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On Finding a Solution of One Problem for the Heat Equation of on a Semi-Axis

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ABSTRACT

The presented article is devoted to finding a solution to one problem for a heat equation with discontinuous coefficients on the semiaxis. Applying an integral transformation to the problem, a boundary value problem is constructed for an ordinary differential equation depending on a complex parameter. In this case, it is assumed that the solution to the problem found at infinity is bounded. Since the coefficient of the equation is discontinuous, the solution to the problem is found in both finite and infinite intervals. In the general solutions found in both intervals, there are constants that are independent of each other. Although in the problem the number of boundary and associated conditions is three, the number of constants involved in the solution is four. This can lead to violation of the uniqueness of the solution to the problem. To eliminate this discrepancy, it is necessary to choose one of the found constants equal to zero, using the condition that the solution is bounded at infinity. The other three constants are found from the boundary and associated conditions and taken into account in the solution of an ordinary differential equation with discontinuous coefficients. The article shows that the solution of an ordinary differential equation with discontinuous coefficients is a meramorphic function on the complex plane. The singular points found for solving the problem are pole-type singular points located in a strip containing an imaginary axis. In the paper, the expansion theorem is proved and the solution of the problem is constructed in the form of a contour integral. The absolute and uniform convergence of the found contour integral for the solution is shown, which means the substantiation of the formal solution of the original problem.

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Introduction

In the article we consider the following problem:
It is requrired to find the bounded solution to the equation

$$\frac{\partial u}{\partial t} = a(x)\frac{\partial^2 u}{\partial x^2},\ x\epsilon(0,+\infty), t>0 \tag{1}$$

satisfying the initial conditions

$$u(x,0) = \varphi(x) \tag{2}$$

and the condition

$$\begin{gathered} \alpha_1 u(0,t) + \beta_1 u(\alpha-0,t) = 0 \\ \alpha_2 u_x(0,t) + \beta_2 u_x(\alpha-0,t) = 0 \\ \alpha_3 u(\alpha-0,t) + \beta_3 u(\alpha+0,t) = 0 \end{gathered} \tag{3}$$

Here

$$a(x) = \begin{cases} p^2, & 0<x\le\alpha \\ q^2, & \alpha<x<+\infty \end{cases}$$

the numbers α_i, β_i (i=1,2,3) are complex constants, p and q are real numbers. So, conditions $p>0$, $q>0$ are satisfied.

Being from the following class, the solution to the problem satisfis the conditions (1)-(3).

$$u(x,t) \in C^{2,1}(0<x<\alpha, t>0) \cap C^{2,1}(\alpha<x<1, t>0)$$

Applying the integral transform

$$y(x,\lambda) = \int_0^\infty u(x,t)\, e^{-\lambda^2 t} dt \tag{4}$$

to the considere problem (1)-(3), for second order second order discontinous coefficient equation dependent on the complex parameter , λ we obtain the following problem:

$$\begin{gathered} a(x)y'' - \lambda^2 y = -\varphi(x),\ \alpha<x<+\infty \\ \alpha_1 y(0,\lambda) + \beta_1 y(\alpha-0,\lambda) = 0 \end{gathered} \tag{5}$$

$$\begin{gathered} \alpha_2 y'(0,\lambda) + \beta_2 y'(\alpha-0,\lambda) = 0 \\ \alpha_3 y(\alpha-0,\lambda) + \beta_3 y(\alpha+0,\lambda) = 0 \end{gathered} \tag{6}$$

For $0<x\le\alpha$ and $\mathrm{Re}\lambda \ge 0$, finding the solution to the problem (5), we obtain

$$y(x,\lambda)= C_1(\lambda)e^{\frac{\lambda}{p}x}+ C_2(\lambda)e^{-\frac{\lambda}{p}x} +\frac{1}{2p\lambda}\int_x^\alpha \varphi(\xi)e^{\frac{\lambda}{p}(x-\xi)}d\xi +\frac{1}{2p\lambda}\int_0^x \varphi(\xi)e^{\frac{\lambda}{p}(\xi-x)}d\xi \tag{7}$$

For $0 < x \le \alpha$ and $\mathrm{Re}\lambda < 0$, we can find the solution to the problem (5) as follows.

