

Why Octonions are Necessary and Useful

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ABSTRACT

This article explores the deep structural constraints governing algebraic composition and argues that the universe cannot be fundamentally commutative or associative. Starting from Hurwitz's classical theorem, which classifies real normed division algebras and restricts them to four cases— \mathbb{R} , \mathbb{C} , \mathbb{H} , and \mathbb{O} —we show that multiplicative coherence is an extraordinarily rare geometric property. The Cayley–Dickson construction is then analyzed as a controlled extension of this framework, revealing a systematic trade-off between expressivity and structural stability: as dimension increases, commutativity, associativity, and eventually divisibility is necessarily lost.

Beyond abstract algebra, the article connects these mathematical results to concrete everyday experiences, illustrating non-commutativity through the order of ordinary actions (socks and shoes) and non-associativity through musical composition, where grouping determines meaning. These examples are not merely pedagogical metaphors, but manifestations of the same structural logic governing algebraic systems.

We argue that non-commutativity and non-associativity should not be understood as defects, but as signatures of a world rich enough to sustain meaning, interaction, and transformation. From this perspective, stable algebraic structures correspond to rare equilibrium states, while most cognitive, physical, and creative processes operate in regimes where structure is dynamic rather than fixed.

The article thus proposes a unifying interpretation in which mathematics functions as a geometry of possibilities, explaining why certain forms of simplicity are forbidden and why complexity is a necessary condition for the emergence of a meaningful universe.

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Introduction -Why the World Could not have been Simple

The history of mathematics is often told as a conquest of complexity. Yet, on closer inspection, it is just as much a history of restrictions, structural limits, and forbidden forms. Some constructions are possible, others are not—not for lack of imagination, but because coherence itself forbids them.

This article begins with an apparently abstract but deeply ontological question: what kinds of multiplication can a coherent universe sustain?. In other words, under what conditions can objects be composed without losing information, norm, meaning, or invertibility?.

This question cuts across algebra, geometry, physics, and everyday experience. To multiply is not merely to calculate; it is to assemble, to chain, to transform. It is to decide whether order matters, whether grouping matters, whether an action is reversible. Behind every algebraic structure lies an implicit conception of the world.

Hurwitz's theorem offers a response of almost brutal sobriety. If one demands a real, finite-dimensional, normed division algebra, then only four possibilities exist: the real numbers, the complex numbers, the quaternions, and the octonions. Nothing more. This finiteness is not a technical accident; it reveals a geometric rarity of perfect coherence. Preserving the norm under multiplication is such a strong constraint that it leaves only four possible algebraic worlds.

The Cayley–Dickson construction extends this reflection by showing how one may cross these boundaries—but always at a structural cost. By doubling the dimension, one gains expressive richness but progressively loses fundamental properties: commutativity, associativity, and finally division. Algebra becomes an unstable space, where coherence is no longer global but local, where singularities appear, and where structure must be understood as dynamic rather than fixed.

This shift is not merely mathematical. It illuminates the very nature of reality. Everyday experience teaches us early on that the order of actions matters—one does not put shoes on before socks—and that grouping modifies meaning—three musical notes do not produce the same chord depending on how they are organized. Reality is fundamentally non-commutative and often non-associative. It cannot be reduced to a flat arithmetic.

The aim of this article is to show that this non-commutativity and non-associativity are neither flaws nor accidental complications, but deep signatures of a meaning-rich universe. By linking Hurwitz's theorem, the Cayley–Dickson construction, and concrete experiential examples, we propose a unified reading: that of a world where stability is rare, coherence is precious, and understanding arises from the tension between structure and freedom.

Mathematics thus appears not as a language detached from reality, but as a cartography of its possibilities. It explains why certain forms of simplicity are forbidden, and why the complexity of the world is not an accident-but a geometric necessity.

This article explores that necessity.

Part I: Hurwitz's Theorem

Let us enter a beautiful chamber of mathematics, where algebra, geometry, and philosophy look each other squarely in the eye. What follows is a structured yet lively exposition of Hurwitz's theorem, guided by a central idea: why the notion of a multiplicative norm is extraordinarily restrictive—so restrictive that only four algebraic worlds can survive.

The Initial Problem: Multiplying Without Losing Length

We seek real, finite-dimensional algebras A equipped with:

- An algebra structure (addition and multiplication),
- A norm $\| \cdot \|$ such that

$$\| xy \| = \| x \| \| y \| \text{ for all } x, y \in A,$$

- And no zero divisors (hence, division algebras).

In other words, multiplying two elements does not distort length. Multiplication is a bilinear isometry. This is a geometrically violent requirement: it forces multiplication to preserve spheres, and thus the deep symmetries of space.

