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Functional Differential Equations and Stability Analysis

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Introduction

This work investigates Functional Differential Equations, which model systems whose dynamics depend on past or future states. Stability results of solutions are studied. The work also includes analytical and numerical methods, with simulations in Python and MATLAB, and applications to delay models.

Objectives

General Objective

- Advance the theory and practice of FDEs, investigating fundamental properties and behaviors of these equations, developing analytical and numerical techniques for their solution and application.

Specific Objectives:

- Study the stability of solutions of FDEs with time delays;
- Implement numerical algorithms, such as the one-step method and discretization methods, to approximate solutions of FDEs;
- Apply FDEs in modeling biological systems and control systems with delay.

Methodology

The work was based on literature review and mathematical modeling using delay differential equations. Computational simulations were performed in Python to visualize the dynamic behavior of the models.

Functional Differential Equations

Definition

A **Functional Differential Equation** is any equation of the form

$$y'(t) = f(t, y(t), y(t-\tau)).$$

where the derivative of y at t depends on its current value $y(t)$ and the past value $y(t-\tau)$, and τ is the delay.

Classification of FDEs

Categories

Time

- **Retarded:** Depend on past time;
- **Advanced:** Depend on future time.

Order

- **1st order:** contain at most first derivatives;
- **2nd order:** contain at most second derivatives;
- **Higher order:** derivatives are of higher order.

Linearity

- **Linear:** The function and its derivatives appear linearly.
- **Nonlinear:** The functions or derivatives do not appear linearly.

Dependent Variables

- **Single-variable:** Have one dependent variable;
- **Multivariable:** Have more than one dependent variable.

Functional Dependence

- **Autonomous:** Do not depend explicitly on t;
- **Non-autonomous:** Depend explicitly on t.

Example

Consider the following FDE : $x''(t) = x^2(t) - x(t + e^{-t})$

Classification: This FDE is classified as:

- **Regarding Time:** Retarded with variable delay
- **Regarding Order:** 2nd order.
- **Regarding Linearity:** Nonlinear;
- **Regarding Number of dependent variables:** Single-variable;
- **Regarding Functional Dependence:** Non-autonomous.

Banach Fixed Point Theorem

Theorem

The theorem ensures that if a function approximates two points of a space (contraction condition), then there exists a unique fixed point, guaranteeing that the solution exists and is unique (see[1,pp. 7-12]).

Stability of Functional Differential Equations

Lyapunov Stability

It is that in which a given system remains near equilibrium if it starts near equilibrium; if the initial condition is far away, there is no guarantee that the system stays near equilibrium, it may move away from the neighborhood of equilibrium.

Asymptotic Stability
The trivial solution $x(t) \equiv 0$ is said to be *asymptotically stable* if it satisfies two criteria:
- **Lyapunov Stability:** solutions that start close to $x(t) = 0$ remain close to it.
- **Asymptotic Convergence:**

$$\lim_{t\to\infty} \|x(t)\| = 0.$$

Exponential Stability
The solution x(t) ≡ 0 is said to be exponentially stable if there exist constants $M > 0$ and $\lambda > 0$ such that

$$\|x_t\| \leq M\, e^{-\lambda t}\, \|\phi\|, \quad \forall t \geq 0,$$

where

$$\|x_t\| = \sup_{\theta\in[-r,0]} \|x(t+\theta)\|.$$

- $\lambda > 0$ controls how fast the solution converges to equilibrium: the larger λ, the faster the convergence speed.
- $M > 0$ bounds the influence of the initial history, showing how much the past can affect the solution before it stabilizes.

Stability Technique
Lyapunov-Krasovskii Functional
The technique is based on constructing a functional of the form

$$V(x_t) = \frac{1}{2}x(t)^2 + \frac{|b|}{2}\int_{t-\tau}^{t} x(s)^2\, ds$$

and following these steps:

- Check positivity:

$$V(x_t) > 0 \quad \text{for } x_t \neq 0, \quad V(x_t) = 0 \iff x_t \equiv 0$$

- Compute the derivative of the functional $V(x_t)$
- Use Young's inequality $|ab| \leq \frac{a^2}{2} + \frac{b^2}{2}$ to eliminate the cross term: $x(t)\,x(t-\tau)$
- Check the sign of the derivative to conclude the type of stability

Conditions for Stability
- If $\dot{V}(x_t) \leq 0 \implies$ **Lyapunov stability**
- If $\dot{V}(x_t) < 0 \implies$ **asymptotic stability**
- If $\dot{V}(x_t) \leq -\alpha\|x(t)\|^2,\ \alpha > 0 \implies$ **exponential stability**