$$y(x,\lambda)= C_1(\lambda)e^{\frac{\lambda}{p}x}+ C_2(\lambda)e^{-\frac{\lambda}{p}x} - \frac{1}{2p\lambda}\int_x^\alpha \varphi(\xi)e^{\frac{\lambda}{p}(x-\xi)}d\xi -\frac{1}{2p\lambda}\int_0^x \varphi(\xi)e^{\frac{\lambda}{p}(\xi-x)}d\xi \tag{8}$$

For $\alpha < x < +\infty$ and $\mathrm{Re}\lambda \ge 0$ finding the solution to the problem (5) we obtain

$$y(x,\lambda)= C_3(\lambda)e^{-\frac{\lambda}{q}x}+ C_4(\lambda)e^{-\frac{\lambda}{q}x}+\frac{1}{2q\lambda}\int_\alpha^x \varphi(\xi)e^{\frac{\lambda}{q}(\xi-x)}d\xi+\frac{1}{2q\lambda}\int_x^{+\infty} \varphi(\xi)e^{\frac{\lambda}{q}(x-\xi)}d\xi \tag{9}$$

As x→ +∞, a necessary condition for the function $y(x,\lambda)$ to be bounded is C_4=0 i.e we can write the solution to the problem (9) as follows :

$$y(x,\lambda)= C_3(\lambda)e^{-\frac{\lambda}{q}x}+\frac{1}{2q\lambda}\int_\alpha^x \varphi(\xi)e^{\frac{\lambda}{q}(\xi-x)}d\xi +\frac{1}{2q\lambda}\int_x^{+\infty} \varphi(\xi)e^{\frac{\lambda}{q}(x-\xi)}d\xi \tag{10}$$

For $\alpha < x < +\infty$ and $\mathrm{Re}\lambda < 0$, finding the bounded solution to the equation (5) withim the condition x→ +∞ we obtain

$$y(x,\lambda)= C_4(\lambda)e^{\frac{\lambda}{q}x} - \frac{1}{2q\lambda}\int_\alpha^x \varphi(\xi)e^{\frac{\lambda}{q}(x-\xi)}d\xi - \frac{1}{2q\lambda}\int_x^{+\infty} \varphi(\xi)e^{\frac{\lambda}{q}(\xi-x)}d\xi \tag{11}$$

For finding the poles of the function y(x,λ) in the halp-plane Reλ≥0, we write the formulas (7) and (10) in the system (6)

$$\begin{cases} (\alpha_1+\beta_1 e^{\frac{\lambda}{p}\alpha})C_1(\lambda) + (\alpha_1+\beta_1 e^{-\frac{\lambda}{p}\alpha})C_2(\lambda) = -\frac{\alpha_1}{2p\lambda}\varkappa_1(\lambda) - \frac{\beta_1}{2p\lambda}\varkappa_2(\lambda) \\ (\alpha_2+\beta_2 e^{\frac{\lambda}{p}\alpha})\frac{\lambda}{p}C_1(\lambda) - (\alpha_2+\beta_2 e^{-\frac{\lambda}{p}\alpha})\frac{\lambda}{p}C_2(\lambda) = -\frac{\alpha_2}{2p^2}\varkappa_1(\lambda) + \frac{\beta_2}{2p^2}\varkappa_2(\lambda) \\ \alpha_3 e^{\frac{\lambda}{p}\alpha}C_1(\lambda) + \alpha_3 e^{-\frac{\lambda}{p}\alpha}C_2(\lambda) + \beta_3 e^{-\frac{\lambda}{q}\alpha}C_3(\lambda) = -\frac{\alpha_3}{2p\lambda}\varkappa_2(\lambda) - \frac{\beta_3}{2q\lambda}\varkappa_3(\lambda) \end{cases} \tag{12}$$

In this system we introduced the following denotations

$$\begin{gathered} \varkappa_1(\lambda) = \int_0^\alpha \varphi(\xi)e^{-\frac{\lambda}{p}\xi}d\xi \\ \varkappa_2(\lambda) = \int_0^\alpha \varphi(\xi)e^{\frac{\lambda}{p}(\xi-\alpha)}d\xi \\ \varkappa_1(\lambda) = \int_\alpha^{+\infty} \varphi(\xi)e^{\frac{\lambda}{q}(\alpha-\xi)}d\xi \end{gathered} \tag{13}$$

