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Problem of Rational Distances from a Point to the Vertices of a Unit Square on a Plane and Its Application to Control Theory

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ABSTRACT

Among the unsolved problems of mathematics there are those whose hidden meaning turns out to be many times more important and instructive than solutions of these problems themselves. Most likely, this is exactly the case with the problem of rational distances from a point to the vertices of a unit square on a plane, if you approach it not quite from the frontal side.

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Introduction

Problems related to finding integer or rational distances from a certain point to the vertices of a figure on a plane are of considerable interest in modern computational mathematics [1-3].

Thus, in 2001, E. Pegg discovered a simplest arbitrary triangle with side lengths of 8, 22, and 19 Figure. 1, containing an internal point with distances of 17, 6, and 4 to each of its vertices [1]. As for the possibility of the existence of a unit square and a certain point on a plane, the distances from which to each of its vertices would be expressed by rational quantities, this problem is still considered an unsolved mathematical problem in the literature [2].

Here we will talk not so much about debunking this problem, although this is also part of the author's intentions, but about one interesting geometric illustration of a four-dimensional phase space, which follows from it, subject to the simplest linear connection between the phase coordinates of the indicated space.

Integer and Rational Distances from A Point to the Vertices of the Simplest Figures

As was said above, in the work the existence of the simplest arbitrary triangle and some point on its plane, the distances to the vertices from which are expressed by integers Figure 1 was shown [1].

Several years later, a new problem was formulated, which requires proving or disproving the possibility of constructing a unit square and some point on a plane, the distances from which to all vertices of which would be expressed by rational numbers Figure 2 [2].

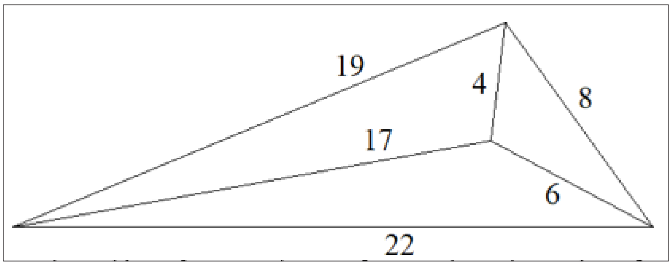

Figure 1: The Problem of Integer Distances from a Point to the Vertices of a Triangle

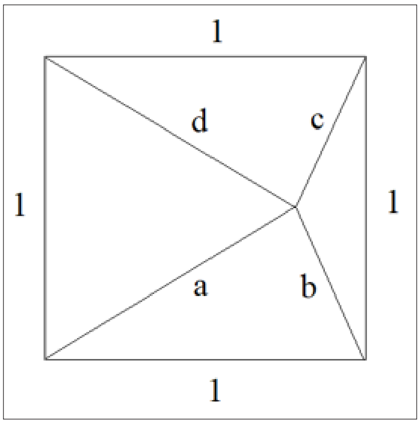

Figure 2: The Problem of Rational Distances from a Point to the Vertices of a Unit Square

Special Interpretation of the Problem without Loss of Generality

The problems of finding distances from certain points to the vertices of geometric figures considered above were not limited to finding a point within the given figures Figure 1, 2. In fact, the desired point, by virtue of the condition of the problem, can be located both inside Figure 3a, and outside Figure 3b, the figure itself, as shown in Figure 3.

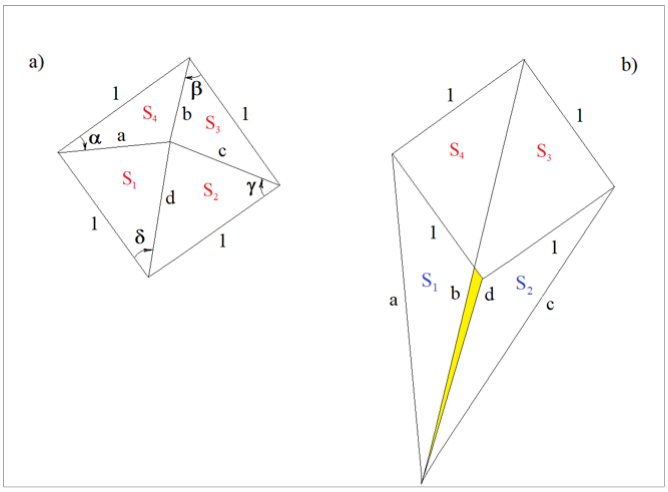

Figure 3: Law Of Conservation of Areas (A) And «Flat Places» (B) In A Unit Square

If in the first case, when we are talking about a certain point inside a unit square, the law of conservation of component areas is obviously observed, namely:

$$S_1 + S_2 + S_3 + S_4 = 1, \qquad (1)$$

then in the second case, when the wandering point ends up outside the unit square, it turns out that the law of conservation of areas is no longer satisfied.