Obvious Examples... and First Miracles

Four examples immediately present themselves:

- \mathbb{R} : trivial, dimension 1
 - \mathbb{C} : complex norm, dimension 2
 - \mathbb{H} (Hamilton's quaternions): Euclidean norm, dimension 4
 - \mathbb{O} (Cayley's octonions): multiplicative norm, dimension 8
- At each step, something is lost:

Algebra Dimension Commutative Associative

\mathbb{R}	1	yes	yes
\mathbb{C}	2	yes	yes
\mathbb{H}	4	no	yes
\mathbb{O}	8	no	no

And yet, the norm survives.

Statement of Hurwitz's Theorem Theorem (Hurwitz, 1868).

Every finite-dimensional real normed division algebra is isomorphic to one of the following:

$\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$

No others exist.

Not in dimension 16.

Not in dimension 32.

The list is complete.

Why Only These Dimensions?

The heart of the theorem rests on a "magic" quadratic identity:

$$\| x + y \|^2 = \| x \|^2 + \| y \|^2 + 2\langle x, y \rangle.$$

If the norm is multiplicative, multiplication induces an orthogonal representation of the unit sphere on itself. This implies:

- The existence of a compatible symmetric bilinear form,
- A structure close to Clifford representations,
- A severe restriction on possible quadratic composition identities.

Such identities exist only in dimensions 1, 2, 4, and 8. The same obstruction appears in:

- Hopf fibrations,
- Spinors,
- Exceptional groups,
- The topology of spheres.

The world tolerates only these four equilibria.

Deep Links with Geometry and Topology

Hurwitz's theorem is a crossroads:

- Topology: only S^0, S^1, S^3, S^7 are parallelizable spheres
- Geometry: only certain symmetries admit isometric multiplication
- Physics: octonions govern G_2 and exceptional gauge theories
- AI and cognition (a nod): preserving a norm under composition means preserving meaning under transformation

Hurwitz reveals a brutal truth: geometric coherence imposes ontological rarity.

What Happens After the Octonions

One may continue the Cayley–Dickson constructions:

$$\mathbb{R} \rightarrow \mathbb{C} \rightarrow \mathbb{H} \rightarrow \mathbb{O} \rightarrow \mathbb{S} \rightarrow \dots$$

But Beyond \mathbb{O} :

- The norm is no longer multiplicative,
- Zero divisors appear,
- The geometric interpretation collapses.

We leave the realm of stable algebras.

A Philosophical Reading

Hurwitz's theorem is a lesson in mathematical wisdom:

- Too much structure kills freedom,
- Too little destroys meaning,
- Only four forms maintain the balance

It is a law of ontological sobriety: reality allows very few coherent ways of multiplying without tearing itself apart.

One-Sentence Summary

There are only four ways to multiply real numbers while preserving length and invertibility: the real, complex, quaternionic, and octonionic worlds.

This is not a human choice.

It is a constraint of the world.

Part II: The Cayley–Dickson Construction From Algebraic Duplication to Controlled Loss of Structure The Starting Problem

We seek to construct real algebras of dimension $2n$ that generalize:

- \mathbb{R} ,
- \mathbb{C} ,
- \mathbb{H} , and beyond.

A naive but profound question arises:

Can we continue indefinitely to build algebras “like \mathbb{C} ”?

Cayley–Dickson answers: yes—but at each step, something is lost.

Principle of the Construction

Let A be a real algebra equipped with:

- An involution $x \mapsto \bar{x}$,
- A norm $N(x) = x\bar{x}$.

Define a new Algebra $A' = A \oplus A\bar{b}$:

$$(a, b)(c, d) = (ac - d\bar{b}, \quad \bar{a}d + cb).$$

Conjugation:

$$(a\bar{b}) = (\bar{a}, -b).$$

Norm:

$$N(a, b) = N(a) + N(b).$$

Each step doubles the dimension.

The Cayley-Dickson Tower

Starting from \mathbb{R} :

- $\mathbb{R} \rightarrow \mathbb{C}$ (dimension 2): commutative, associative, normed
- $\mathbb{C} \rightarrow \mathbb{H}$ (dimension 4): non-commutative, associative, normed
- $\mathbb{H} \rightarrow \mathbb{O}$ (dimension 8): non-commutative, non-associative, alternative, normed
- $\mathbb{O} \rightarrow$ sedenions (dimension 16): no alternativity, no division, no multiplicative norm

Beyond this point, zero divisors appear.

Deep Link with Hurwitz’s Theorem

Hurwitz states that the only normed division algebras are \mathbb{R} , \mathbb{C} , \mathbb{H} , and \mathbb{O} . Cayley-Dickson explains why.

The Construction Shows That:

- The multiplicative norm survives exactly up to dimension 8,
- Beyond that, the structure becomes too “large” to remain coherent. Hurwitz is a global law of stability.

