is a p-blank of G then $[H{:}\langle x\rangle] \ge |G|$ and that hence by **Lagrange** if **some** element of H is a p-blank of G then $[H{:}G] \ge p$. We show how this observation depends on a **coset principle** discovered by **Cauchy**.

Coset Principle (Avoidance of p-blanks).
Let $G \subseteq H$ and $\boldsymbol{x} \in H$ have prime order p.
If for some $t \in H$ the right coset $\langle \boldsymbol{x}\rangle t$ or the left coset $t\langle \boldsymbol{x}\rangle$ of $\langle \boldsymbol{x}\rangle$ in H contains at least two elements of G, then $\boldsymbol{x} \in G$ and therefore $\langle \boldsymbol{x}\rangle \subseteq G$.
If $\boldsymbol{x} \notin G$ (that is, $\boldsymbol{x}$ is a p-blank of G in H) then every right coset $\langle \boldsymbol{x}\rangle t$ and every left coset $t\langle \boldsymbol{x}\rangle$ of $\langle \boldsymbol{x}\rangle$ in H $(t \in H)$ contains at most one element from G.

PROOF: Suppose $y, z \in G \cap \langle \boldsymbol{x}\rangle t$ and let $w := y\,z^{-1}$. Then $1 \ne w = \boldsymbol{x}^k\, t\, t^{-1}(\boldsymbol{x}^l)^{-1} \in \langle \boldsymbol{x}\rangle$, and so $\langle w\rangle = \langle \boldsymbol{x}\rangle$ since the order p of $\langle \boldsymbol{x}\rangle$ is prime. But $w \in G$, and thus $\boldsymbol{x} \in G$ and $\langle \boldsymbol{x}\rangle \subseteq G$. Similarly, if $y, z \in G \cap t\langle \boldsymbol{x}\rangle$ and $v := y^{-1} z$, then $1 \ne v = (\boldsymbol{x}^k)^{-1}\, t^{-1} t\, \boldsymbol{x}^l \in \langle \boldsymbol{x}\rangle$, and so $\langle v\rangle = \langle \boldsymbol{x}\rangle$ since $|\langle \boldsymbol{x}\rangle| = p$ is prime, whence $\boldsymbol{x} \in G$ and $\langle \boldsymbol{x}\rangle \subseteq G$ since $v \in G$. ❑

Proposition 1 (Influence of a single p-blank).
Let G "contain" a p-blank in the finite overgroup H which contains elements of prime order p. Then $[H{:}G] \geq p$. If $\boldsymbol{x} \in H$ has order p and $\boldsymbol{x} \notin G$ then $[H{:}<\boldsymbol{x}>] \geq |G|$.

PROOF: $<\boldsymbol{x}>$ partitions H into $[H{:}<\boldsymbol{x}>]$, say right, cosets. By the **Coset Principle** each right coset of $<\boldsymbol{x}>$ in H contains at most one element from G. Consequently there are at least $|G|$ right cosets of $<\boldsymbol{x}>$ in H; so $[H{:}<\boldsymbol{x}>] \geq |G|$. But $[H{:}<\boldsymbol{x}>] = |H| / p$ and $|G| = |H| / [H{:}G]$ by **Lagrange**'s theorem. Therefore $|H| / p \geq |H| / [H{:}G]$, that is to say, $[H{:}G] \geq p$. ❑

PROOF of **Cauchy**'s published *first* group theorem: **a)** $(1\ 2\ 3 \ldots p)(p{+}1)(p{+}2)\ldots(n)$ is an element of H of order p. Thus **a)** follows from **Proposition 1**: the group G can "contain" *not a single* p-blank.
b) If $x \in H$ has order p then the complete cycle notation of x consists of at least one p-cycle and further p-cycles and 1-cycles but no other cycles since p is prime and the order of x is the least common multiple of the orders of its cycles. So we need only assume that G contains all p-cycles of H. If $p = 3$ then **b)** is trivially true. If $p = 2$ then $G = H$ since every element of H is a product of cycles and any m-cycle is a product of m-1 transpositions: $(i_1\ i_2\ i_3 \ldots i_m) = (i_1\ i_2)(i_1\ i_3)\ldots(i_1\ i_{m-1})$.
Let $p \geq 5$ and $(i_1\ i_2\ i_3)$ be any 3-cycle of H. Then, as **Cauchy** shows on Page 17 of his paper of 1815,

$$(i_1\ i_2\ i_3) = \begin{pmatrix} i_1\ i_2\ i_3\ i_4\ i_5 \ldots i_{p-1}\ i_p \\ \\ i_2\ i_3\ i_1\ i_4\ i_5 \ldots i_{p-1}\ i_p \end{pmatrix} = \begin{pmatrix} i_1\ i_2\ i_3\ i_4 \ldots i_{p-2}\ i_{p-1}\ i_p \\ \\ i_3\ i_4\ i_2\ i_5 \ldots i_{p-1}\ i_p\quad i_1 \end{pmatrix}\begin{pmatrix} i_3\ i_4\ i_2\ i_5 \ldots i_{p-1}\ i_p\quad i_1 \\ \\ i_2\ i_3\ i_1\ i_4 \ldots i_{p-2}\ i_{p-1}\ i_p \end{pmatrix} =$$

$(i_1\ i_3\ i_2\ i_4\ i_5 \ldots i_{p-1}\ i_p)(i_3\ i_2\ i_1\ i_p\ i_{p-1}\ i_{p-2} \ldots i_5\ i_4)$ is a product of just two p-cycles (belonging to different subgroups of H of order p). Since G is a sub***group*** of H, it follows that $(i_1\ i_2\ i_3) \in G$.
c) If $x \in H$ is even, it is a product of pairs of transpositions. Each such pair is either of the form $(i_1\ i_2)(i_2\ i_3) = (i_1\ i_2\ i_3) \in G$ or of the form $(i_1\ i_2)(i_3\ i_4) = (i_1\ i_2)(i_2\ i_3)(i_2\ i_3)(i_3\ i_4) = (i_1\ i_3\ i_2)(i_2\ i_4\ i_3) \in G$.
Hence every element of $\underline{A}^n$ is contained in G.
d) It follows from **a)**, **b)** and **c)** that G contains $\underline{A}^n$. Hence $[H{:}G] \leq 2$ since $[H{:}\underline{A}^n] = 2$.
e) If p is the ***least*** prime divisor of the order of a finite group H and $U \subseteq H$ with $[H{:}U] \leq p$ then **either** $U = <1>$ (in which case $|H| = p$) **or** $U = \mathbf{core}_{\boldsymbol{H}}\boldsymbol{(U)} := \bigcap \{ x^{-1} U x \mid x \in H \} \neq <1>$ (in which case $[H{:}U] = p$), since $[H{:}\mathrm{core}_H(U)]$ divides $|H|$; therefore U is normal in H.
If $[H{:}G] = 2$ then $G = \underline{A}^n$ because of G containing $\underline{A}^n$ by **c)**, **b)** and **a)** for $p = 3$. ❑

In **Cauchy**'s coset principle, G has the rôle of a **transversal for cosets with *group elements of prime order***. Interchanging the rôles of G and the prime elements we get **conjugates of *prime elements* as transversals for G** and a ***dual* coset principle** for p-blanks, also due to **Cauchy**. This viewpoint enables iteration capabilities which allow the construction of all cosets of G in H. Such an iteration ability cannot be offered by the original coset principle which allows the conclusion $p \leq [H{:}G]$, but the *dual* coset principle will allow to draw even the conclusion $p \mid [H{:}G]$ thanks to its iteration ability.