And the point is not at all in the area of the small shaded subregion, which in one case enters into balance with a negative sign, and in the other-with a positive one, which ultimately is zeroed. The point is that in mathematics there is no concept of negative area due to the fact that *area as a measure of space is a strictly positive value.* However, at the same time, there is no doubt that with negative values of angles *α,β,γ,δ*, negative values of areas are formed according to known geometric relationships, which should be called not areas, but «flat places» of a flat two-dimensional space, taking both positive and negative values:

$$\begin{cases} \overline{S}_1 = \frac{1}{2} d \operatorname{Sin}\delta = \frac{1}{2} a \operatorname{Cos}\alpha, \\ \overline{S}_2 = \frac{1}{2} c \operatorname{Sin}\gamma = \frac{1}{2} d \operatorname{Cos}\delta, \\ \overline{S}_3 = \frac{1}{2} b \operatorname{Sin}\beta = \frac{1}{2} c \operatorname{Cos}\gamma, \\ \overline{S}_4 = \frac{1}{2} a \operatorname{Sin}\alpha = \frac{1}{2} b \operatorname{Cos}\beta. \end{cases} \qquad (2)$$

Then formula (1), valid only for areas, is naturally generalized to the case of «flat places» and turns out to be applicable to any mutual arrangement of a wandering point and a unit square on a plane:

$$\overline{s}_1 + \overline{s}_2 + \overline{s}_3 + \overline{s}_4 = 1. \qquad (3)$$

It is interesting to note that the discovered four-dimensional phase space, subject to constraint (3), degenerates into three-dimensional phase spaces on the sides of the unit square and on their extensions, and at the vertices of the same square it is reduced, of course, into two dimen sional phase spaces.

Proof of the Impossibility of the Existence of Rational Distances

We present a proof of the impossibility of the existence of rational distances from a certain point to the vertices of a unit square on a plane.

Let us assume the existence of two rational distances b and d Figure 3 out of four. Then we will have the simplified calculation scheme shown in Figure 4 Let us write the cosine theorem for all sides for the arbitrary triangle shown:

$$\begin{cases} b^2 = 2 + d^2 - 2\sqrt{2}\, d \operatorname{Cos}\xi, \\ d^2 = 2 + b^2 - 2\sqrt{2}\, b \operatorname{Cos}\eta, \\ 2 = b^2 + d^2 - 2bd \operatorname{Cos}\zeta = b^2 + d^2 + 2bd \operatorname{Cos}(\xi + \eta). \end{cases} \qquad (4)$$

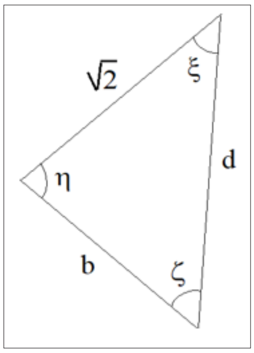

Figure 4: Simplified Calculation Scheme of a Triangle

From the first two algebraic equations of system (4) it follows that the cosines of the angles ξ and η determined by the following expressions:

$$\operatorname{Cos}\xi = \frac{p}{\sqrt{2}} = \frac{d^2 - b^2 + 2}{2\sqrt{2}d}, \qquad (A)$$

$$\operatorname{Cos}\eta = \frac{q}{\sqrt{2}} = \frac{b^2 - d^2 + 2}{2\sqrt{2}b}. \qquad (B)$$

From Expressions (A) and (B) we can Conclude:

$$\frac{p}{q} = \frac{b(d^2 - b^2 + 2)}{d(b^2 - d^2 + 2)} = \lambda. \qquad (C)$$

From the first Equation (C) it Follows that

$$P = \lambda q, \qquad (D)$$

From where, taking into account (A) and (B), we will have the following relationship between the cosines of the angles *ξ* and η:

$$\operatorname{Cos}\xi = \frac{\lambda q}{\sqrt{2}} = \lambda \operatorname{Cos}\eta. \qquad (E)$$