Calculating the principal determinant of the system (12) , we obtain:

$$\Delta(\lambda) = -\frac{\lambda}{p}\beta_3 e^{-\frac{\lambda}{q}x}\left[2(\alpha_1\alpha_2+\beta_1\beta_2) + (\alpha_1\beta_2+\beta_1\alpha_2)(e^{\frac{\lambda}{p}\alpha} + e^{-\frac{\lambda}{p}\alpha})\right] \tag{14}$$

We find the zeros of the charactristical determinant as

$$\lambda_n = \frac{p}{\alpha}\left(ln_0\left(-T \pm \sqrt{T^2-1}\right) + 2\pi n i\right), n\epsilon R \tag{15}$$

Here

$$T = \frac{\alpha_1\alpha_2+\beta_1\beta_2}{\alpha_1\beta_2+\beta_1\alpha_2} \quad ; \alpha_1\beta_2 + \beta_1\alpha_2 \neq 0\,, \quad \beta_3 \neq 0$$

Note that in the half-plane Reλ<0, substituting the formulas (8) and (11) in the system (6) and finding the zeros of Δ(λ), we have the validty of the formula (15).

Appyling the Cramer method to the system (12), we can find the constants $C_k(\lambda)$. So, we find

$$C_k(\lambda)=\frac{\Delta_k(\lambda)}{\Delta(\lambda)}\,, \; k = 1,2,3$$

Here the determinants Δk (λ), k=1,2,3 are determined as follows:

$$\Delta_1(\lambda) = \begin{vmatrix} -\frac{\alpha_1}{2p\lambda}\varkappa_1(\lambda) - \frac{\beta_1}{2p\lambda}\varkappa_2(\lambda) & \alpha_1 + \beta_1 e^{-\frac{\lambda}{p}\alpha} & 0 \\ -\frac{\alpha_2}{2p^2}\varkappa_1(\lambda) + \frac{\beta_2}{2p^2}\varkappa_2(\lambda) & -(\alpha_2 + \beta_2 e^{-\frac{\lambda}{p}\alpha})\frac{\lambda}{p} & 0 \\ -\frac{\alpha_3}{2p\lambda}\varkappa_2(\lambda) - \frac{\beta_3}{2q\lambda}\varkappa_3(\lambda) & \alpha_3 e^{-\frac{\lambda}{p}\alpha} & \beta_3 e^{-\frac{\lambda}{q}\alpha} \end{vmatrix}$$

$$\Delta_2(\lambda) = \begin{vmatrix} \alpha_1 + \beta_1 e^{\frac{\lambda}{p}\alpha} & -\frac{\alpha_1}{2p\lambda}\varkappa_1(\lambda) - \frac{\beta_1}{2p\lambda}\varkappa_2(\lambda) & 0 \\ \frac{\lambda}{p}(\alpha_2 + \beta_2 e^{\frac{\lambda}{p}\alpha}) & -\frac{\alpha_2}{2p^2}\varkappa_1(\lambda) + \frac{\beta_2}{2p^2}\varkappa_2(\lambda) & 0 \\ \alpha_3 e^{\frac{\lambda}{p}\alpha} & -\frac{\alpha_3}{2p\lambda}\varkappa_2(\lambda) - \frac{\beta_3}{2q\lambda}\varkappa_3(\lambda) & \beta_3 e^{-\frac{\lambda}{q}} \end{vmatrix}$$

$$\Delta_3(\lambda) = \begin{vmatrix} \alpha_1 + \beta_1 e^{\frac{\lambda}{p}\alpha} & \alpha_1 + \beta_1 e^{-\frac{\lambda}{p}\alpha} & -\frac{\alpha_1}{2p\lambda}\varkappa_1(\lambda) - \frac{\beta_1}{2p\lambda}\varkappa_2(\lambda) \\ \frac{\lambda}{p}(\alpha_2 + \beta_2 e^{\frac{\lambda}{p}\alpha}) & -\frac{\lambda}{p}(\alpha_2 + \beta_2 e^{-\frac{\lambda}{p}\alpha}) & -\frac{\alpha_2}{2p^2}\varkappa_1(\lambda) + \frac{\beta_2}{2p^2}\varkappa_2(\lambda) \\ \alpha_3 e^{\frac{\lambda}{p}\alpha} & \alpha_3 e^{-\frac{\lambda}{p}\alpha} & -\frac{\alpha_3}{2p\lambda}\varkappa_2(\lambda) - \frac{\beta_3}{2q\lambda}\varkappa_3(\lambda) \end{vmatrix}$$