Dual Coset Principle (Impact of p-blanks).
Let $G \subseteq H$ and $\boldsymbol{x} \in H$ have prime order p.
a) Suppose $\boldsymbol{x} \notin G$ (that is, $\boldsymbol{x}$ is a p-blank of G in H).
If $G\boldsymbol{x}^k = G\boldsymbol{x}^l$ or $\boldsymbol{x}^k G = \boldsymbol{x}^l G$ for certain k and l $(0 \leq k, l \leq p\text{-}1)$, then $k = l$.
b) Let $x_1 := y_1 \boldsymbol{x} y_1^{-1}$ and $x_2 := y_2 \boldsymbol{x} y_2^{-1}$ with some $y_1, y_2 \in H$. Iteration of right cosets:
If $G x_1^k y_1 = G x_2^l y_2$ for certain $0 \leq k, l \leq p\text{-}1$, then $y_2 \in G x_1^{k-l} y_1$ with $k - l$ to be read modulo p.
c) Let $x_1 := y_1^{-1} \boldsymbol{x} y_1$ and $x_2 := y_2^{-1} \boldsymbol{x} y_2$ with some $y_1, y_2 \in H$. Iteration of left cosets:
If $y_1 x_1^k G = y_2 x_2^l G$ for certain $0 \leq k, l \leq p\text{-}1$, then $y_2 \in y_1 x_1^{k-l} G$ with $k - l$ to be read modulo p.

PROOF: **a)** Suppose $0 \leq k, l \leq p\text{-}1$ such that $G\boldsymbol{x}^k = G\boldsymbol{x}^l$. Then $\boldsymbol{x}^k (\boldsymbol{x}^l)^{-1} = \boldsymbol{x}^l (\boldsymbol{x}^k)^{-1} \in <\boldsymbol{x}> \cap G$. But $<\boldsymbol{x}> \cap G = <1>$ by hypothesis since the order p of $<\boldsymbol{x}>$ is prime, and thus $\boldsymbol{x}^k = \boldsymbol{x}^l$. Similarly, if $\boldsymbol{x}^k G = \boldsymbol{x}^l G$ for certain $0 \leq k, l \leq p\text{-}1$, then $(\boldsymbol{x}^k)^{-1} \boldsymbol{x}^l = (\boldsymbol{x}^l)^{-1} \boldsymbol{x}^k \in <\boldsymbol{x}> \cap G$, and so $\boldsymbol{x}^k = \boldsymbol{x}^l$ because $<\boldsymbol{x}> \cap G = <1>$ by hypothesis since $|<\boldsymbol{x}>| = p$ is prime.
b) $G x_1^k y_1 = G x_2^l y_2 \;\Rightarrow\; y_2 \boldsymbol{x}^l y_2^{-1} y_2 y_1^{-1} y_1 \boldsymbol{x}^{-k} y_1^{-1} = x_2^l y_2 (x_1^k y_1)^{-1} \in G$
$\Rightarrow\; y_2 (x_1^{k-l} y_1)^{-1} = y_2 (y_1 \boldsymbol{x}^{k-l})^{-1} = y_2 \boldsymbol{x}^{l-k} y_1^{-1} \in G \;\Rightarrow\; y_2 \in G x_1^{k-l} y_1$.

c) $y_1 x_1^{k} G = y_2 x_2^{l} G \;\Rightarrow\; y_1^{-1} \boldsymbol{x}^{-k} y_1 y_1^{-1} y_2 y_2^{-1} \boldsymbol{x}^{l} y_2 = (y_1 x_1^{k})^{-1} y_2 x_2^{l} \in G$
$\Rightarrow\; (y_1 x_1^{k-l})^{-1} y_2 = (\boldsymbol{x}^{k-l} y_1)^{-1} y_2 = y_1^{-1} \boldsymbol{x}^{l-k} y_2 \in G \;\Rightarrow\; y_2 \in y_1 x_1^{k-l} G$. ❑

> **Cauchy's concealed *second* group theorem.**
> Let G be p-blank for the prime p in the finite overgroup H which contains elements of order p. Then p divides $[H{:}G]$ (and therefore $[H{:}G] \geq p$).

PROOF: Let $\boldsymbol{x_1}$ be an element of H of order p and put $\boldsymbol{X} := <\boldsymbol{x_1}>$. Let $x_{1k} := \boldsymbol{x_1}^{k}$ $(0 \leq k \leq p\text{-}1)$ be the powers of $\boldsymbol{x_1}$. If $1 \leq k \leq p\text{-}1$, then x_{1k} has order p whence $x_{1k} \notin G$ by hypothesis, and therefore $\{ 1 = x_{10}, \boldsymbol{x_1}, x_{12}, \ldots, x_{1\,p\text{-}1} \} = \boldsymbol{X}$ is a subgroup of H of order p with $G \cap \boldsymbol{X} = <1>$.

If $G x_{1k} = G x_{1l}$ for certain k and l, then $x_{1k} x_{1l}^{-1} \in G \cap \boldsymbol{X}$ whence $x_{1k} x_{1l}^{-1} = 1$ whence $x_{1k} = x_{1l}$ whence $k = l$. Consequently, the right cosets $G x_{1k}$ of G in H are *all distinct* $(0 \leq k \leq p\text{-}1)$.
This is **Cauchy**'s **Dual Coset Principle a)**. It follows that $[H{:}G] \geq p$ which is **Proposition 1**.
If $G + G\boldsymbol{x_1} + G x_{12} + G x_{13} + \ldots + G x_{1\,p\text{-}1} = H$, then $[H{:}G] = p$, and the theorem is proved.

Alternatively, if $<\boldsymbol{x_1}> s = <\boldsymbol{x_1}> t$ for certain $s, t \in G$ and $u := s\,t^{-1}$, then $u \in G \cap <\boldsymbol{x_1}>$ whence $u = 1$ whence $s = t$. Consequently, the right cosets $<\boldsymbol{x_1}> t$ of $<\boldsymbol{x_1}>$ in H are *all distinct* $(t \in G)$.
This is **Cauchy**'s **Coset Principle**. Therefore $|H| \,/\, p = [H{:}<\boldsymbol{x_1}>] \geq |G|$ which is **Proposition 1**.
If $\sum \{ <\boldsymbol{x_1}> t \mid t \in G \} = H$, then $p \cdot |G| = |H|$ whence $[H{:}G] = p$, and the theorem is proved.

Otherwise let $t_1 := 1$ and pick $t_2 \in H$ with $t_2 \notin G x_{1k}$, in particular $t_2 \neq x_{1k}$, for all $0 \leq k \leq p\text{-}1$.

The further proof is based on our arranging similar to **Lagrange**'s theorem – after swapping $\boldsymbol{x_1}$ for 1 – the right cosets of G in H in the form of a rectangle of p columns and $\mathrm{S} := [H{:}G] \,/\, p$ rows :

set of ***certain*** orbits of H under G acting by left translation	the **first row** consists of ***all*** right cosets $G\boldsymbol{x_1}^{k}$ of G in H $(0 \leq k \leq p\text{-}1)$ with the powers of some p-blank $\boldsymbol{x_1}$ of G in H; the **following rows** consist of right cosets of G in H with the powers of left conjugates of $\boldsymbol{x_1}$				correspondence	$\boldsymbol{X} := <\boldsymbol{x_1}>$; set of ***all*** orbits of H under G ↺ ↻ $\boldsymbol{X}$, the **simultaneous** actions of G by left translation and of $\boldsymbol{X}$ by right translation
$G\boldsymbol{x_1}^{0} t_1 = G$	$G\boldsymbol{x_1}$	$G\boldsymbol{x_1}^{2}$	…	$G\boldsymbol{x_1}^{p\text{-}1}$	↔	cosets $G <\boldsymbol{x_1}> = G\boldsymbol{X}$ = double coset $G\,1\,\boldsymbol{X}$
$G\boldsymbol{x_2}^{0} t_2 = G t_2$	$G\boldsymbol{x_2} t_2$	$G\boldsymbol{x_2}^{2} t_2$	…	$G\boldsymbol{x_2}^{p\text{-}1} t_2$	↔	cosets $G <\boldsymbol{x_2}> t_2$ = double coset $G t_2 \boldsymbol{X}$
$G\boldsymbol{x_3}^{0} t_3 = G t_3$	$G\boldsymbol{x_3} t_3$	$G\boldsymbol{x_3}^{2} t_3$	…	$G\boldsymbol{x_3}^{p\text{-}1} t_3$	↔	cosets $G <\boldsymbol{x_3}> t_3$ = double coset $G t_3 \boldsymbol{X}$
…	…	…	…	…	…	…
$G\boldsymbol{x_{\mathrm{S}}}^{0} t_{\mathrm{S}} = G t_{\mathrm{S}}$	$G\boldsymbol{x_{\mathrm{S}}} t_{\mathrm{S}}$	$G\boldsymbol{x_{\mathrm{S}}}^{2} t_{\mathrm{S}}$	…	$G\boldsymbol{x_{\mathrm{S}}}^{p\text{-}1} t_{\mathrm{S}}$	↔	cosets $G <\boldsymbol{x_{\mathrm{S}}}> t_{\mathrm{S}}$ = double coset $G t_{\mathrm{S}} \boldsymbol{X}$