Example: FDE with Delay
Equation and Lyapunov-Krasovskii Functional
Consider the FDE:

$$x'(t) = -2x(t) - x(t-1)$$

Lyapunov-Krasovskii functional:

$$V(x_t) = \frac{1}{2}x(t)^2 + \frac{1}{2}\int_{t-1}^{t} x(s)^2\, ds$$

Steps to Determine Stability
- **Check Positivity:**

$$V(x_t) > 0 \quad \text{for } x_t \neq 0, \qquad V(x_t) = 0 \iff x_t \equiv 0$$

- **Compute the Derivative:**

$$\dot{V}(x_t) = x(t)x'(t) + \frac{1}{2}\left(x(t)^2 - x(t-1)^2\right)$$

Substituting into the FDE and solving we get:

$$\dot{V}(x_t) = -x(t)^2 < 0$$

Since $\dot{V}(x_t) < 0$ for all $x_t \neq 0$, the solution x(t) ≡ 0 is

Asymptotically Stable

Methods for Solving RDDEs
Step Method
This method is based on the idea of dividing the time interval into steps of length $t = \tau$ and solving the delayed differential equation in each step, using the solution from the previous step to obtain the solutions for the following steps.

Recursive Representation of the Method
Step Method
Let $x_k(t)$ be the solution in the interval $I_k = [k\tau, (k+1)\tau]$.

For all $k \geq 0$:

$$x_k(t) = \begin{cases} \text{Solution of:} \begin{cases} x'(t) = f\left(t, x(t), \phi(t-\tau)\right), & t \in [0,\tau] \\ x(0) = \phi(0) \end{cases} \\ \text{Solution of:} \begin{cases} x'(t) = f\left(t, x(t), x_{k-1}(t-\tau)\right), & t \in I_k \\ x(k\tau) = x_{k-1}(k\tau) \end{cases} \end{cases}$$

Hutchinson Equation (Step Method)
Example
Consider the RDDE

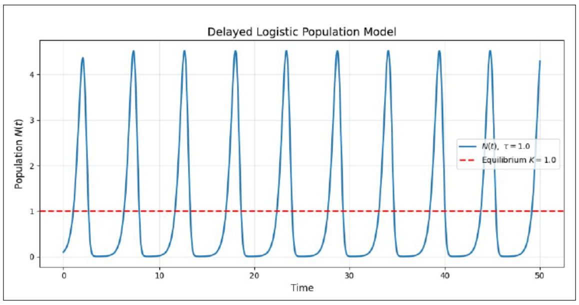
$$\frac{dx(t)}{dt} = r\,x(t)\left(1 - \frac{x(t-\tau)}{K}\right) \tag{1}$$

Model Parameters:
- x(t) — population at time t;
- $r = 2.5$ — intrinsic growth rate;
- $K = 1.0$ — carrying capacity of the environment;
- τ = 1.0 — time delay;
- $\phi(t) = 0.1$ — initial function.

Recursive Solution of the Hutchinson Model
Step Method: Hutchinson Model
The solution of the problem by the step method is given by the recursive form:

$$x_k(t) = \begin{cases} \text{Sol. of:} \begin{cases} \dot{x}(t) = 2.5x(t)\left[1 - \phi(t-1)\right], & t \in [0,1] \\ x(0) = 0,1 \end{cases}, & k = 0 \\ \text{Sol. of:} \begin{cases} \dot{x}(t) = 2.5x(t)\left[1 - x_{k-1}(t-1)\right], & t \in I_k \\ x(k) = x_{k-1}(k) \end{cases}, & k \geq 1 \end{cases}$$

Graphical Simulation of the Solution



Comment

- The growth coefficient is high ($r = 2.5$). In the theory of FDEs, the classical asymptotic stability threshold for this model is given by the product $r\tau = \frac{\pi}{2} \approx 1.57$.
- Since the product $r\tau > 1.57$, the stationary equilibrium point $x^* = 1$ loses its stability. The system fails to converge to a fixed value, giving rise to periodic oscillations.

Taylor Polynomial Expansion Method

Formalization of the Method

This method is based on the Taylor approximation in the interval $I_k = [t_0 + k\tau,\ t_0 + (k+1)\tau]$, with expansion point $t_k = t_0 + k\tau$.

