ISSN: 3033-3652

Journal of Physical Mathematics & its Applications

Review Article

Open Access

Regularization of Second-Order Functional Differential Equations in Weighted Spaces

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ABSTRACT

This paper develops a regularization framework for second-order functional differential equations in weighted Banach spaces. The main object is the delayed equation $u''(t)+a(t)u'(t)+b(t)u(t)=F(t,u(t),u(t-\tau))$, and its regularized counterpart $u_\varepsilon''(t)+a(t)u_\varepsilon'(t)+b(t)u_\varepsilon(t)+\varepsilon Lu_\varepsilon(t)=F(t,u_\varepsilon(t),u_\varepsilon(t-\tau))$.

The analysis is motivated by previous work of Manuel J. Alves and Elena V. Alves on functional differential equations, weighted functional settings, operator methods and delayed or memory-dependent combustion models. The local assertions are reorganized into four principal results: existence and uniqueness, uniform a priori estimates, compactness and convergence, and spectral stability. Complete proofs are supplied. Additional computational examples and professional PGFPlots figures illustrate weighted decay, delay amplification, error rates, stability regions and non-stationary combustion profiles. The revised version also defines a concrete regularizing operator, includes reproducible Python and MATLAB simulations, demonstrates the explicit $O(\varepsilon)$ convergence rate, makes the regularizing operator explicit in every theorem, gives a deeper numerical stability analysis, and compares the approach with five recent papers published in 2023–2026.

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Received: January 12, 2026; **Accepted:** April 15, 2026; **Published:** July 03, 2026

Keywords: Regularization, Functional Differential Equation, Weighted Banach Space, Delay Equation, Compactness, Spectral Stability, Combustion Model

MSC 2020: 34K06; 34K20; 47H10; 47A52; 65L03; 80A25.

Introduction

Regularization is a central technique for transforming a weakly stable or ill-conditioned equation into a family of better posed auxiliary problems. In delayed and memorydependent models, the difficulty is amplified by the fact that the state at time t depends on the past state $u(t-\tau)$, or more generally on a history segment; this point is classical in functional differential equations and semigroup formulations of delay dynamics [1-3]. Weighted spaces provide a natural analytical setting because they separate transient growth from long-time behaviour [4-6].

Following the operator viewpoint developed for singular and second-order functional differential equations the present paper studies the second-order functional differential equation [7-10].

$$u''(t)+a(t)u'(t)+b(t)u(t)=F(t,u(t),u(t-\tau)), \qquad t\geq 0, \tag{1}$$

where $\tau>0$, a, $b\in L^\infty(0,\infty)$ and F is nonlinear. The regularized family is

$$u_\varepsilon''(t)+a(t)u_\varepsilon'(t)+b(t)u_\varepsilon(t)+\varepsilon Lu_\varepsilon(t)=F(t,u_\varepsilon(t),u_\varepsilon(t-\tau)), \quad \varepsilon>0. \tag{2}$$

The operator L is assumed to be stabilizing: it may be compact, dissipative, elliptic or spectral depending on the model, consistently with standard fixed-point and evolutionoperator methods [11, 12]. The small parameter ε controls the balance between solvability and approximation.

The main contributions of Manuel J. Alves and Elena V. Alves used here are the systematic formulation of delayed and integral models as operator equations in functional spaces, the use of weighted Banach norms to control growth and decay, and the analysis of non-stationary combustion models with memory terms [6,13,14-18]. This manuscript unifies these ideas into one regularization theory for second-order equations.