Note that the first column of this $p \times \mathrm{S}$ rectangle is a ***somehow*** **shortened** version of the $\mathrm{M} \times \mathrm{R}$ rectangle associated to **Lagrange**'s theorem which represents the case $\boldsymbol{X} = <1>$ and $\mathrm{S} = \mathrm{R}$:

$G = {}_{1}\mathrm{Orb}(G)$	↔	$t_1 := 1 =: z_1$	z_2	z_3	…	z_{M}
$G t_2 = {}_{t_2}\mathrm{Orb}(G)$	↔	t_2	$z_2 t_2$	$z_3 t_2$	…	$z_{\mathrm{M}} t_2$
$G t_3 = {}_{t_3}\mathrm{Orb}(G)$	↔	t_3	$z_2 t_3$	$z_3 t_3$	…	$z_{\mathrm{M}} t_3$
…	…	…	…	…	…	…
$G t_{\mathrm{R}} = {}_{t_{\mathrm{R}}}\mathrm{Orb}(G)$	↔	t_{R}	$z_2 t_{\mathrm{R}}$	$z_3 t_{\mathrm{R}}$	…	$z_{\mathrm{M}} t_{\mathrm{R}}$

The first row of the new rectangle is composed of the p right cosets of G with the elements of $\boldsymbol{X}$; t_2 is an element of H not found among those contained in cosets of the first row, $\boldsymbol{x_2} := t_2\,\boldsymbol{x_1}\,t_2^{-1}$ **is the left conjugate of $\boldsymbol{x_1}$ with** t_2, and the second row is composed of the p right cosets of G with the elements $\boldsymbol{x_2}^k\,t_2 = t_2\,\boldsymbol{x_1}^k$ $(0 \le k \le p\text{-}1)$; t_3 is an element of H not found among those contained in cosets of the first two rows, $\boldsymbol{x_3}$ **is the left conjugate of $\boldsymbol{x_1}$ with** t_3, and the third row is composed of the p right cosets of G with the elements $\boldsymbol{x_3}^k\,t_3 = t_3\,\boldsymbol{x_1}^k$ $(0 \le k \le p\text{-}1)$; etc. … .

New rows are added in this manner **until every element of H has been accounted for**. The resulting rectangle is a multiset (or a family), that is, there could be one or more *repetitions* of the same coset. It therefore remains for us to prove that the $p{\cdot}$S cosets obtained are *all distinct*.

First, ***the cosets in any one row are distinct***. For the assumption $G\,\boldsymbol{x_i}^k\,t_i = G\,\boldsymbol{x_i}^l\,t_i$ gives $\boldsymbol{x_i}^k = \boldsymbol{x_i}^l$ since $<\boldsymbol{x_i}> \cap\, G = t_i\,\boldsymbol{X}\,t_i^{-1} \cap G = <1>$ because of $\boldsymbol{x_i} \notin G$ by hypothesis and $|<\boldsymbol{x_i}>| = p$ is prime, and therefore $k = l$; this is the direct use of **Cauchy**'s **Dual Coset Principle a)** for the ith row.

Second, ***the cosets from two different rows are distinct***. For if, say, $G\,\boldsymbol{x_i}^k\,t_i = G\,\boldsymbol{x_j}^l\,t_j$, $i < j$, we should have $t_j\,\boldsymbol{x_1}^l\,t_j^{-1}\,t_j\,t_i^{-1}\,t_i\,\boldsymbol{x_1}^{-k}\,t_i^{-1} = \boldsymbol{x_j}^l\,t_j\,(\boldsymbol{x_i}^k\,t_i)^{-1} \in G$, that is, $t_j\,\boldsymbol{x_1}^{l\text{-}k}\,t_i^{-1} = t_j\,(t_i\,\boldsymbol{x_1}^{k\text{-}l})^{-1}$ $= t_j\,(\boldsymbol{x_i}^{k\text{-}l}\,t_i)^{-1} \in G$, and therefore t_j ***belongs to*** $G\,\boldsymbol{x_i}^{\boldsymbol{k\text{-}l}}\,t_i$ (see the **Dual Coset Principle b)**); but then t_j would occur in a coset of a previous row (namely the ith), ***contrary to*** **hypothesis**; this is the iteration capability of **Cauchy**'s **Dual Coset Principle b)**.

It follows that the arranged $p \times$ S rectangle contains *all* the elements of $\{\, Z \subseteq H \;\big|\; |Z| = |G| \,\}$ that are a right coset of G in H, *once each*. Therefore $p{\cdot}\text{S} = [\,H : G\,]$ as claimed. ❑

If n is an integer and p is a prime, then n_p denotes the **highest power of p dividing** n and is called the **p-part of** n. For example, if n = 120 = 5!, then $n_2 = 2^3$, $n_3 = 3$, $n_5 = 5$, and $n_p = 1 = p^0$ for $p \ge 7$, and if n = 808,017,424,794,512,875,886,459,904,961,710,757,005,754,368,000,000,000 = $2^{46} \cdot 3^{20} \cdot 5^9 \cdot 7^6 \cdot 11^2 \cdot 13^3 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 41 \cdot 47 \cdot 59 \cdot 71 \approx 8{\cdot}10^{53}$ (see https://en.wikipedia.org/wiki/Monster_group), then $n_2 = 2^{46}$, $n_3 = 3^{20}$, $n_5 = 5^9$, $n_7 = 7^6$, $n_{11} = 11^2$, $n_{13} = 13^3$, $n_p = p$ for $17 \le p \le 71$, and $n_p = 1 = p^0$ for $p > 71$.

Cauchy's group theorem for $|H|_p = p$. Let $G \subseteq H$ with H being finite. If H contains elements of order p and $|H|_p = p$, then G contains elements of order p if (and only if) p divides $|G|$.

PROOF: If G contains elements of order p then p divides $|G|$ by **Lagrange**'s theorem. If p divides $|G|$ then p does not divide $[H{:}G]$, since p is the highest power of p dividing $|H|$, and therefore G is not p-blank in H by **Cauchy's concealed *second* group theorem.** ❑

The Monster:

A 'huge great symmetrical snowflake', with

$2^{46} \cdot 3^{20} \cdot 5^9 \cdot 7^6 \cdot 11^2 \cdot 13^3 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 41 \cdot 47 \cdot 59 \cdot 71$

or 808,017,424,794,512,875,886,459,904,961,710,757,005,754,368,000,000,000 symmetries, that can only be 'seen' once you get to 196,883-dimensional space.