For all $k \geq 0$, we have:

$$x_k(t) = \begin{cases} \displaystyle\sum_{j=0}^{n} \frac{x^{(j)}(t_0)}{j!}(t - t_0)^j, & \text{if } k = 0,\ t \in I_0 \\ \displaystyle\sum_{j=0}^{n} \frac{x^{(j)}(t_k)}{j!}(t - t_k)^j, & \text{if } k \geq 1,\ t \in I_k \end{cases}$$

Derivatives in the Taylor Method

where the derivatives $x^{(j)}(t_k)$ are defined recursively by:

$$x^{(j)}(t_k) = \begin{cases} \phi^{(j)}(t_0), & \text{if } k = 0,\ j = 0 \\ f(t_0,\ \phi(t_0),\ \phi(t_0 - \tau)), & \text{if } k = 0,\ j = 1 \\ \dfrac{d^{j-1}}{dt^{j-1}} f(t,\ x(t),\ \phi(t - \tau))\Big|_{t=t_0}, & \text{if } k = 0,\ j \geq 2 \\ x_{k-1}(t_k), & \text{if } k \geq 1,\ j = 0 \\ f(t_k,\ x_{k-1}(t_k),\ x_{k-1}(t_k - \tau)), & \text{if } k \geq 1,\ j = 1 \\ \dfrac{d^{j-1}}{dt^{j-1}} f(t,\ x(t),\ x_{k-1}(t - \tau))\Big|_{t=t_k}, & \text{if } k \geq 1,\ j \geq 2 \end{cases}$$

Example: Taylor Polynomial Method

Delayed Differential Equation

Consider

$$\begin{cases} x'(t) = -x(t-1) + t^2, & t \in [0, 2], \\ x(t) = 0, & t \in [-1, 0] \end{cases}$$

Interval Division

$$[0, 2] = [0, 1] \cup [1, 2]$$

1st Step: Determine all Derivatives

$$x'(t) = -x(t-1) + t^2$$
$$x''(t) = -x'(t-1) + 2t$$
$$x'''(t) = -x''(t-1) + 2$$
$$x^{(4)}(t) = -x'''(t-1)$$
$$\vdots$$
$$x^{(n)}(t) = -x^{(n-1)}(t-1), \quad n \geq 4$$

2nd Step: Values of the Derivatives at $t_0 = 0$

For t ∈ [−1, 0], we have $x(t) = 0$ and $x'(t) = 0$:

$$x'(0) = x''(0) = \cdots = x^{(n)}(0) = 0, \quad n \neq 3$$
$$x'''(0) = -x''(0) + 2 = 2$$

3rd Step: Build the Approximate Solution on [0, 1]

Taylor polynomial expansion around $t_0 = 0$:

$$x(t) = \frac{x'''(0)}{3!}t^3 = \frac{2}{6}t^3 = \frac{t^3}{3}$$

$$x(t) = \frac{t^3}{3}, \quad t \in [0, 1]$$

4th Step: Values of the Derivatives at $t_0 = 1$

Considering

$$x(1) = \frac{1}{3} \text{ and } x(0) = 0:$$

$$x'(1) = -x(0) + 1^2 = 1$$

$$x''(1) = x'''(1) = 2$$

$$x^{(4)}(1) = -x'''(0) = -2$$

$$x^{(n)}(1) = -x^{(n-1)}(0) = 0, \quad n \geq 5$$

5th Step: Approximate Solution on [1, 2]

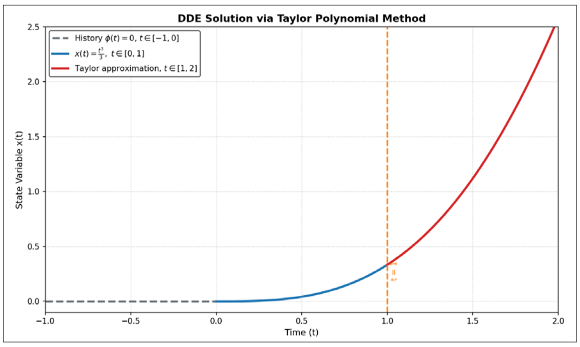
Taylor polynomial around $t_0 = 1$:

$$x(t) = x(1) + \frac{x'(1)}{1!}(t-1) + \frac{x''(1)}{2!}(t-1)^2 + \frac{x'''(1)}{3!}(t-1)^3 + \frac{x^{(4)}(1)}{4!}(t-1)^4$$

$$x(t) = \frac{1}{3} + (t-1) + (t-1)^2 + \frac{1}{3}(t-1)^3 - \frac{1}{12}(t-1)^4$$