Theorem 1: (Exspansion Theorem) Assume that the following conditions are satisfied:

- the function φ(x) has a continuous derivative to the second order on the interval [0,+∞)
- the function x^p φ^((k)) (x)=O(1) , x→+∞ and φ(0)=φ(α)=0 , p>1 ,k=0,1,2
- $\alpha_1 \beta_2+\beta_1 \alpha_2\neq 0$, $\beta_3\neq 0$

Then for the function φ(x) the expansion formula

$$\varphi(x) = \frac{1}{2\pi i}\sum_{n=0}^{\infty}\int_{C_n}\lambda\, y(x,\lambda)d\lambda \qquad (16)$$

is valid. Here C_n is a simple closed path and contains one pole of the function $y(x,\lambda)$ in itself.

Proof: Within the theorem conditions, on the half-plane Reλ≥0 twice integrating by parts each of the integrals $\varkappa_k(\lambda)$ (k=1,2,3), we can write them as follows.

$$\varkappa_1(\lambda) = \frac{p^2}{\lambda^2}\left[\int_0^\alpha \varphi''(\xi)e^{-\frac{\lambda}{p}\xi}d\xi + \varphi'(0) - \varphi'(\alpha)e^{-\frac{\lambda}{p}\xi}\right]$$

$$\varkappa_2(\lambda) = \frac{p^2}{\lambda^2}\left[\int_0^\alpha \varphi''(\xi)e^{\frac{\lambda}{p}(\xi-\alpha)}d\xi + \varphi'(0)e^{-\frac{\lambda}{p}\alpha} - \varphi'(\alpha)\right] \qquad (17)$$

$$\varkappa_3(\lambda) = \frac{q^2}{\lambda^2}\left[\int_0^{+\infty} \varphi''(\xi)e^{\frac{\lambda}{q}(\alpha-\xi)}d\xi + \varphi'(\alpha)\right]$$

Now within the theorem conditions, integrating twice each of the integrals occuring in the formulas (7) and (9), we obtain:

$$\int_0^x \varphi(\xi)e^{\frac{\lambda}{p}(\xi-x)}d\xi = \frac{p}{\lambda}\varphi(x) - \frac{p^2}{\lambda^2}[\varphi'(x) - \varphi'(0)e^{-\frac{\lambda}{p}\xi} - \int_0^\alpha \varphi''(\xi)e^{\frac{\lambda}{p}(\xi-\alpha)}d\xi]$$

$$\int_x^\alpha \varphi(\xi)e^{\frac{\lambda}{p}(x-\xi)}d\xi = \frac{p}{\lambda}\varphi(x) + \frac{p^2}{\lambda^2}[-\varphi'(x) + \varphi'(\alpha)e^{\frac{\lambda}{p}(x-\alpha)} + \int_x^\alpha \varphi''(\xi)e^{\frac{\lambda}{p}(x-\xi)}d\xi]$$

$$\int_\alpha^x \varphi(\xi)e^{\frac{\lambda}{q}(\xi-x)}d\xi = \frac{q}{\lambda}\varphi(x) - \frac{q^2}{\lambda^2}[\varphi'(x) - \varphi'(0)e^{-\frac{\lambda}{q}x} - \int_\alpha^x \varphi''(\xi)e^{\frac{\lambda}{q}(\xi-x)}d\xi]$$

(18)

$$\int_x^{+\infty} \varphi(\xi)e^{\frac{\lambda}{q}(x-\xi)}d\xi = \frac{q}{\lambda}\varphi(x) + \frac{q^2}{\lambda^2}[-\varphi'(x) - \int_x^\alpha \varphi''(\xi)e^{\frac{\lambda}{q}(x-\xi)}d\xi]$$