— John Conway quote from Marcus du Sautoy. (2008). *Symmetry.*

Cauchy's group theorem for *permutation groups of prime degree* p.

a) Let p be a prime. Then $|\underline{S}^p| = p!$ and therefore $|\underline{S}^p|_p = p$, and $\underline{S}^p$ contains $(p\text{-}1)!$ many p-cycles and $(p\text{-}2)!$ subgroups of order p. If $x \in \underline{S}^p$ then x has order p if and only if x is a p-cycle.

Let G be a permutation group of prime degree p, that is, $G \precsim \underline{S}^p$.

b) If p divides $|G|$ then G contains at least one subgroup of order p, and conversely.

c) If G is transitive (on $\{1, 2, 3, \dots, p\}$) then p divides $|G|$.

d) G is transitive if and only if G contains subgroups of order p.

PROOF. The cases $p = 2$ and $p = 3$ are trivial: $\{ <1>, \{1, (1\ 2)\} \}$ are all subgroups of $\underline{S}^2$ and $\{ <1>, \{1, (1\ 2)\}, \{1, (1\ 3)\}, \{1, (2\ 3)\}, \{1, (1\ 2\ 3), (1\ 3\ 2)\}, \{1, (1\ 2), (1\ 3), (2\ 3), (1\ 2\ 3), (1\ 3\ 2)\} \}$ are all subgroups of $\underline{S}^3$. Now let $p \ge 5$ be arbitrarily chosen.

a) Put $\Omega := \{1, 2, \dots, p\}$ and $H := \mathrm{Sym}(\Omega)$. An **arrangement** of Ω is a list (:= an ordered set) $i_1, i_2, i_3, \dots, i_n$ of all the elements of Ω with no repetitions; it defines a permutation $x \in H$ by $\alpha^x := i_\alpha$ for all $\alpha \in \Omega$; the set of arrangements of Ω is denoted by $\mathrm{Arr}(\Omega)$. Then $|H| = |\mathrm{Arr}(\Omega)|$ by dint of $x \mapsto (1^x, 2^x, \dots, p^x)$ $(x \in H)$ and $|\mathrm{Arr}(\Omega)| = p!$, since an arbitrary $(i_1, i_2, i_3, \dots, i_p) \in \mathrm{Arr}(\Omega)$ is obtained by picking first i_1, which has p choices, then i_2, which has p-1 choices, so $\{i_1, i_2\}$

has $p\,(p\text{-}1)$ choices, etc., until one is left with $\{i_p\}$. Hence H has $p!$ elements. An arbitrary p-cycle $x = (\,i_1\; i_2\; i_3 \dots i_p\,)$ is obtained in the same way but the p-cycles $(\,i_2\; i_3\; i_4 \dots i_p\; i_1\,)$, $(\,i_3\; i_4\; i_5 \dots i_p\; i_1\; i_2\,)$, etc., also represent x, that is, there are p different cycle representations of the same p-cycle x (and convention is to let $i_1 := 1$ as a rule). Hence the set **Cyc(*H*, *p*)** := $\{x \in H \mid x$ is a p-cycle$\}$ has $p!\,/\,p = (p\text{-}1)!$ elements. The equivalence relation $\wp$ on Cyc(H, p) defined by $x_1 \wp x_2 :\Leftrightarrow \langle x_1 \rangle = \langle x_2 \rangle$ $(x_1, x_2 \in \mathrm{Cyc}(H, p))$ partitions Cyc(H, p) into classes of equal sizes p-1. Hence there are $|\mathrm{Cyc}(H, p)|\,/\,(p\text{-}1) = (p\text{-}2)!$ subgroups of H of order p. Any $x \in \mathrm{Cyc}(H, p)$ has order p. If $x \in H$ has order p then the complete cycle notation of x consists of at least one p-cycle and further p-cycles and 1-cycles, since p is prime, but there is no room for further cycles since $|\Omega| = p$; hence x is a p-cycle (see **Cauchy**'s picture on **Page 24**).

b) follows from **a)** and **Cauchy**'s group theorem for $|H|_p = p$.

c) Let $i \in \{1, 2, \dots, p\}$ be a point, $\mathbf{Orb}_i(\boldsymbol{G}) = \boldsymbol{i}^{\boldsymbol{G}} := \{\, i^x \mid x \in G\,\}$ be the orbit of i under G, and $\mathbf{Stab}_{\boldsymbol{G}}(\boldsymbol{i}) = \boldsymbol{G_i} := \{\, x \in G \mid i^x = i\,\}$ be the stabiliser of i in G. Then $|\mathrm{Orb}_i(G)| = |i^G| = [\,G : G_i\,] = [\,G : \mathrm{Stab}_G(i)\,]$ since the mapping $i^x \longmapsto G_i\, x$ $(x \in G)$ defines an isomorphism of the G-spaces i^G and $G_i \backslash G$. If G is transitive then $p = |\{1, 2, \dots, p\}| = |\mathrm{Orb}_i(G)| = [\,G : \mathrm{Stab}_G(i)\,] = |G|/|\mathrm{Stab}_G(i)|$ by **Lagrange**'s theorem, that is, $|G| = p \cdot |\mathrm{Stab}_G(i)|$.

d) If G contains a subgroup X of order p then by **a)** every element $\neq 1$ of X is a p-cycle which can't help to act transitively on $\{1, 2, \dots, p\}$. If G is transitive on $\{1, 2, \dots, p\}$ then p divides $|G|$ by **c)** and hence G contains subgroups of order p by **b)**. ❑

> **Cauchy's concealed *third* group theorem.**
> Let G be p-blank in the finite overgroup H for the prime p.
> If H contains a subgroup of order $|H|_p$ (that is, a Sylow p-subgroup),
> then $[\,H{:}G\,]$ is divisible by $|H|_p$.

PROOF: The proof consists in arranging the (right) cosets of G in H in a rectangle with $|H|_p$ columns and $\mathrm{T} := [\,H{:}G\,]/|H|_p$ rows, very similar to the arranging in **Cauchy**'s concealed *second* group theorem of the right cosets of G in H as a rectangle of p columns and $\mathrm{S} := [\,H{:}G\,]\,/\,p$ rows, and then showing that the cosets are distinct which requires a generalisation of the **Dual Coset Principle** from p-elements to p-subgroups.

Define the integer b by $p^b := |H|_p$. If $b = 0$ then the statement is certainly true irrespective of the existence of prime p-elements of H. If $b = 1$ then we have **Cauchy**'s concealed *second* group theorem since all p-elements $\neq 1$ of H have order p by **Lagrange**'s theorem. Therefore suppose $b \geq 2$. Let $\boldsymbol{X}$ be a subgroup of H of order p^b and define $\boldsymbol{X} =: \{1 =: \boldsymbol{x}_{10}, \boldsymbol{x}_{11}, \boldsymbol{x}_{12}, \dots, \boldsymbol{x}_{1\,p^b\text{-}1}\} = \{\boldsymbol{x}_{1c} \mid 0 \leq c \leq p^b\text{-}1\}$. By **Lagrange**'s theorem each element of $\boldsymbol{X}$ is a p-element. The proof is based on our arranging – just as for **Cauchy**'s *second* group theorem – the right cosets of G in H in the form of a rectangle of p^b columns and $\mathrm{T} := [\,H{:}G\,]\,/\,p^b$ rows :