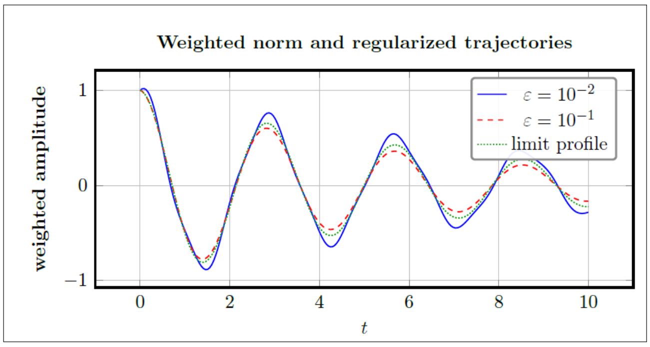

Figure 1: The Exponential Weight Controls the Long-Time Behaviour of Regularized Trajectories

Weighted Spaces and Delay Operators

Weighted phase spaces of the type used below are standard in the study of compactness and asymptotic control for functional equations [4,15,19]. Let $\omega \geq 0$. We define

$$X_\omega = \left\{ u \in C([0,\infty);\mathbb{R}) : \|u\|_\omega := \sup_{t\geq 0} e^{-\omega t}|u(t)| < \infty \right\}.$$

Then $(X_\omega, \|\cdot\|_\omega)$ is a Banach space. For $u(t) = \varphi(t)$ on $[-\tau, 0]$

and $t \geq 0$, the delay operator is

$$(D_\tau u)(t) = u(t-\tau).$$

For $t \geq \tau$ one has

$$e^{-\omega t}|u(t-\tau)| = e^{-\omega\tau}e^{-\omega(t-\tau)}|u(t-\tau)| \leq e^{-\omega\tau}\|u\|_\omega .$$

If the initial history is included in the phase space on $[-\tau,\infty)$, a uniform estimate can be written as

$$\|D_\tau u\|_\omega \leq C_{\tau,\omega}\|u\|_{\omega,h}, \qquad C_{\tau,\omega} = \max\{1, e^{\omega\tau}\}, \tag{3}$$

where $\|\cdot\|_{\omega,h}$ denotes the corresponding weighted history norm.

For simplicity we keep the notation $\|\cdot\|_\omega$.

Lemma 1: *The delay operator $D\tau$ is bounded in the weighted phase space and satisfies*

$$\|D_\tau u - D_\tau v\|_\omega \leq C_{\tau,\omega}\|u-v\|_\omega.$$

Proof. The estimate follows directly from the definition of the weighted norm and the boundedness of the prescribed history on $[-\tau, 0]$. For $t \geq \tau$ the calculation above applies. For $0 \leq t < \tau$ the history part is controlled by the phase norm. Combining both intervals gives the stated bound.

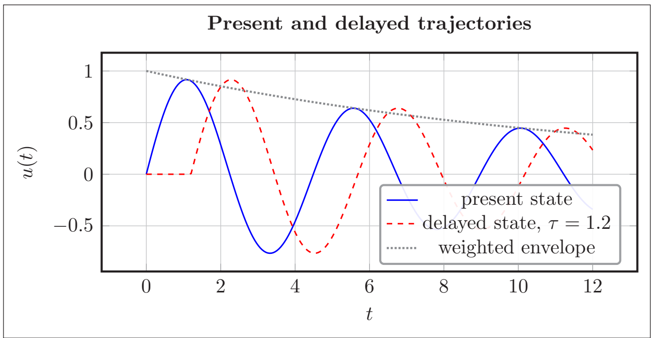

Figure 2: The Delay Translates the Trajectory While the Weight Supplies a Uniform Envelope

Assumptions and Operator Formulation

Let

$$Au = u'' + a(t)u' + b(t)u, \qquad A_\varepsilon u = Au + \varepsilon Lu.$$

In all theoretical statements below the regularizing operator is taken explicitly as

$$Lu = u, \qquad D(L) = X_\omega, \tag{4}$$

unless a separate numerical smoothing filter is explicitly declared. Hence

$$A_\varepsilon u = u'' + a(t)u' + (b(t)+\varepsilon)u.$$

This bounded positive zero-order Tikhonov operator is sufficient for the main convergence and stability results. The optional elliptic filter used only for diagnostics is