Taking into account the expressions for the sines of the angles *ξ* and η

$$\operatorname{Sin}\xi = \sqrt{1 - \frac{p^2}{2}}, \qquad (F)$$

$$\mathrm{Sin}\,\eta=\sqrt{1-\frac{q^2}{2}}, \qquad (G)$$

We Transform Expression (F) taking into Account (D) as Follows:

$$\mathrm{Sin}\,\xi=\sqrt{1-\frac{\lambda^2 q^2}{2}}=\hat{\mathrm{S}}\mathrm{in}\,\eta\neq\mathrm{Sin}\,\eta. \qquad (H)$$

Then the basic Trigonometric Identity for the angle ξ is Written:

$$\mathrm{Cos}^2\,\xi+\mathrm{Sin}^2\,\xi=\lambda^2\mathrm{Cos}^2\,\eta+\hat{\mathrm{S}}\mathrm{in}^2\,\eta=1. \qquad (I)$$

At the same time, it is well known that the basic trigonometric identity for the angle η can be valid only in one form of writing:

$$\mathrm{Cos}^2\,\eta+\mathrm{Sin}^2\,\eta=1. \qquad (J)$$

Comparing (I) and (J), we come to the conclusion that the equalities in (I) are true if and if $\lambda=1$,only, for which $\hat{\mathrm{S}}\mathrm{in}\,\eta=\mathrm{Sin}\,\eta$..

Therefore, as a result of (D), we establish that the triangle shown in Fig. 4 is isosceles with equal lateral sides b = d. Then, expressions (A) and (B) can be replaced by the following simplified analogue for both angles ξ and η

$$\mathrm{Cos}\,\xi=\mathrm{Cos}\,\eta=\frac{p}{\sqrt{2}}=\frac{1}{\sqrt{2}d}. \qquad (K)$$

Let us now turn to the third algebraic equation of system (4):

$$d^2(1-\mathrm{Cos}\,\zeta)=1, \qquad (L)$$

which allows rational solutions to the problem: d = b ≥ 1, since 0 ≤ Cosξ <1, then by virtue of (K) and (D) we will also have p = q ≤ 1.

Thus, *the possible solutions of the quadratic equation (L) are represented by a set of points lying on straight lines passing through opposite vertices of the unit square*, while being outside the square. Consequently, under the assumption that the distance A from a wandering point to the third, least distant vertex is rational, it turns out that the fourth, most distant distance in this case will represent an irrational quantity A+√(2)and conversely, if the distance B from the wandering point to the third, most distant vertex is rational, then it turns out that the fourth, least distant distance will nevertheless be expressed by an irrational quantity B-√2.

Based on the above, we conclude that *it is impossible for four rational distances to exist from a given point to the vertices of a unit square on a plane.*

Four-Dimensional Phase Space in the Theory of Manipulator Control

Let us consider the four-dimensional phase space shown in Figure 5, the phase coordinates of which are subject to the following constraint equation:

$$S_1+S_2+S_3+S_4=\pi/2. \qquad (5)$$

We will assume that on the phase space ABCD Figure 5 the representative point O makes the simplest motion *rectilinear and uniform* – from point A to point C. Consequently, the *phase trajectory* of the point's motion in the phase space under consideration will be the *phase segment* AC.

Then at $a(t)=t,\ t\in[0;\sqrt{\pi}\,s]$ we will have laws of change of phase coordinates in the form of the law of change of areas S_1, S_2, S_3, S_4 over time:

$$\begin{cases} S_1=S_2=\frac{1}{2}a(t)\sqrt{\frac{\pi}{2}}\mathrm{Sin}\,\frac{\pi}{4}=\frac{\sqrt{\pi}}{4}t, \\ S_3=S_4=\frac{\pi/2-2S_1}{2}=\frac{\pi}{4}-\frac{\sqrt{\pi}}{4}t. \end{cases} \qquad (6)$$

The laws of change of phase coordinates (6) in connection with (5) in the dynamics of manipulation mechanisms can correspond to the movement of the four-link manipulator from the initial position to the final position, shown in Fig. 6, under the condition of identifying the generalized coordinates of the motion of the four-link manipulator with the areas of the phase space of the corresponding dimension, i.e. $q_1\equiv S_1$, $q_2\equiv S_2$, $q_3\equiv S_3$, $q_4\equiv S_4$.