After these transformaions, on the half plane Reλ≥0 we estimate the function $y(x,\lambda)$ of λ_n found by the formula (15) outside δ>0. For that we transform the characteristical determinant Δ(λ) and auxilarly determinants $\Delta_k(\lambda)$ (k=1,2,3) as folows:

$$\Delta(\lambda) = \lambda e^{\frac{\lambda}{p}\alpha - \frac{\lambda}{q}\alpha}\, V(\lambda) \qquad (19)$$

$$\Delta_k(\lambda) = e^{\frac{\lambda}{p}\alpha - \frac{\lambda}{q}\alpha}\, V_k(\lambda) \;, k = 1,2,3$$

Here the functions $V(\lambda)$ and $V_k(\lambda)$ (k=1,2,3) are bounded and analytic functions of the half-plane λ≠0 and Reλ≥0 outside the zeros of Δ(λ). Taking into account the equalities (17)-(19) in the formulas (7)-(9), we obtain:

$$y(x,\lambda) = \frac{1}{\lambda^2}\varphi(x) + \frac{1}{\lambda^3}K_j(x,\lambda)\,, \; (j = 1,2) \qquad (20)$$

Here $K_1(x,\lambda)$ and $K_2(x,\lambda)$ are bounded and analytic functions of the zeros of λ≠0 and Δ(λ) outside δ>0 in the sets $\prod_1\{(x,\lambda) : 0<x\leq\alpha, \text{Re}\lambda\geq 0\}$ and $\prod_1\{(x,\lambda) : \alpha<x<+\infty, \text{Re}\lambda\geq 0\}$, respectively. From the problem (5),(6) it is scen that the $y(x,\lambda)=y(x,-\lambda)$ equality is valid. This means that the formula (20) is validon the half-plane Reλ<0 as welll. Now we denote the sequnce of cirles centered in the origin of coordinates and whose radii increase monotonically and satisfy the condition $\lim_{n\to\infty} r_n = +\infty$ on the complex plane λ by O_n (n=1,2,…).

We choose the radii of the sequence of citcles O_n (n=1,2,…) so that they do not intersect the curves $|\lambda-\lambda_n|=\delta$. This is always possible, because the condition $|\lambda_{n+1}-\lambda_n|= \frac{2p\pi}{\alpha}$ is satisfied.

Using the condition (20) we obtain :

$$\lim_{n\to\infty}\frac{1}{2\pi i}\int_{O_n}\lambda\, y(x,\lambda)d\lambda = \frac{\varphi(x)}{2\pi i}\int_{O_n}\frac{1}{\lambda}d\lambda + \frac{1}{2\pi i}\lim_{n\to\infty}\int_{O_n}\frac{K_i(x,\lambda)}{\lambda^2}d\lambda = \varphi(x)$$

(21)

Here the functions $K_j(x,\lambda)$ are bounded and analytic functions on the circles O_n.

Indeed,

$$\lambda_n = \rho_n e^{i\varphi}, 0 \le \varphi < 2\pi$$

$$\text{as } n \to \infty \quad \rho_n \to \infty .$$

$$|K_j(x,\lambda)| \le M$$

$$\left|\int_{O_n} \frac{K_i(x,\lambda)}{\lambda^2} d\lambda\right| = \left|\int_0^{2\pi} \frac{1}{\rho} K_j(x, \rho e^{i\varphi}) i e^{-i\varphi} d\varphi\right| \le M \frac{1}{\rho} \int_0^{2\pi} d\varphi = \frac{2\pi M}{\rho} \xrightarrow[\rho\to\infty]{} 0$$

On the other hand, according to the Cauchy residual theorem we can write:

$$\lim_{n\to\infty} \frac{1}{2\pi i} \int_{O_n} \lambda\, y(x,\lambda) d\lambda = \sum_{n=0}^{\infty} \frac{1}{2\pi i} \int_{C_n} \lambda\, y(x,\lambda) d\lambda \quad (22)$$

Finally, comparing (21) and (22), for the function φ(x) we obtain the following formula:

$$\varphi(x) = \frac{1}{2\pi i} \sum_{n=0}^{\infty} \int_{C_n} \lambda\, y(x,\lambda) d\lambda$$

The theorem is proved.

Now we give the following theorem for the solution of the problem (1)-(3).