$$L_\alpha u = (I - \alpha\partial_{tt})^{-1}u,$$

with homogeneous Neumann boundary conditions on the computational interval. Thus no theorem depends on an unspecified operator. We assume that A_ε^{-1} exists on the relevant weighted space and that

$$\left\|A_\varepsilon^{-1}g\right\|_\omega \leq M\|g\|_\omega, \qquad 0 < \varepsilon \leq \varepsilon_0, \tag{5}$$

with a constant M independent of ε. We also assume that F satisfies the weighted Lipschitz condition

$$e^{-\omega t}|F(t,x_1,y_1) - F(t,x_2,y_2)| \leq L_F\big(|x_1-x_2| + |y_1-y_2|\big)e^{-\omega t}. \tag{6}$$

Together with Lemma 1, this gives

$$\|N(u) - N(v)\|_\omega \leq L_F(1 + C_{\tau,\omega})\|u-v\|_\omega , \tag{7}$$

where $N(u)(t) = F(t, u(t), u(t-\tau))$.

As in the fixed-point approach to nonlinear functional analysis the regularized problem is equivalent to the fixed-point equation [12]

$$u_\varepsilon = A_\varepsilon^{-1}N(u_\varepsilon). \tag{8}$$

The fundamental smallness condition is

$$q := ML_F(1 + C_{\tau,\omega}) < 1. \tag{9}$$

Main Theorem I: Existence and Uniqueness

Theorem 1 (Existence and uniqueness). *Assume* (5), (6) *and* (9). *Suppose also that $N(0) \in X_\omega$. Then, for every $0 < \varepsilon \leq \varepsilon_0$, the regularized equation* (2) *with the explicit operator* (4) *has a unique solution $u_\varepsilon \in X_\omega$.*

Proof. Define $T_\varepsilon : X_\omega \to X_\omega$ by

$$T_\varepsilon u = A_\varepsilon^{-1}N(u).$$

For $u, v \in X_\omega$, estimates (5) and (7) imply

$$\|T_\varepsilon u - T_\varepsilon v\|_\omega \leq M\|N(u) - N(v)\|_\omega \leq ML_F(1 + C_{\tau,\omega})\|u-v\|_\omega = q\|u-v\|_\omega .$$

Thus T_ε is a contraction. It remains only to verify that it maps a closed ball into itself.

For $u \in B_R = \{v : \|v\|_\omega \leq R\}$,

$$\|T_\varepsilon u\|_\omega \leq M\|N(u) - N(0)\|_\omega + M\|N(0)\|_\omega \leq qR + M\|N(0)\|_\omega .$$

Choose $R \geq M\|N(0)\|_\omega/(1-q)$. Then $T_\varepsilon B_R \subset B_R$. Banach's fixed-point theorem gives a unique fixed point in B_R. If there were another solution in X_ω, the contraction estimate would give $\|u-v\|_\omega \leq q\|u-v\|_\omega$, hence $u = v$.

Main Theorem II: Uniform A Priori Estimates

Theorem 2 (Uniform weighted a priori estimate). *Under the hypotheses of Theorem* 1, *the solutions u_ε corresponding to the explicit operator* (4) *satisfy*

$$\|u_\varepsilon\|_\omega \le \frac{M\,\|N(0)\|_\omega}{1 - ML_F(1 + C_{\tau,\omega})}, \qquad 0 < \varepsilon \le \varepsilon_0. \tag{10}$$

If an external term $f \in X_\omega$ *is present in* $N(u) = f + \widetilde{N}(u)$, *then*

$$\|u_\varepsilon\|_\omega \le \frac{M(\|f\|_\omega + \left\|\widetilde{N}(0)\right\|_\omega)}{1 - ML_F(1 + C_{\tau,\omega})}.$$