General Solution

$$x(t) = \begin{cases} \dfrac{t^3}{3}, & t \in [0, 1], \\ \dfrac{1}{3} + (t-1) + (t-1)^2 + \dfrac{1}{3}(t-1)^3 - \dfrac{1}{12}(t-1)^4, & t \in [1, 2] \end{cases}$$



Graphical Simulation of the Solution

Mathematical Modeling Applied to Gonorrhea

Basic Concepts

- **Gonorrhea:** is a sexually transmitted infection (STI) caused by the bacterium **Neisseria gonorrhoeae**;
- **Incidence**: number of new cases of gonorrhea recorded in a population during a given period;
- **Prevalence**: total number of infected individuals at a given time.

SIS Model with Delay Applied to Gonorrhea

SIS Model with Delay

The SIS model with delay is an epidemiological model that divides the population into two classes: susceptible and infected, assuming that recovered individuals return to the susceptible class.

The Model Describes Two Main Processes:

- **Transmission:** A susceptible individual can become infected by coming into contact with an infected individual.
- **Recovery:** An infected individual recovers and returns to the susceptible class (no permanent immunity).

System of the SIS Model with Delay

$$\begin{cases} \dfrac{dS(t)}{dt} = -\beta S(t)I(t-\tau) + \gamma I(t-\tau), \\ \dfrac{dI(t)}{dt} = \beta S(t)I(t-\tau) - \gamma I(t), \end{cases} \qquad t \geq 0.$$

Parameter Description

- $\beta > 0$: represents the disease transmission rate;
- $\gamma > 0$: represents the recovery rate of infected individuals;
- $\tau > 0$: time delay (incubation period) associated with the time needed for infected individuals to become effectively contagious.

Basic Reproduction Number

$$R_0 = \frac{\beta N}{\gamma}.$$

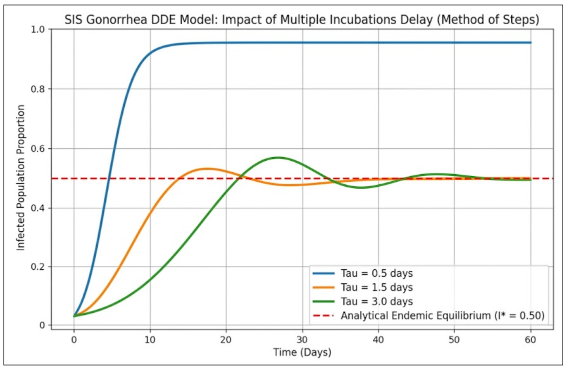
- If $R_0 > 1$, the disease persists in the population, leading to an endemic equilibrium (numbers of infected and susceptible no longer change over time);
- If $R_0 \leq 1$, the disease tends to disappear over time.

Influence of Parameters on the SIS Model with Delay

Parameter Analysis

- Increase in β: Intensifies disease transmission;
- Increase in γ: Accelerates recovery, making disease persistence more difficult;
- Delay τ introduces a memory effect in the SIS model, causing the epidemic evolution to depend on the past state of the system, giving rise to oscillations in the number of infected individuals.

Graphical Simulation of the Model



SIRS Model with Delay

Definition of the SIRS Model

It is an epidemiological model that divides the population into three classes: Susceptible, Infected, and Recovered.

- **Transmission with Delay:** A susceptible individual becomes infected after contact with an infected individual;
- **Recovery (Immunity):** An infected individual recovers acquiring temporary protection against the pathogen;
- **Loss of Immunity:** An individual loses their immunity and returns to the susceptible class.

System of the SIRS Model with Delay

SIRS Model with delay

Assuming a constant total population $N = S(t) + I(t) + R(t)$, the system of Delay Differential Equations is given by:

$$\begin{cases} \dfrac{dS(t)}{dt} = -\beta S(t)I(t-\tau) + \delta R(t), \\ \dfrac{dI(t)}{dt} = \beta S(t)I(t-\tau) - \gamma I(t), \\ \dfrac{dR(t)}{dt} = \gamma I(t) - \delta R(t), \end{cases} \qquad t \geq 0.$$

Parameters of the Model

Parameter Description

- $\beta > 0$: Infection transmission rate;
- $\gamma > 0$: Recovery rate of infected individuals;
- $\delta > 0$: Rate of loss of immunity (where $1/\delta$ is the average protection time);
- $\tau > 0$: Time delay associated with the biological incubation period.