set of ***certain*** orbits of H under G acting by left translation	the **first row** consists of ***all*** right cosets $G\,\boldsymbol{x}_{1c}$ of G in H $(0 \leq c \leq |H|_p\text{-}1)$ with the elements of some Sylow p-subgroup $\boldsymbol{X}$ of H, all of whose **elements of order** p are p-blanks of G in H; the **following rows** consist of right cosets of G in H with the elements of left conjugates of $\boldsymbol{X}$				correspondence	$\lvert\boldsymbol{X}\rvert = \lvert H\rvert_p = p^b$; set of ***all*** orbits of H under $G \circlearrowleft \circlearrowright \boldsymbol{X}$, the **simultaneous** actions of G by left translation and of $\boldsymbol{X}$ by right translation
$G\,x_{10}\,t_1 = G$	$G\,\boldsymbol{x}_{11}$	$G\,\boldsymbol{x}_{12}$	…	$G\,\boldsymbol{x}_{1\,p^b\text{-}1}$	↔	cosets $G\,\{\boldsymbol{x}_{1c} \mid 0 \leq c \leq p^b\text{-}1\}$ $= G\boldsymbol{X} =$ double coset $G\,1\,\boldsymbol{X}$
$G\,x_{20}\,t_2 = G\,t_2$	$G\,\boldsymbol{x}_{21}\,t_2$	$G\,\boldsymbol{x}_{22}\,t_2$	…	$G\,\boldsymbol{x}_{2\,p^b\text{-}1}\,t_2$	↔	cosets $G\,\{\boldsymbol{x}_{2c} \mid 0 \leq c \leq p^b\text{-}1\}\,t_2$ = double coset $G\,t_2\,\boldsymbol{X}$
$G\,x_{30}\,t_3 = G\,t_3$	$G\,\boldsymbol{x}_{31}\,t_3$	$G\,\boldsymbol{x}_{32}\,t_3$	…	$G\,\boldsymbol{x}_{3\,p^b\text{-}1}\,t_3$	↔	cosets $G\,\{\boldsymbol{x}_{3c} \mid 0 \leq c \leq p^b\text{-}1\}\,t_3$ = double coset $G\,t_3\,\boldsymbol{X}$
…	…	…	…	…	…	…
$G\,x_{\mathrm{T}0}\,t_{\mathrm{T}} = G\,t_{\mathrm{T}}$	$G\,\boldsymbol{x}_{\mathrm{T}1}\,t_{\mathrm{T}}$	$G\,\boldsymbol{x}_{\mathrm{T}2}\,t_{\mathrm{T}}$	…	$G\,\boldsymbol{x}_{\mathrm{T}\,p^b\text{-}1}\,t_{\mathrm{T}}$	↔	cosets $G\,\{\boldsymbol{x}_{\mathrm{T}c} \mid 0 \leq c \leq p^b\text{-}1\}\,t_{\mathrm{T}}$ = double coset $G\,t_{\mathrm{T}}\,\boldsymbol{X}$

The first column is a ***somehow* shortened** version of the rectangle associated to **Lagrange**'s theorem. Thus necessarily $\mathrm{T} \le \mathrm{R}$. But we are going to prove even $\mathrm{T} = \mathrm{R} / p^b$. Put $t_1 := 1$. The first row of the rectangle is composed of the p^b right cosets of G with the elements of $\boldsymbol{X}$; t_2 is an element of H not found among those contained in cosets of the first row, $\boldsymbol{x}_{2c} := t_2\, \boldsymbol{x}_{1c}\, t_2^{-1}$ **is the left conjugate of $\boldsymbol{x}_{1c}$ with** t_2 $(0 \le c \le p^b\text{-}1)$, and the second row is composed of the p^b right cosets of G with the elements $\boldsymbol{x}_{2c}\, t_2 = t_2\, \boldsymbol{x}_{1c}$ $(0 \le c \le p^b\text{-}1)$; t_3 is an element of H not found among those contained in cosets of the first two rows, $\boldsymbol{x}_{3c}$ **is the left conjugate of $\boldsymbol{x}_{1c}$ with** t_3 $(0 \le c \le p^b\text{-}1)$, and the third row is composed of the p^b right cosets of G with the elements $\boldsymbol{x}_{3c}\, t_3 = t_3\, \boldsymbol{x}_{1c}$ $(0 \le c \le p^b\text{-}1)$; if $\bigcup\{\, G\, \boldsymbol{x}_{rc}\, t_r \mid 1 \le r \le 3;\ 0 \le c \le p^b\text{-}1 \,\} \ne H$ then we may chose t_4 as an element of H not found among those contained in this union, that is, not in cosets of the first three rows, $\boldsymbol{x}_{4c} := t_4\, \boldsymbol{x}_{1c}\, t_4^{-1}$ $(0 \le c \le p^b\text{-}1)$, and the fourth row will be composed of the p^b right cosets of G with the elements $\boldsymbol{x}_{4c}\, t_4 = t_4\, \boldsymbol{x}_{1c}$ $(0 \le c \le p^b\text{-}1)$; etc. …; since H is finite this line-by-line construction of the p^b-columns table will stop after a finite number T of steps.

New rows were added in this manner **until every element of H has been accounted for**. The resulting rectangle is a multiset (or a family), that is, there could be one or more *repetitions* of the same coset. It therefore remains for us to prove that the $p^b \cdot \mathrm{T}$ cosets obtained are *all distinct*.

Since $\boldsymbol{x}_{1c} \in \boldsymbol{X}$ is a p-element, $\boldsymbol{x}_{rc} = t_r\, \boldsymbol{x}_{1c}\, t_r^{-1} \in t_r\, \boldsymbol{X}\, t_r^{-1}$ is also a p-element $(2 \le r \le \mathrm{T};\ 1 \le c \le p^b\text{-}1)$. Since G is p-blank in H, it is even ***strongly* p*-blank*** in H, that is, G does not contain any p-element $\ne 1$ of H, because of G being a sub***group*** of H: if $x \in H$ has order p^k with $k \ge 2$ then $y := x^q$ with $q := p^{k-1}$ has order p and $x \in G$ implies $y \in G$. This implies the following: $\boldsymbol{x}_{rc} \notin G$ for all $1 \le r \le \mathrm{T}$ and all $1 \le c \le p^b\text{-}1$, that is, $G \cap t_r\, \boldsymbol{X}\, t_r^{-1} = \langle 1 \rangle$ for all $1 \le r \le \mathrm{T}$.

First, ***the cosets in any one row are distinct***. For the assumption $G\, \boldsymbol{x}_{rc}\, t_r = G\, \boldsymbol{x}_{rd}\, t_r$ gives $\boldsymbol{x}_{rc} = \boldsymbol{x}_{rd}$ and therefore $c = d$ since $\boldsymbol{x}_{rc}\, \boldsymbol{x}_{rd}^{-1} = t_r\, \boldsymbol{x}_{1c}\, \boldsymbol{x}_{1d}^{-1}\, t_r^{-1} \in G \cap t_r\, \boldsymbol{X}\, t_r^{-1}$ because $\boldsymbol{x}_{1c}\, \boldsymbol{x}_{1d}^{-1}$ is an element of the sub***group*** $\boldsymbol{X}$ of H; this is a generalisation of **Dual Coset Principle a)** applied to the rth row. Second, ***the cosets from two different rows are distinct***. For if, say, $G\, \boldsymbol{x}_{rc}\, t_r = G\, \boldsymbol{x}_{sd}\, t_s$, $r < s$, there is *a unique* $0 \le e \le p^b\text{-}1$ such that $\boldsymbol{x}_{1e} = \boldsymbol{x}_{1c}\, \boldsymbol{x}_{1d}^{-1}$ again since $\boldsymbol{X}$ is a sub***group*** of H; then also $\boldsymbol{x}_{re} = \boldsymbol{x}_{rc}\, \boldsymbol{x}_{rd}^{-1}$; we should now have $t_s\, (\boldsymbol{x}_{re}\, t_r)^{-1} = t_s\, (\boldsymbol{x}_{rc}\, t_r\, \boldsymbol{x}_{1d}^{-1}\, t_r^{-1}\, t_r)^{-1}$ $= t_s\, (t_r\, \boldsymbol{x}_{1c}\, t_r^{-1}\, t_r\, \boldsymbol{x}_{1d}^{-1})^{-1} = t_s\, (t_r\, \boldsymbol{x}_{1c}\, \boldsymbol{x}_{1d}^{-1})^{-1} = t_s\, \boldsymbol{x}_{1d}\, \boldsymbol{x}_{1c}\, t_r^{-1} = t_s\, \boldsymbol{x}_{1d}\, t_s^{-1}\, t_s\, t_r^{-1}\, t_r\, \boldsymbol{x}_{1c}\, t_r^{-1} = \boldsymbol{x}_{sd}\, t_s\, t_r^{-1}\, \boldsymbol{x}_{rc}^{-1} =$ $\boldsymbol{x}_{sd}\, t_s\, (\boldsymbol{x}_{rc}\, t_r)^{-1} \in G$, and therefore t_s ***belongs to*** $G\, \boldsymbol{x}_{re}\, t_r$; but then t_s would occur in a coset of a previous row (namely the rth), ***contrary to* hypothesis**; this is a generalisation of the iteration capability of **Dual Coset Principle b)** applied to previous rows. It follows that the arranged $p^b \times \mathrm{T}$ rectangle contains *all* the elements of $\{\, Z \subseteq H \mid |Z| = |G| \,\}$ that are a right coset of G in H, *once each*. Therefore $p^b \cdot \mathrm{T} = [H{:}G]$ as claimed. $\square$