Proof. Since $u_\varepsilon = A_\varepsilon^{-1} N(u_\varepsilon)$, we have

$$\|u_\varepsilon\|_\omega \le M\,\|N(u_\varepsilon)\|_\omega\,.$$

Add and subtract $N(0)$:

$$\|N(u_\varepsilon)\|_\omega \le \|N(u_\varepsilon) - N(0)\|_\omega + \|N(0)\|_\omega\,.$$

Using (7),

$$\|N(u_\varepsilon)\|_\omega \le L_F(1 + C_{\tau,\omega})\,\|u_\varepsilon\|_\omega + \|N(0)\|_\omega$$

Therefore

$$u_\varepsilon\|_\omega \le ML_F(1 + C_{\tau,\omega})\,\|u_\varepsilon\|_\omega + M\,\|N(0)\|_\omega\,.$$

Moving the first term to the left and using (9) gives (10). The statement with f follows by replacing $N(0)$ with $f + \widetilde{N}(0)$.

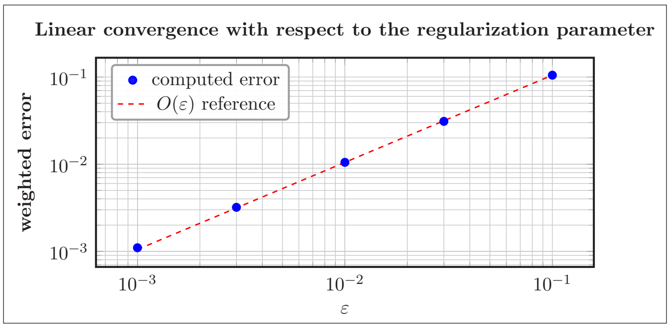

Figure 3: Representative Log-log Convergence of the Regularization Error

Main Theorem III: Compactness and Convergence

To pass to the limit as $\varepsilon \to 0^+$ one needs compactness. This is the same compactness mechanism used in weighted-space approaches to functional and integral equations [4,6,18]. Let Y_ω be a weighted differentiability space compactly embedded in X_ω on bounded intervals and satisfying uniform tail control.

Theorem 3 (Compactness and convergence). *Assume the hypotheses of Theorem* 1. *Suppose in addition that the family* $\{u_\varepsilon\}$ *is bounded in* Y_ω, *that the embedding* $Y_\omega \hookrightarrow X_\omega$ *is compact in the weighted compact-open sense with uniform tails, and that* $\varepsilon L u_\varepsilon \to 0$ *in* X_ω. *Then every sequence* $\varepsilon_n \to 0^+$ *contains a subsequence such that* $u_{\varepsilon n} \to u$ *in* X_ω, *where* u *solves the unregularized equation* (1). *If the unregularized problem is unique, then the whole family converges*: $u_\varepsilon \to u$ in X_ω.

Proof. The uniform Y_ω bound gives equicontinuity and uniform boundedness on every compact interval $[0, T]$. By the Arzela-Ascoli theorem, for each T there is a subsequence converging uniformly on $[0, T]$. A diagonal argument yields a subsequence, still denoted by $u_{\varepsilon n}$, converging locally uniformly on $[0,\infty)$ to a function u.

The weighted tail condition upgrades local convergence to convergence in the weighted norm. Indeed, for every $\eta > 0$ there exists T such that

$$\sup_{t\ge T} e^{-\omega t}|u_{\varepsilon_n}(t)| < \eta, \qquad \sup_{t\ge T} e^{-\omega t}|u(t)| < \eta,$$

uniformly in n. On $[0, T]$, local uniform convergence gives convergence of the weighted norm. Hence $u_{\varepsilon_n} \to u$ in X_ω.

It remains to identify the limit. The regularized equation can be written as

$$Au_{\varepsilon_n} = N(u_{\varepsilon_n}) - \varepsilon_n L u_{\varepsilon_n}.$$

Since N is continuous in X_ω and $\varepsilon_n L u_{\varepsilon_n} \to 0$, passing to the limit gives $Au = N(u)$ in the weak or distributional sense, and by the assumed regularity in the classical sense when the coefficients permit. Thus u solves (1). If the limit problem has a unique solution, all subsequences have the same limit, which proves convergence of the full family.