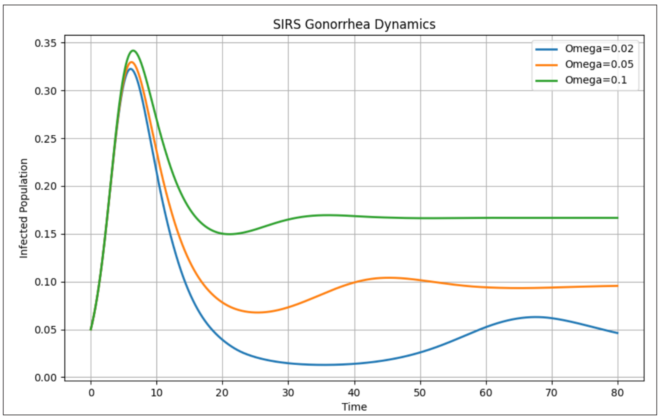
Asymptotic Behavior

If $R_0 \leq 1$: The Disease-Free Equilibrium $E_0 = (N, 0, 0)$ is globally asymptotically stable for any delay $\tau \geq 0$.

If R0 > 1: E0 loses stability and a unique **Endemic Equilibrium** $E^* = (S^*, I^*, R^*)$ appears, where the disease persists:

$$S^* = \frac{\gamma}{\beta}, \qquad I^* = \frac{\delta(R_0 - 1)}{\beta + \frac{\delta\beta}{\gamma}}, \qquad R^* = \frac{\gamma I^*}{\delta}.$$

Graphical Simulation of the Model



Neuroscience Model with Delay

Definition of the Neuronal Model

The model describes the activity dynamics of excitatory and inhibitory neuronal populations, incorporating synaptic delays associated with neural signal propagation.

- **Excitatory Activity:** Represents neurons that increase the firing rate of other neurons;
- **Inhibitory Activity:** Represents neurons that reduce the global activity of the network;
- **Synaptic Interaction:** Communication between neurons occurs with delay due to synaptic transmission time.

System of the Neuronal Model with Delay
Wilson–Cowan Model with Synaptic Delay
Assuming average populations of excitatory $E(t)$ and inhibitory $I(t)$ neurons, the system is given by:

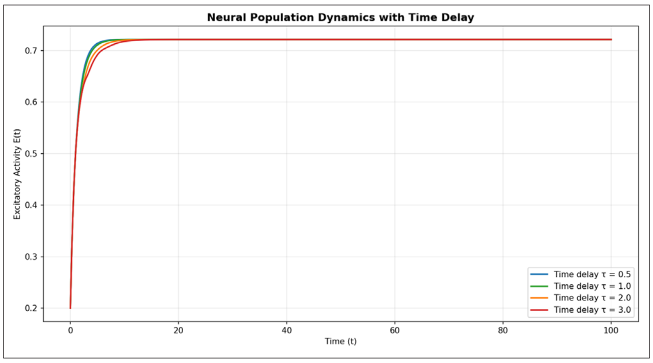
$$\begin{cases} \dfrac{dE(t)}{dt} = -E(t) + S_E\left(w_{EE}E(t-\tau) - w_{EI}I(t-\tau) + P_E\right), \\ \dfrac{dI(t)}{dt} = -I(t) + S_I\left(w_{IE}E(t-\tau) - w_{II}I(t-\tau) + P_I\right), \end{cases} \quad t \geq 0.$$

Parameters of the Model
Parameter Description
- w_{EE} : excitatory connection strength between excitatory neurons;
- w_{EI} : inhibitory influence on excitatory neurons;
- w_{IE} : excitatory influence on inhibitory neurons;
- w_{II} : self-inhibition of the inhibitory population;
- P_E , P_I : external stimuli (sensory or cortical input);
- $\tau > 0$: synaptic delay in neuronal transmission;
- S_E , S_I : sigmoid activation functions (neuronal response).

Dynamic Behavior of the System
Dynamic Regimes
- **Low Excitation:** The system converges to a low neuronal activity steady state;
- **Oscillatory Regime:** For certain parameter values and delay τ , periodic oscillations (neural rhythms) appear;
- **High Excitation:** Persistent activity or pathological states (hyperexcitation) may occur.

Biological Interpretation
The delay τ can induce instabilities that are associated with phenomena such as alpha, beta rhythms and pathological oscillations in epilepsy.

Graphical Simulation of the Model



Mechanical System with PD Control and Delay
Model Description
The system represents a rotational mass with inertia, friction and stiffness, controlled by a proportional-derivative (PD) regulator with time delay.
- **Physical System:** pendulum/rotational mechanism subject to external torque;
- **PD Control:** acts to stabilize the system at $\theta = 0$;
- **Delay τ :** represents processing/sensing time of the controller.