> **Cauchy's group theorem for *suitable overgroups*.**
> Let $G \subseteq H$ with H being finite and containing a subgroup of order $|H|_p$ for the prime p. If p divides $|G|$, then G contains an element (equivalently, a subgroup) of order p.

PROOF: Define the integer b by $p^b := |H|_p$. The case $b = 0$ is impossible since $|H|_p \ge |G|_p$ $\ge p$ by **Lagrange**'s theorem: if p does not divide $|H|$ then p divides neither $|G|$ nor $[H{:}G]$. The case $b = 1$ follows already from **Cauchy's concealed *second* group theorem** and the case $b \ge 2$ follows from **Cauchy's concealed *third* group theorem**: the overgroup H of G contains elements of order p and if G would be p-blank in H then, again by an application of **Lagrange**'s theorem, **p could not divide $|G|$** since $|H|_p$ divides $[H{:}G]$, a contradiction. $\square$

Ueberhaupt aber wird zur Entdeckung der *wichtigsten* Wahrheiten nicht die Beobachtung der seltenen und verborgenen, nur durch Experimente darstellbaren Erscheinungen führen; sondern die der offen daliegenden, Jedem zugänglichen Phänomene. Daher ist die Aufgabe nicht sowohl, zu sehn was noch Keiner gesehn hat, als, bei Dem, was Jeder sieht, zu denken, was noch Keiner gedacht hat.

Arthur Schopenhauer (2 February 1788 until 9 September 1860)
Parerga und Paralipomena: kleine philosophische Schriften
(Dezember 1850[1]; November 1861[2]). Zweiter Band (§ 76[1]; § 77[2]).

Speaking generally, however, it is not the observation of rare and hidden phenomena that can be produced only by experiments, but the study of those that are obvious and accessible to everyone, which will lead to the discovery of the *most important* truths. Therefore the problem is not so much that of seeing what no one has yet seen, but rather of thinking in the case of something seen by everyone that which no one has yet thought.

We do consider *not* the statement of **Cauchy**'s concealed *third* group theorem *but*, for historical reasons, the presented **Cauchy**-style proof a "discovery of a *most important* truth" in the spirit of **Schopenhauer**. This proof relates **Cauchy**'s contributions to group theory of 1812/1815 to those of 1845-1846 just by extending his arguments from cyclic arrangement groups of prime order to arbitrary finite groups (of, or not of, permutations). Hence the Corollary "**Cauchy**'s group theorem for *suitable overgroups*" could well be considered known a good 30 years earlier.

Let H be an overgroup of the group G (whence H is any group if $G = \langle 1 \rangle$) and let $U, V \subseteq H$ be subgroups of H. Then $UxV := \{\, uxv \mid u \in U, v \in V \,\}$ for all $x \in H$ denotes the **double cosets** of U and V in H. The **double coset space** $U\backslash H/V$ of U and V in H is the set of all double cosets of U and V in H. The double cosets of U and V in H are the classes of the equivalence relation $x \,\mathbf{\Delta}\, y \;:\Leftrightarrow\; ux = yv \;\; (\Leftrightarrow x = u^{-1}yv \Leftrightarrow y = uxv^{-1})$ for some $u \in U$ and some $v \in V$ $(\Leftrightarrow UxV = UyV \Leftrightarrow Ux \cap yV \neq \varnothing)$ $(x, y \in H)$ on H. Whilst all Uz resp. zV have the same cardinality as U resp. V, the cardinality of UzV depends on U, V, $z^{-1}Uz$ and zVz^{-1} $(z \in H)$ (see below). This dependency has *considerable* influence on finite groups and their subgroups. $UzV = zV$ if and only if $z^{-1}Uz \subseteq V$ and $UzV = Uz$ if and only if $zVz^{-1} \subseteq U$ $(z \in H)$. The **double index** $[U:H:V]$ of U and V in H is the cardinality of the set $U\backslash H/V$, that is, $[U:H:V] := |U\backslash H/V|$. A **transversal** $\mathcal{T}$ for U and V in H is a subset of H of cardinality $[U:H:V]$ which contains one and only one element from each double coset of U and V in H, that is, a complete system of representatives for the equivalence classes of $\mathbf{\Delta}$; convention is to let $\mathcal{T}$ contain $1 \in U \cap V$ as a rule; if H is finite, we usually put $n := [U:H:V]$ and pick any transversal $\mathcal{T} = \{\, t_1, t_2, t_3, \ldots, t_n \,\} \subseteq H$ of U and V in H with $t_1 = 1$.

Proposition 2 (Double cosets as unions of cosets and sizes of double cosets).
Let U and V be subgroups of the group H and $z \in H$.
Consider $U\backslash H$ as a V-space by right translation. Then $\mathbf{Stab}_V(Uz) = z^{-1}Uz \cap V$ and $UzV = \bigcup \{\, Ux \mid Ux \in \mathbf{Orb}_{Uz}(V) \,\}$. Hence $Uz \in \mathbf{Fix}_{U\backslash H}(V) \Leftrightarrow V \subseteq z^{-1}Uz$.
If H is finite then $|\mathrm{Orb}_{Uz}(V)| = |V| \,/\, |z^{-1}Uz \cap V| = [V : z^{-1}Uz \cap V]$ is the number of right cosets of U in UzV and $|UzV| = |U| \cdot |V| \,/\, |z^{-1}Uz \cap V|$.
In particular, $|z^{-1}Uz \cap V|$ divides $|U| \cdot |V|$.

PROOF: H acts on $U\backslash H$ by right translation with $\mathbf{Orb}_{Ux}(H) = U\backslash H$ and $\mathbf{Stab}_H(Ux) = x^{-1}Ux$ $(x \in H)$. When this action is restricted to V, then V acts on $U\backslash H$ with $\mathbf{Orb}_U(V) = UV$ and $\mathbf{Stab}_V(U) = U \cap V$. Since $\mathrm{Orb}_{Uz}(V) = \{ Uzv \mid v \in V \} \subseteq U\backslash H$ by the definition of orbits, $\bigcup \{ Ux \mid Ux \in \mathrm{Orb}_{Uz}(V) \} = \bigcup \{ Uzv \mid v \in V \} = UzV$. Hence $|\mathrm{Orb}_{Uz}(V)|$ is the cardinality of the set of right cosets of U which are contained in UzV and each of which has cardinality $|U|$. Thus $|UzV| = |U|\cdot|\mathrm{Orb}_{Uz}(V)|$. But $|\mathrm{Orb}_{Uz}(V)| = [V{:}\mathrm{Stab}_V(Uz)]$, since $Uzv \leftrightarrow\mapsto V_{Uz}v$ $(v \in V)$ defines a bijection between $\mathrm{Orb}_{Uz}(V)$ and $\mathrm{Stab}_V(Uz)\backslash V$, and $V_{Uz} = \mathrm{Stab}_V(Uz) = \mathrm{Stab}_H(Uz) \cap V = z^{-1}Uz \cap V$. Therefore $|\mathrm{Orb}_{Uz}(V)| = [V : z^{-1}Uz \cap V] = |V| / |z^{-1}Uz \cap V|$. Finally, $Uzv = Uz \Leftrightarrow zvz^{-1} \in U \Leftrightarrow v \in z^{-1}Uz$ for all $v \in V$. ❑

Proposition 2 shows that the sizes of the double cosets UzV depend substantially on the sizes of the intersections of the right conjugates of U with V. Consideration of the set of all such intersections ***of a given size*** leads to a quite influential ***observation*** which we now state and prove. For intersections of U with left conjugates of V a completely analogous statement is valid.