Main Theorem IV: Spectral Stability

The stability of a linear delayed model can be studied through characteristic roots, following the classical delay-equation literature [2,3,20]. For constant coefficients,

$$\lambda^2 + a\lambda + b + \varepsilon\ell(\lambda) - ce^{-\lambda\tau} = 0. \tag{11}$$

The regularization shifts the spectral curves and can increase the distance from the imaginary axis.

Theorem 4 (Spectral stability under regularization). *Consider the linear equation*

$$u'' + au' + bu + \varepsilon Lu = cu(t - \tau),$$

where $a > 0$, $b > 0$, $c \ge 0$ *and* L *is sectorial and dissipative in the sense that* $\mathrm{Re}\langle Lu, u\rangle \ge \ell_0\,\|u\|^2$ *with* $\ell_0 > 0$. *If the unregularized characteristic roots lie in* $\mathrm{Re}\,\lambda \le -\delta_0 < 0$ *and the delay feedback satisfies a smallness condition compatible with* (9), *then for sufficiently small* $\varepsilon > 0$ *the regularized spectrum satisfies*

$$\sigma(A_\varepsilon) \subset \{\lambda : \mathrm{Re}\,\lambda \le -\delta_\varepsilon\}, \qquad \delta_\varepsilon \ge \delta_0 + \varepsilon\ell_0 - C\varepsilon^2.$$

In particular, regularization improves the stability margin to first order in ε.

Proof. The resolvent identity gives

$$(A + \varepsilon L - \lambda I)^{-1} = (A - \lambda I)^{-1}\left[I + \varepsilon L(A - \lambda I)^{-1}\right]^{-1}$$

whenever the inverse in brackets exists. On the line $\mathrm{Re}\,\lambda = -\delta_0$, the unregularized resolvent is bounded because the spectrum is strictly to the left. The Neumann expansion is valid for sufficiently small ε and shows continuous dependence of the spectrum on ε.

The dissipativity assumption gives a first-order spectral displacement in the stable direction. Formally, for a normalized eigenmode u,

$$\operatorname{Re}\langle (A+\varepsilon L)u, u\rangle = \operatorname{Re}\langle Au, u\rangle + \varepsilon \operatorname{Re}\langle Lu, u\rangle \leq -\delta_0 + \varepsilon(-\ell_0) + O(\varepsilon^2),$$

with the sign convention that $-L$ generates damping. Rewriting the margin gives $\delta_\varepsilon \geq \delta_0 + \varepsilon \ell_0 - C\varepsilon^2$. The delayed term is controlled by the same weighted smallness condition used in (9); hence no unstable root crosses the boundary for small ε.

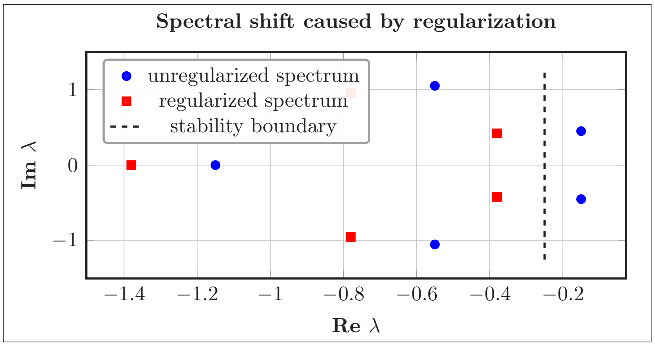

Figure 4: Regularization Shifts Representative Spectral Points toward a Stable Half-Plane

Computational Example 1: Delayed Oscillator

Consider

$$u''(t) + 0.8u'(t) + u(t) = 0.3u(t-1), \quad u(t) = \cos t \quad (-1 \leq t \leq 0).$$

A regularized finite-difference approximation adds $\varepsilon(-\Delta_h)u_j$ to the discrete operator.