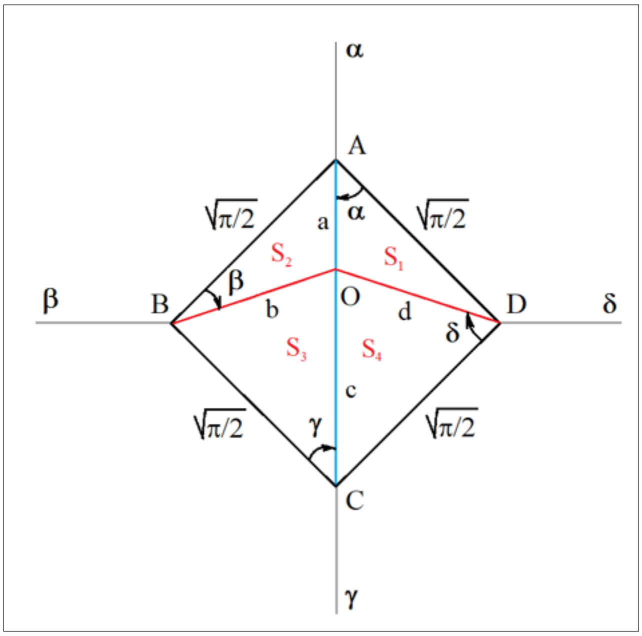

Figure 5: Four-Dimensional phase Space ABCD and Phase Trajectory AC

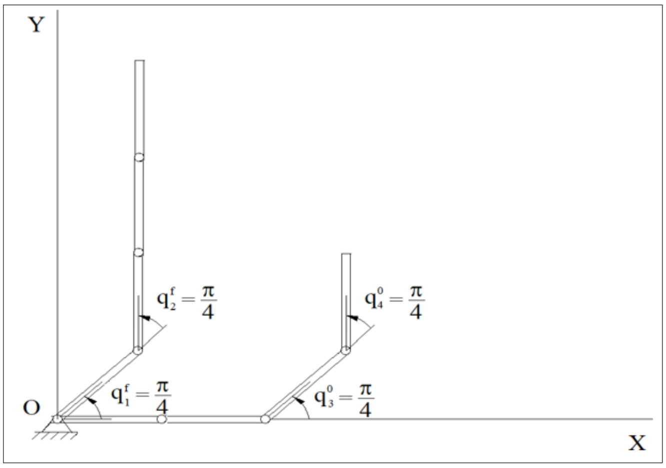

Figure 6: Movement of a Four-Link Manipulator from the Initial Position to the Final Position

Let the laws of motion of a four-link manipulator be given by the dependencies

$$\begin{cases} q_1 = q_2 = \frac{\sqrt{\pi}}{4} t, \\ q_3 = q_4 = \frac{\pi}{4} - \frac{\sqrt{\pi}}{4} t, \end{cases} \quad t \in [0; \sqrt{\pi}\, s] \qquad (7)$$

under initial conditions of motion:

$q_1 = q_2 = 0, q_3 = q_4 = \pi / 4$ and $\dot{q}_1 = \dot{q}_2 = \sqrt{\pi} / 4, \dot{q}_3 = \dot{q}_4 = -\sqrt{\pi} / 4.$

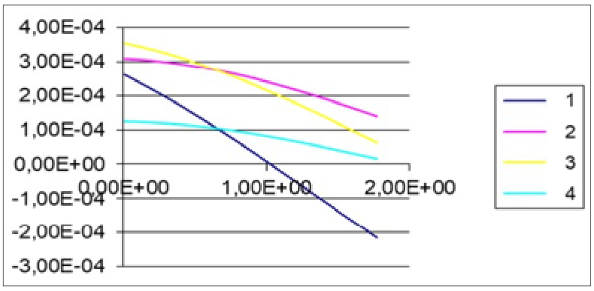

Figure 7: Dependence of Drive Torques on time

Then, having also specified the geometric and inertial characteristics of the manipulator links, for example, the lengths of the links $l_1 = l_2 = l_3 = l_4 = 0.1$ m , the radius of the cross-sections of the links $r_1 = r_2 = r_3 = r_4 = 3.10^{-3}$ m and the same density of their material (steel) $P_1 = P_2 = P_3 = P_4 = 7850 kg / m^3$, it is possible to formulate the first problem of the dynamics of the four-link manipulator under study Figure 6, for which, due to existing methods of dynamic analysis, it is possible to determine the drive moments in the articulations of the manipulator, which is illustrated in Figure 7 [4].

Conclusion

The article presents an original interpretation of the four-dimensional phase space in the form of an area model. A specific scope of application of the proposed model in the control of manipulation mechanisms is resented, which creates good prospects for using this innovation in the field of drone control.

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