Dynamic Model with Delay
System Equation

$$J\ddot{\theta}(t) + B\dot{\theta}(t) + K\theta(t) = u(t)$$

where the delayed control is

$$u(t) = -K_p\theta(t-\tau) - K_d\dot{\theta}(t-\tau)$$

First-Order System

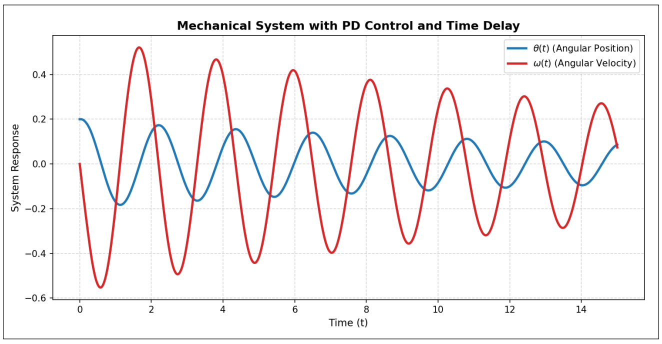
$$\begin{cases} \dot{\theta}(t) = \omega(t), \\ \dot{\omega}(t) = \dfrac{1}{J}\left[-B\omega(t) - K\theta(t) - K_p\theta(t-\tau) - K_d\omega(t-\tau)\right] \end{cases}$$

System Parameters
- J: inertia of the system (resistance to acceleration);
- B: viscous friction coefficient;
- K: spring stiffness (restoration);
- K_p: proportional gain of the controller;
- K_d : derivative gain (artificial damping);
- τ : delay in the control signal.

Dynamic Behavior
Effects of Delay
- $\tau \approx 0$: stable and well-damped system;
- Moderate τ : damped oscillations;
- Large τ : instability and growing oscillations.

Physical Interpretation
The delay in the controller reduces the real-time correction capability, potentially generating dynamic instability.

Graphical Simulation of the Model



Mackey–Glass Model
Definition of the Model
The Mackey–Glass model is a delay differential equation used to describe the dynamics of blood cell production in the body.

- It was developed to model the production of white blood cells;
- Production depends on past values of the system;
- The delay represents the biological response time;
- It can generate stable, periodic or chaotic behaviors.

Dynamic System with Delay
The Mackey–Glass model is described by the equation:

$$\frac{dx(t)}{dt} = \frac{\beta\, x(t-\tau)}{1 + x(t-\tau)^n} - \gamma x(t)$$

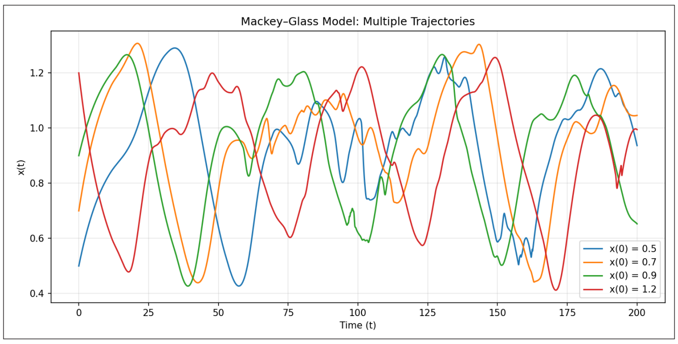
- $x(t)$: amount of cells at time t;
- $x(t-\tau)$: past value of the system;
- The delay represents the biological production time;
- The nonlinear term limits growth.

Parameters of the Model
Parameter meanings
- β: cell production rate;
- γ: cell elimination or death rate;
- n: controls system saturation;
- τ : physiological delay (response time of the organism).

System Behavior
Effects of Delay
- Small delay: stable system;
- Intermediate delay: regular oscillations;
- Large delay: irregular and chaotic behavior.

The delay makes the system depend on the past, potentially generating instability and complex patterns.

Graphical Simulation of the Model



Conclusions and Recommendations
- The work advanced the theory and practice of the study of FDEs;
- Established conditions for existence and uniqueness of solutions;
- Implemented techniques for the study of stability, as well as;
- Presented models applied to retarded functional differential equations;
- For future work, we recommend investigating FDEs where the delay depends on the solution itself, implementing higher-order numerical methods to obtain more accurate solutions, and Hopf bifurcation analysis in stability studies [1-8].

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