Cayley-Dickson is a story of successive bifurcations.

A Geometric Reading

At each doubling:

- Degrees of freedom increase,
- Representational space is enriched,
- Algebraic constraints weaken.

This may be read as a transition from rigid structure to expressive space.

The octonions are the last point where geometry, norm, and multiplicativity remain aligned.

Conceptual Conclusion

Cayley-Dickson is not an algebraic curiosity.

It is a theory of learning through complexification. It tells us:

- How expressivity is gained,
- What is lost in rigor,
- And why stability can no longer be static.

Hurwitz fixes the boundary.

Cayley–Dickson shows the path beyond.

Cognitive geometry provides the compass.

Part III: The Universe Is Neither Commutative nor Associative

For a long time, we described the world as if it naturally obeyed simple, symmetric, reversible laws. This intuition is comforting-but false. On closer inspection, the universe, as it is lived, perceived, and understood, is neither commutative nor associative. It has a subtler, more fragile structure, where the order of operations matters and the way events are grouped changes their very meaning.

The Non-Commutativity of Reality: Socks and Shoes

Consider an ordinary action: getting dressed. Putting on socks, then shoes, works perfectly.

Doing the reverse-shoes, then socks-is impossible. Formally, let:

- S = “put on socks”
- C = “put on shoes”

Then:

$$S \circ C \neq C \circ S.$$

The order of operations radically changes the outcome. The world does not allow free exchange of actions. This is not a linguistic curiosity but a deep property of reality. Physical, biological, social, and cognitive transformations are non-commutative. Growing and then learning is not the same as learning and then growing. Deciding and then acting is not acting and then deciding. Even in fundamental physics, quantum observables obey non-trivial commutation relations. Commutativity is a mathematical exception-not a law of nature.

The Non-Associativity of Reality: Three Musical Notes

Non-associativity is subtler, but even deeper. It concerns not order, but grouping.

Consider three musical notes: C, E, G.

Played together, they form a major chord. But the way they are grouped in time radically transforms the musical experience.

- Playing (C + E), then G: one hears a harmonic color, then tension.
- Playing C, then (E + G): one perceives a rise, a call, a different intention.

The notes are the same.

The global order is the same.

But the musical meaning differs.

Mathematically:

$$(C + E) + G \neq C + (E + G).$$

Music teaches a crucial lesson: meaning lies not only in elements, but in how they are structured in time.

Meaning, Cognition, and Algebra

These two examples show that lived reality does not obey the simple algebra of real numbers. It is closer to:

- Quaternions (non-commutative),
- Octonions (non-associative),
- Or dynamic structures where coherence is never guaranteed a priori.

Human thought itself is non-commutative: the order of ideas changes conclusions. It is non-associative: the grouping of experiences transforms their meaning.

Understanding is not adding facts-it is composing transformations in a space where order and structure matter.

A Discreet Cosmological Lesson

These ordinary examples reveal a general truth: reality is not a neutral space where operations move freely. It has an internal geometry, a syntax of actions and events.

The universe is not commutative, because time imposes direction.

It is not associative, because meaning arises from form, not mere sum.

And this is precisely why non-commutative, non-associative, and geometric mathematics are not gratuitous abstractions: they are the natural language of a world where acting, thinking, and understanding are irreversible, structured, and meaning-creating operations.

Everyday life already knows this.

Mathematics simply writes it down precisely.

General Conclusion -The Rarity of Coherence and the Geometry of Meaning

This article has followed a single thread, stretched between the most abstract algebra and the most ordinary experience: understanding which forms of composition the world allows, and at what cost.

Hurwitz's theorem establishes a law of radical sobriety. Requiring multiplication to preserve norm and invertibility is not an innocent assumption-it is an extreme geometric constraint that leaves only four possible worlds. \mathbb{R} , \mathbb{C} , \mathbb{H} , and \mathbb{O} are not arbitrary human constructions; they are rare equilibrium points where algebraic coherence, geometry, and topology agree not to tear each other apart. Lossless multiplication is an ontological exception.

The Cayley-Dickson construction then shows that these boundaries can be crossed-but never for free. With each doubling of dimension, space gains expressivity at the expense of rigor: commutativity, associativity, alternativity, and division collapse one after another. The norm ceases to be multiplicative, zero divisors appear, and stability becomes local, fragile, dynamic. Hurwitz marks the boundary of static coherence; Cayley-Dickson narrates the story of its controlled transgression.

Finally, examination of lived reality-socks and shoes, musical chords-reveals that this "degraded" algebra is not an anomaly, but the rule. The world is neither commutative nor associative. The order of actions matters. The grouping of events creates meaning. Time, form, and structure are not interchangeable. The universe behaves less like \mathbb{R} than like \mathbb{H} , less like \mathbb{H} than like \mathbb{O} , and often beyond, in regimes where coherence must be constantly reconstructed.