Proposition 3 (The subsets $\mathrm{M}(U,V,n)$).
Let U and V be subgroups of the finite group H and n be some positive integer.
Define $\mathbf{M}(\boldsymbol{U},\boldsymbol{V},\boldsymbol{n}) := \{ x \in H \mid |x^{-1}Ux \cap V| = n \} = \{ x \in H \mid |\mathrm{Stab}_V(Ux)| = n \}$.
If $\mathrm{M}(U,V,n) \neq \varnothing$, then $\mathrm{M}(U,V,n)$ is the union of some double cosets of U and V in H each of which has size $|U|\cdot|V| / n$, and therefore $|\mathrm{M}(U,V,n)|$ is a multiple of $|U|\cdot|V| / n$.

PROOF: Let $x \in \mathrm{M}(U,V,n)$. If $y \in UxV$ then $y = uxv$ for some $u \in U$ and some $v \in V$ whence $y^{-1}Uy \cap V = (xv)^{-1}Uxv \cap v^{-1}Vv = v^{-1}(x^{-1}Ux \cap V)v$ whence $y \in \mathrm{M}(U,V,n)$. Thus $UxV \subseteq \mathrm{M}(U,V,n)$. If $\mathrm{M}(U,V,n) \neq UxV$ put $x_1 := x$ and pick $x_2 \in \mathrm{M}(U,V,n) \setminus Ux_1V$. Then as for x_1 we conclude that $Ux_2V \subseteq \mathrm{M}(U,V,n)$. If $\mathrm{M}(U,V,n) \neq Ux_1V + Ux_2V$ pick $x_3 \in H$ with $x_3 \notin Ux_1V$ and $x_3 \notin Ux_2V$. Then again $Ux_3V \subseteq \mathrm{M}(U,V,n)$. This iteration process will stop after a finite number T of steps and then $\mathrm{M}(U,V,n)$ has been partitioned into T double cosets of U and V in H, that is, $\mathrm{M}(U,V,n) = Ux_1V + Ux_2V + Ux_3V + \ldots + Ux_{\mathrm{T}}V$. In particular, $|x_i^{-1}Ux_i \cap V| = n$ for all $1 \leq i \leq \mathrm{T}$. Now **Proposition 2** gives $|Ux_iV| = |U|\cdot|V| / |x_i^{-1}Ux_i \cap V| = |U|\cdot|V| / n$ for all $1 \leq i \leq \mathrm{T}$. It follows that $|\mathrm{M}(U,V,n)| = \mathrm{T}\cdot(|U|\cdot|V| / n)$ as claimed. ❑

We now can supplement by a ***modern*** proof of **Cauchy**'s concealed *third* group theorem the ***classical*** **Cauchy**-style proof which is claiming to constitute the "Entdeckung einer w i ch t i g st e n Wahrheit" because it has been "thinking in the case of something seen by everyone that which" scarcely any "one has yet thought". It is to be noticed that the *modern* proof first constructs a rectangle with T rows of size $|S|\cdot|G|$ and then removes $|G|$ columns to be left with **the very same rectangle** as the *classical* proof. While the "modernisation" of "Lagrange" required only cosets (see **Page 25**), "modernising" of "Cauchy" requires double cosets.

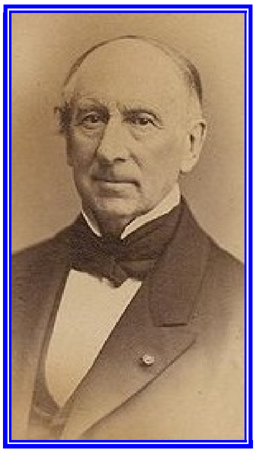
Corollary (Confirmation of Cauchy's concealed *third* group theorem).
Let p be a prime and $G \subseteq H$ with H being finite. If H contains a subgroup S of order $|H|_p$ (that is, a Sylow p-subgroup) and $x^{-1}Sx \cap G = \langle 1 \rangle$ for all $x \in H$ (for example, when G is p-blank in H), then $[H{:}G]$ is divisible by $|H|_p$.

PROOF: Apply **Proposition 3** with $U := S$ and $V := G$ and $n := 1$. Then $\mathrm{M}(S,G,1) = H$ is a union of some double cosets of S and G in H each of which has size $|S|\cdot|G|$ whence $|H| = \mathrm{T}\cdot|S|\cdot|G|$ with some positive integer T. Since also $|H| = [H{:}G]\cdot|G|$ by good old **Lagrange**'s theorem, it follows after cancellation by $|G|$ that $[H{:}G] = \mathrm{T}\cdot|S| = \mathrm{T}\cdot|H|_p$. ❑

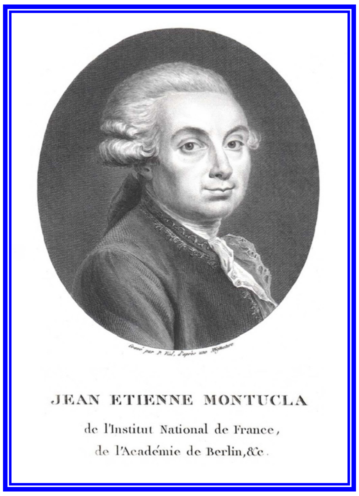
Citation: Felix F. Flemisch (2026) Characterising locally finite groups satisfying the strong Sylow Theorem for the prime p – Mathematics edition – Second edition. Journal of Mathematical & Computer Applications. SRC/JMCA-284. DOI: doi.org/10.47363/JMCA/2026(5)239

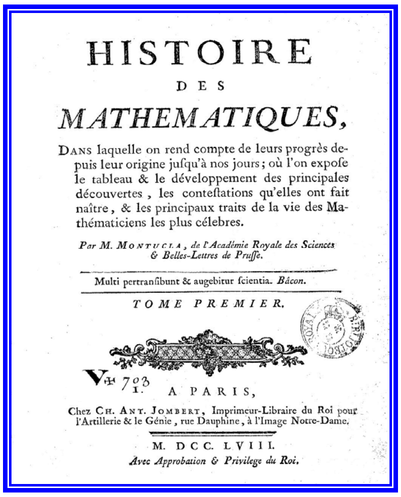
d **Michael Meo** in his otherwise quite remarkable contribution "*The mathematical life of Cauchy's group theorem*" (see *Historia Mathematica*, Volume **31**, Issue **2**, May 2004, pages 196-221 at https://www.sciencedirect.com/science/article/pii/S031508600300003X) claims firmly that "The initial proof by Cauchy, however, was unprecedented in its complex computations involving permutational group theory and contained an egregious error." and in French "La démonstration originale donnée par Cauchy comportait, en revanche, des calculs complexes de groupes de permutations et contenait au fond une erreur." This is *not true, really not* ☹. His quotation (with the wrong year 1763, however) "Celui qui s'engage le premier dans une région inconnue, mérite quelque indulgence pour les fausses routes qu'il y peut faire." from the mathematician **Jean-Étienne Montucla**, however, is *really true* ☺ as the author knows very very well (see *Histoire des mathématiques* (Éd.1758), Tome **premier**, avec Approbation & Privilege du Roi, p. xv, http://gallica.bnf.fr/ark:/12148/bpt6k5401071b and https://www.amazon.fr/Histoire-math%C3%A9matiques-%C3%89d-1758-Jean-%C3%89tienne-Montucla/dp/2012888763).