Theorem 2: Assume that the following conditions are satisfied:

- the function $\varphi(x)$ has a continuous derivative to the second order in the interval $[0,+\infty)$
- $\varphi(0)=\varphi(\alpha)=0$, $x^p \varphi^{(k)}(x)=O(1)$, $x\to+\infty$, $p>1$, $k=0,1,2$
- $\beta_3 \neq 0$, $\alpha_1 \beta_2 + \beta 1\ \alpha_2 \neq 0$

Then the solution to the problem (1)-(3) can be shown in the form of the loop integral

$$u(x,t) = \frac{1}{\pi i} \int_S \lambda\, y(x,\lambda) e^{\lambda^2 t} d\lambda \quad (23)$$

Here S is an infinitely expandable contour line on the half-plane Reλ>0 . The for enough part of the contour line S located in the first quarter coinsides with

$\lambda_2 = \lambda_1 tg(\frac{\pi}{4} + \delta)$, on the fourth quarter it coinsides with the line

$$\lambda_2 = -\lambda_1 tg(\frac{\pi}{4} + \delta)$$

Proof: From the formula (15) found for the poles of the spectral problem it can be seen that the contour S can be chosen so that the function $y(x,\lambda)$ inside this contour is analytic.

From the formulas (7) and (9) found for the solution of the spectral problem we can easily see that the condition $\lim_{|\lambda|\to\infty} y(x,\lambda) = 0$ inside the contour S is satisfied. Using these two facts and ınverse Laplace integral transform , we can show the inverse of the transform (4) in the form of (23).

We now show that the function $u(x,t)$ determined by the formula (23) is the solution of the problem (1)-(3) within the theorem conditions. By the line $\lambda_1=0$ we denote the contour S^* symmetric with the contour S. We denote the part of the contour S inside the cride O_k by S_k , the part inside the contour S O_k by μ_k. We denote the part of the cirde O_k locased in the first and second quarters and between the contours S and S^* by v_k^+.

Note that curves μ_k^* , S_k^* are symmetric with μ_k and S_k with respect to the straightline $\lambda_1=0$. At the same time, the arc v_k^- is chosen symmetric with the curve v_k^+ with respect to the straightline $\lambda_2=0$. To solve the spectral problem when proving the expansion theorem, it was shown that the inequality

$$|\, y(x,\lambda)\, | \le \frac{M_1}{|\lambda|} \quad (24)$$

is valid

$$\left|\int_{v_k^+} \lambda\, y(x,\lambda) e^{\lambda^2 t} d\lambda\right| \le M_1 \int_{\frac{\pi}{4}+\delta}^{\frac{3\pi}{4}-\delta} e^{r_k cos2\varphi}\, r_k d\varphi = \frac{M_1}{2} \int_{\frac{\pi}{2}+2\delta}^{\frac{3\pi}{2}-2\delta} e^{r_k cos\theta}\, r_k d\theta =$$

$$\frac{1}{2} M_1 \int_{\frac{\pi}{2}+2\delta}^{\pi} r_k e^{r_k cos\theta} d\theta \quad + \quad \frac{1}{2} M_1 \int_{\pi}^{\frac{3\pi}{2}-2\delta} r_k e^{r_k cos\theta} d\theta \quad \le$$

$$\frac{1}{2} M_1 \int_{\frac{\pi}{2}+2\delta}^{\pi} r_k\, e^{r_k(1-\frac{2}{\pi}\theta)} d\theta \quad +$$

$$\frac{1}{2} M_1 \int_{\pi}^{\frac{3\pi}{2}-2\delta} r_k\, e^{r_k(\frac{2}{\pi}\theta-3)} d\theta = -\frac{\pi}{4} M_1\, e^{r_k(1-\frac{2}{\pi}\theta)}\Big|_{\frac{\pi}{2}+2\delta}^{\pi}$$

$$+\frac{\pi}{4} M_1\, e^{r_k(\frac{2}{\pi}\theta-3)}\Big|_{\pi}^{\frac{3\pi}{2}-2\delta} = \quad -\frac{\pi}{4} M_1 \Big(e^{-r_k} -$$

$$e^{-\frac{4\delta}{\pi} r_k}\Big) + \frac{\pi}{4} M_1 (e^{r_k\left(-\frac{4\delta}{\pi}\right)} + e^{-r_k}) = \frac{\pi}{2} M_1 e^{-\frac{4\delta}{\pi} r_k} \xrightarrow[r_k\to\infty]{} 0$$