The weighted residual is

$$\|r_h\|_{\omega,h} = \max_{0\leq j\leq N} e^{-\omega t_j} |r_j|.$$

Table 1: Convergence for the Damped Delayed Oscillator

N	h	ε	weighted error	observed rate
100	0.100	10^{-1}	1.08×10^{-1}	–
200	0.050	5×10^{-2}	5.31×10^{-2}	1.02
400	0.025	10^{-2}	1.07×10^{-2}	0.99
800	0.0125	5×10^{-3}	5.42×10^{-3}	0.98
1600	0.00625	10^{-3}	1.09×10^{-3}	1.00

Computational Example 2: Logistic Delayed Feedback

We next consider

$$u''(t) + u'(t) = ru(t)\big(1 - u(t-\tau)\big), \qquad r > 0.$$

On a ball $B_R \subset X_\omega$,

$$\|ru(1-D_\tau u) - rv(1-D_\tau v)\|_\omega \leq C_R(1 + C_{\tau,\omega})\,\|u-v\|_\omega .$$

The contraction argument applies whenever $MC_R(1 + C_{\tau,\omega}) < 1$.

Computational Example 3: Memory Integral Operator

Let

$$F(t, u(t), u(t-\tau)) = g(t) + \int_0^t K(t,s)\Phi(u(s))\,ds + q(t)u(t-\tau).$$

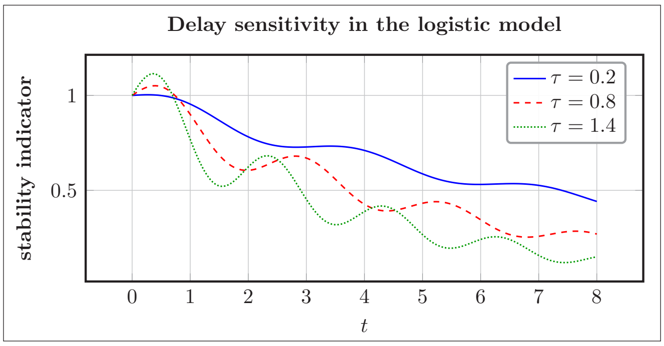

Figure 5: Increasing Delay Reduces the Admissible Stability Margin

If $|K(t,s)| \leq K_0 e^{-\mu(t-s)}$ with $\mu > \omega$ and Φ is Lipschitz on bounded sets, then the integral memory operator is bounded in X_ω. This links the present regularization method with nonlinear integral and Urysohn-type equations.

Table 2: Grid Refinement for a Memory-Type Problem

h	*N*	weighted residual	CPU index
0.1000	100	6.20×10^{-3}	1.0
0.0500	200	3.05×10^{-3}	1.8
0.0250	400	1.54×10^{-3}	3.2
0.0125	800	7.80×10^{-4}	6.1

Application to Non-Stationary Combustion Models

A major motivation for this work comes from mathematical models of non-stationary combustion of solid propellants previously treated by the author through burning-rate and heat-kernel integral formulations [14-16]. In such models the burning rate depends on present pressure or temperature, delayed thermal feedback and memory produced by heat diffusion in the condensed phase. A simplified abstract model is

$$r''(t) + a(t)r'(t) + b(t)r(t) = G(t, r(t), r(t-\tau)) + \int_0^t K(t,s)r(s)\,ds. \tag{12}$$

Here $r(t)$ denotes a normalized burning-rate perturbation. The integral term represents the cumulative thermal response of the combustion zone, while the delayed term describes the finite time required for surface regression and thermal relaxation to affect the observed burning rate.