Nous montrons celui qui s'engageait vraiment le premier sur un terrain en grande partie inconnu – dans la théorie des groupes de permutations – et celui qui aurait été indulgent avec lui pour les fausses routes qu'il aurait pu y faire s'il y en avait eu, ce qui n'était pas le cas:

Augustin-Louis Cauchy (21 August 1789 until 23 May 1857)
(see https://fr.wikipedia.org/wiki/Augustin_Louis_Cauchy)





HISTOIRE
DES
MATHEMATIQUES,

DANS laquelle on rend compte de leurs progrès depuis leur origine jusqu'à nos jours; où l'on expose le tableau & le développement des principales découvertes, les contestations qu'elles ont fait naître, & les principaux traits de la vie des Mathématiciens les plus célebres.

Par M. MONTUCLA, de l'Académie Royale des Sciences & Belles-Lettres de Prusse.

Multi pertransibunt & augebitur scientia. *Bâcon.*

TOME PREMIER.

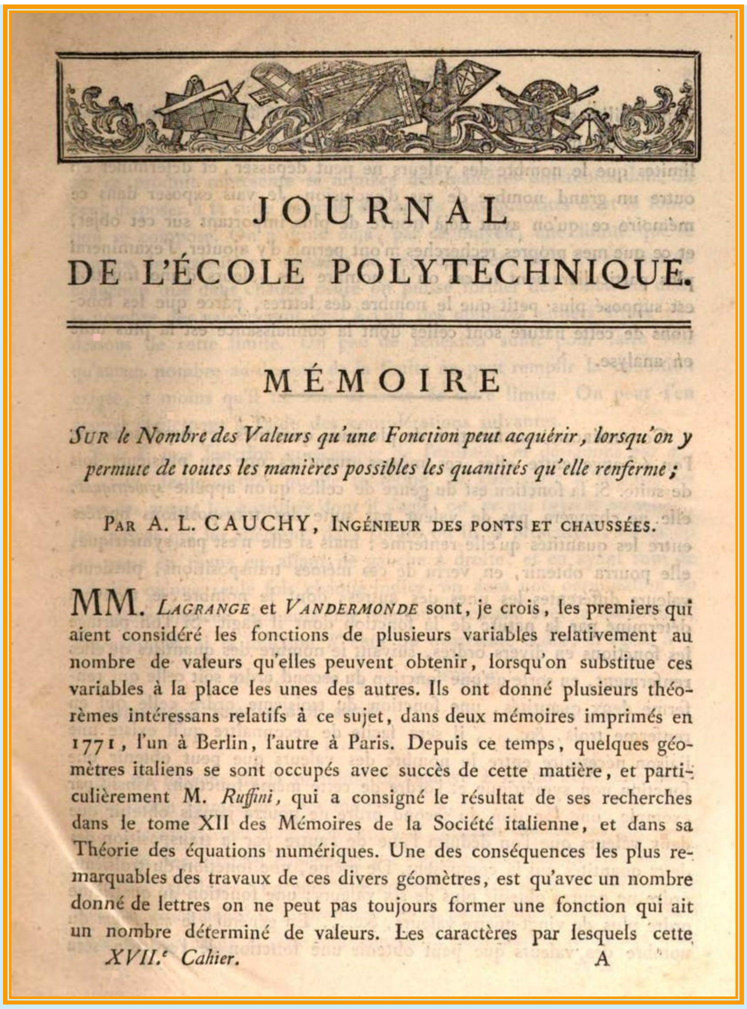
A PARIS,

Chez CH. ANT. JOMBERT, Imprimeur-Libraire du Roi pour l'Artillerie & le Génie, rue Dauphine, à l'Image Notre-Dame.

M. DCC. LVIII.
Avec Approbation & Privilege du Roi.

Jean-Étienne Montucla (5 September 1725 until 18 December 1799)
(see https://fr.wikipedia.org/wiki/Jean-%C3%89tienne_Montucla)

Cauchy in the succession of Joseph-Louis de Lagrange (Giuseppe Luigi Lagrangia), Alexandre-Théophile Vandermonde and Paolo Ruffini



JOURNAL

DE L'ÉCOLE POLYTECHNIQUE.

MÉMOIRE

Sur le Nombre des Valeurs qu'une Fonction peut acquérir, lorsqu'on y permute de toutes les manières possibles les quantités qu'elle renferme;

PAR A. L. CAUCHY, INGÉNIEUR DES PONTS ET CHAUSSÉES.

MM. *LAGRANGE* et *VANDERMONDE* sont, je crois, les premiers qui aient considéré les fonctions de plusieurs variables relativement au nombre de valeurs qu'elles peuvent obtenir, lorsqu'on substitue ces variables à la place les unes des autres. Ils ont donné plusieurs théorèmes intéressans relatifs à ce sujet, dans deux mémoires imprimés en 1771, l'un à Berlin, l'autre à Paris. Depuis ce temps, quelques géomètres italiens se sont occupés avec succès de cette matière, et particulièrement M. *Ruffini*, qui a consigné le résultat de ses recherches dans le tome XII des Mémoires de la Société italienne, et dans sa Théorie des équations numériques. Une des conséquences les plus remarquables des travaux de ces divers géomètres, est qu'avec un nombre donné de lettres on ne peut pas toujours former une fonction qui ait un nombre déterminé de valeurs. Les caractères par lesquels cette

XVII.e Cahier. A

Mémoire Sur le Nombre des Valeurs qu'une Fonction peut acquérir, lorsqu'on y permute de toutes les manières possibles les quantités qu'elle renferme ;
Par **A. L. Cauchy**, Ingénieur des ponts et chaussées.
Journal de l'École Polytechnique, XVII^{e} Cahier, Tome dixième, Janvier 1815, 1-28
(see http://gallica.bnf.fr/ark:/12148/bpt6k433673r.image
and https://opacplus.bsb-muenchen.de/title/6409549 and
http://nonagon.org/ExLibris/cauchys-memoire-sur-le-nombre-des-valeurs)

For **Lagrange**'s theorem **Cauchy** considers (on Page 5 and Page 6) "une fonction quelconque" K "de l'ordre n" and puts N := n!, R := "le nombre total des valeurs essentiellement différentes de K" (which he calls "*indice* de K") and M := "le nombre total des valeurs de K que l'on suppose égales entre elles" (which he calls "*diviseur indicatif* par lequel on doit diviser N pour obtenir l'indice de K"). By then collecting repeatedly "nouvelles valeurs de K" **until they are all used up** and having started from M, he argues that "on aura donc R M = N". He could not consider abstract groups and considered not even permutation groups but substitutions of variables of polynomials: $\underline{S}^n$ acts on the set $F[X_1,X_2,...,X_n]$ of polynomials with coefficients from a field F (usually the complex numbers) by permuting the variables: if π is a polynomial and x is some permutation then $\pi^x(X_1, X_2, \dots, X_n) := \pi(X_{1^{x^{-1}}}, X_{2^{x^{-1}}}, \dots, X_{n^{x^{-1}}})$.

The published *first* group theorem by **Cauchy** states that R ≤ 2 if R $< p$ for the largest prime $p \leq n$ (resp. the largest prime p dividing n!). The major step of the proof consists of verifying the following (see Page 9) : "On fait voir que si l'on suppose R $< p$, chaque valeur de K ne pourra être changée par aucune substitution du degré p." This means that $[H:G] < p$ for $G \subseteq H := \underline{S}^n$ implies that G **does "contain"** ***not a single p*****-blank**.