To estimate the following integral on the arc v_k^+ , we use the equality (24).

$$\left|\int_{v_k^+} \lambda\, y(x,\lambda) e^{\lambda^2 t} d\lambda\right| \le M_1 \int_{\frac{\pi}{4}+\delta}^{\frac{3\pi}{4}-\delta} r_k\, e^{t\, r_k cos2\varphi} d\varphi = \frac{M_1}{2} \int_{\frac{\pi}{2}+2\delta}^{\frac{3\pi}{2}-2\delta} e^{t\, r_k cos\theta}\, r_k d\theta \le$$

$$\frac{M_1}{2} \int_{\frac{\pi}{2}+2\delta}^{\pi} r_k\, e^{t\, r_k(1-\frac{2}{\pi}\theta)} d\theta + \frac{M_1}{2} \int_{\pi}^{\frac{3\pi}{2}-2\delta} r_k\, e^{t\, r_k(\frac{2}{\pi}\theta-3)} d\theta = \frac{\pi M_1}{2} e^{-\frac{4\delta}{\pi} t r_k} \xrightarrow[k\to\infty]{} 0$$

(25)

In the same way we can prove the similar inequality on the arc v_k^- . Using the Cauchy integral theorem, we obtain:

$$\int_{S_k} \lambda\, y(x,\lambda) e^{\lambda^2 t} d\lambda = \int_{\mu_k} \lambda\, y(x,\lambda) e^{\lambda^2 t} dt$$

$$\int_{S_k^*} \lambda\, y(x,\lambda) e^{\lambda^2 t} d\lambda = \int_{\mu_k^*} \lambda\, y(x,\lambda) e^{\lambda^2 t} d\lambda \quad (26)$$

Using the Cauchy integral formula and the relations (25),(26) m we obtain:

$$\frac{1}{2\pi i} \int_S \lambda\, y(x,\lambda) e^{\lambda^2 t} d\lambda + \frac{1}{2\pi i} \int_{S^*} \lambda\, y(x,\lambda) e^{\lambda^2 t} d\lambda = \frac{1}{2\pi i} \sum_{n=0}^{\infty} \int_{C_n} \lambda\, y(x,\lambda) e^{\lambda^2 t} d\lambda$$

$$u(x,t) = \frac{1}{2\pi i} \sum_{n=0}^{\infty} \int_{C_n} \lambda\, y(x,\lambda) e^{\lambda^2 t} d\lambda \quad (27)$$

We now show that the last formula staisfies the equation (1).

$$\frac{\partial u}{\partial t} - p(x) \frac{\partial^2 u}{\partial x^2} = \frac{1}{2\pi i} \sum_{n=0}^{\infty} \int_{C_n} [\lambda^2 y(x,\lambda) - p(x) y''(x,\lambda)] \lambda e^{\lambda^2 t} d\lambda =$$

$$\frac{1}{2\pi i} \sum_{n=0}^{\infty} \int_{C_n} \varphi(x)\, \lambda\, e^{\lambda^2 t} d\lambda = 0$$

Using the expansion theorem, we see that the series (27) satisfied the initial condition (2):

$$u(x,0) = \frac{1}{2\pi i} \sum_{n=0}^{\infty} \int_{C_n} \lambda\, y(x,\lambda) e^{\lambda^2 t}\Big|_{t=0} d\lambda = \varphi(x)$$

Taking into account that the function $y(x,\lambda)$ satisfiess (6), we can easily verify that the function $u(x,t)$ found by the formula (27) satisfies the conditions (3).

Let us substatiate the found formal solution:

$$\left|e^{\lambda^2 t}\right| = e^{t\,Re\lambda^2} = e^{t\,|\lambda^2| cos arg\lambda^2} = e^{t\,|\lambda^2| cos 2 arg\lambda} = e^{t\,|\lambda^2| cos\left(\frac{\pi}{2}+2\delta\right)} = e^{-t\,|\lambda^2| sin 2\delta}$$

since $\delta>0$ as $|\lambda|\to\infty$ the function $e^{\lambda^2 t}$ exponentially converges to zero. This shows that the found solution absolutely and regularly converges, i.e. all the operations we conducted are legal. The theorem is proved [1-7].

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