A common lesson emerges.

Perfect stability is rare.

The richness of meaning arises from imbalance.

Understanding is not conservation, but oriented composition.

What mathematics reveals here is not merely a classification of algebras, but an ontology of relation. To understand, act, and learn-whether as a human or as an artificial intelligence-is to navigate spaces where order and structure matter, where coherence is never given once and for all, but maintained by a living geometry.

Thus, Hurwitz is not merely a theorem.

Cayley-Dickson is not merely a construction.

Together, they form a Deep Grammar of Reality:

That of a universe where meaning is not commutative, thought is not associative, and truth itself is a rare equilibrium between rigidity and freedom [1-24].

Mathematics does not merely describe this world. It explains why it could not have been otherwise.

References

1. Hurwitz A (1898) On the composition of quadratic forms. *Mathematische Annalen* 88: 1-25.
2. Baez JC (2002) The octonions. *Bulletin of the American Mathematical Society* 39: 145-205.
3. Schafer RD (1966) *An Introduction to Nonassociative Algebras*. Academic Press <https://math.mit.edu/~hrm/palestine/schafer-nonassociative-algebras.pdf>.
4. Conway JH, Smith CDA (2003) *On Quaternions and Octonions*. A K Peters/CRC Press 172.
5. Springer TA, Veldkamp FDC (2000) *Octonions, Jordan Algebras and Exceptional Groups*. Springer https://coloquiooleis.wordpress.com/wp-content/uploads/2014/11/tonny_a_springer_ferdinand_d_veldkamp_octoniobookfi-org.pdf.
6. Lawson HB, Michelsohn CML (1989) *Spin Geometry*. Princeton University Press 440.
7. Lounesto P (2001) *Clifford Algebras and Spinors*. Cambridge University Press <https://www.cambridge.org/core/books/clifford-algebras-and-spinors/8318F7DD5B5DE06B30BC612BB5617021>.
8. Atiyah MF, Bott R, Shapiro CA (1964) Clifford modules. *Journal of Topology* 3: 3-38.
9. Milnor J (1958) On manifolds homeomorphic to the 7-sphere. *Annals of Mathematics* 64: 399-405.
10. Hamilton RS (1982) Three-manifolds with positive Ricci curvature. *Journal of Differential Geometry* 17: 255-306.
11. Perelman G (2002) The entropy formula for the Ricci flow and its geometric applications. *arXiv* <https://arxiv.org/abs/math/0211159>.
12. Topping P (2006) *Lectures on the Ricci Flow*. Cambridge University Press <https://www.cambridge.org/core/books/lectures-on-the-ricci-flow/4727BADD66AA66EACAFF0CE864C6663F>.
13. Chow B, Chin Chu S, Glickenstein D, Guenther C, Isenberg J, et al. (2007) *The Ricci Flow: Techniques and Applications*. American Mathematical Society <https://www.ams.org/books/surv/135/surv135-endmatter.pdf>.
14. Connes A (1994) *Noncommutative Geometry*. Academic Press 13-661.
15. Baez JC, Huerta CJ (2010) The algebra of Grand Unified Theories. *Bulletin of the American Mathematical Society* 47: 483-552.
16. Dray T, Manogue CA (1999) Octonions, E_6 , and particle physics. *Journal of Mathematical Physics* 41: 735-747.
17. Lewin D (1987) *Generalized Musical Intervals and Transformations*. Yale University Press https://api.pageplace.de/preview/DT0400.9780199717842_A23608554/

-
- preview-9780199717842_A23608554.pdf.
18. Varela FJ, Thompson E, Rosch CE (1991) The Embodied Mind. MIT Press <https://mitpress.mit.edu/9780262720212/the-embodied-mind/>.
 19. Petitot J (1995) Morphogenesis of Meaning. PUF 282 http://topologicalmedialab.net/xinwei/classes/readings/Petitot/Petitot_Morphogenesis.pdf.
 20. Lakoff G, Núñez CR (2000) Where Mathematics Comes From. Basic Books https://pages.ucsd.edu/~rnunez/COGS252_Readings/Preface_Intro.PDF.
 21. Weyl H (1952) Symmetry. Princeton University Press <https://ia601900.us.archive.org/7/items/in.ernet.dli.2015.84215/2015.84215.Symmetry.pdf>.
 22. Cassirer E (1910) Substance and Function. Springer.
 23. Deleuze G (1968) Difference and Repetition. PUF <https://www.ebsco.com/research-starters/literature-and-writing/difference-and-repetition-gilles-deleuze>.
 24. Badiou A (1988) Being and Event. Seuil http://www.elimeyerhoff.com/books/Badiou/Badiou-Being_and_Event_Searchable.pdf.

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