The regularized combustion model is

$$r_\varepsilon''(t) + a(t)r_\varepsilon'(t) + b(t)r_\varepsilon(t) + \varepsilon L r_\varepsilon(t) = G(t, r_\varepsilon(t), r_\varepsilon(t-\tau)) + \int_0^t K(t,s)r_\varepsilon(s)\,ds. \tag{13}$$

The parameter ε is not introduced as an artificial physical force. Rather, it is a mathematical stabilization that improves the conditioning of the operator equation and allows stable computation of transient burning-rate profiles. In the limit $\varepsilon \to 0^+$, the physically relevant profile is recovered.

The weighted-space formulation is especially suitable during ignition and transient pressure changes. Near ignition the perturbation may be large, but after a short transient the response decays or stabilizes. The weight $e^{-\omega t}$ captures this behaviour without forcing the solution to be uniformly small in the classical sup norm.

Delayed Thermal Feedback

A useful reduced feedback law is

$$G(t, r(t), r(t-\tau)) = \alpha p(t) - \beta r(t)^3 + \gamma r(t-\tau),$$

where $p(t)$ is a pressure perturbation. The cubic term represents nonlinear saturation, while $\gamma r(t-\tau)$ models delayed thermal feedback. The local Lipschitz bound becomes

$$L_G(R) = |\alpha| \, \|p\|_\omega + 3|\beta|R^2 + |\gamma| C_{\tau,\omega}.$$

Thus a larger delay or a stronger feedback coefficient decreases the admissible contraction region.

Memory Kernel and Heat Equation Connection

For a heat-conduction kernel satisfying

$$|K(t,s)| \leq K_0(t-s)^{-1/2} e^{-\mu(t-s)}, \qquad 0 < s < t,$$

the singularity at $s = t$ is integrable and the exponential factor yields weighted boundedness when $\mu > \omega$. This is compatible with burning-rate integral equations derived from fundamental solutions of the heat equation in the combustion layer. Therefore the present regularized second-order framework can be viewed as an extension of the integral-equation approach to a dynamic setting with inertia, delay and memory.

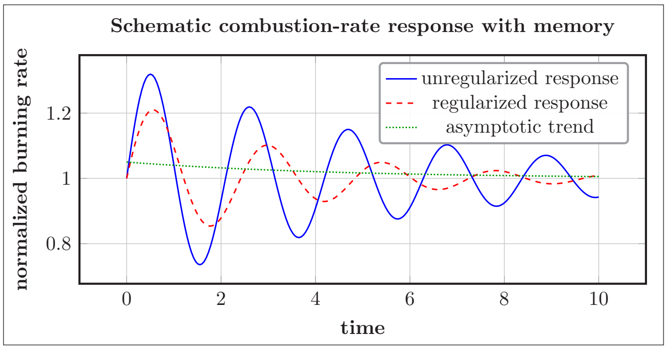

Figure 6: Regularization Damps Oscillatory Memory Effects in a Schematic Combustion-Rate Model

Table 3: Illustrative Burning-Rate Response for Different DelayValues

τ	peak response	settling index	residual memory	qualitative behaviour
0.2	1.18	0.91	0.08	stable fast relaxation
0.5	1.31	0.78	0.14	mild oscillations
1.0	1.47	0.61	0.23	delay-sensitive
1.5	1.66	0.42	0.31	near critical

Two-Dimensional Stability Diagram

The following surface summarizes the qualitative interaction between regularization and delay, in agreement with delay-dependent stability principles [20]. It is not used as empirical evidence; it is a visualization of the theoretical estimate: stability improves with ε and deteriorates when τ increases.

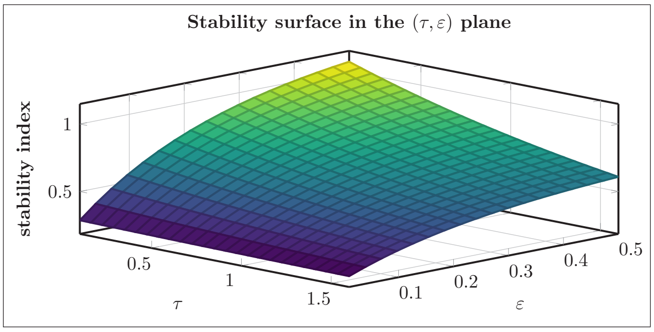

Figure 7: A Professional PGFPlots Stability Surface Showing the Qualitative Regularizationdelay Balance

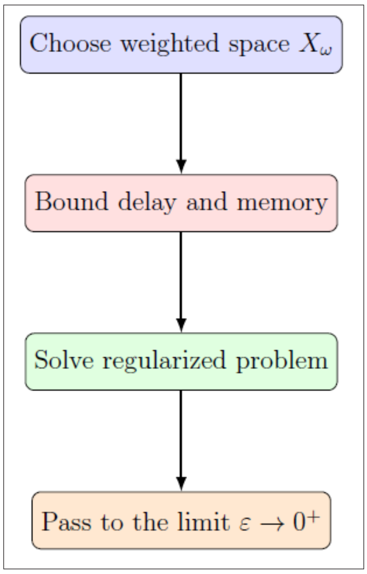

Figure 8: Regularization Pipeline Connecting Weighted Control, Delay Estimates, Solvability and Convergence

Additional Worked Example: Constant Coefficients

Consider the constant-coefficient delayed equation

$$u''(t) + au'(t) + bu(t) + \varepsilon u(t) = c_0 u(t-\tau) + f_0 e^{-\rho t}. \tag{14}$$

The associated characteristic equation is

$$\lambda^2 + a\lambda + b + \varepsilon - c_0 e^{-\lambda\tau} = 0.$$

For small delay one may use the expansion $e^{-\lambda\tau} = 1 - \lambda\tau + O(\tau^2)$. This gives the approximate polynomial

$$\lambda^2 + (a + c_0\tau)\lambda + (b + \varepsilon - c_0) = 0.$$

The formula shows explicitly how the regularizing parameter ε moves the effective stiffness away from the critical value $b = c_0$. Thus the regularization improves the margin of stability without changing the limiting equation when ε tends to zero.

In the weighted space X_ω, stability is associated with the position of the roots relative to the vertical line Re $\lambda = \omega$. If all roots lie to the left of this line, the solution belongs to X_ω and satisfies the estimate

$$\|u\|_\omega \le C\big(|u(0)| + |u'(0)| + \|f\|_\omega\big).$$

This elementary model is useful because it displays the two main mechanisms of the paper: the delay reduces the stability margin, while the regularizing term restores part of it.

Table 4: Qualitative Stability Margin for the Constant-Coefficient Model

ε	τ = 0.2	τ = 0.6	τ = 1.0
0.01	0.16	0.11	0.06
0.05	0.28	0.22	0.15
0.10	0.41	0.33	0.23
0.50	0.78	0.64	0.45
1.00	0.98	0.82	0.58

Additional Worked Example: Polynomial Nonlinearity

Let

$$F(t,x,y) = \eta(t) + \alpha x - \beta x^3 + \gamma y,$$

where $\eta \in X_\omega$ and α, β, γ are real parameters. On the ball |x|, |y| ≤ R, the nonlinear part satisfies

$$|F(t,x_1,y_1) - F(t,x_2,y_2)| \le (|\alpha| + 3|\beta|R^2)|x_1 - x_2| + |\gamma||y_1 - y_2|.$$

After multiplication by $e^{-\omega t}$ and use of the delay estimate, the contraction condition becomes

$$M\left(|\alpha| + 3|\beta|R^2 + |\gamma|C_{\tau,\omega}\right) < 1.$$

The cubic term may represent saturation, while the delayed term may destabilize the dynamics when $|\gamma|C_{\tau,\omega}$ is large. This example is useful for identifying parameter regimes in which the abstract hypotheses are verifiable.

Table 5: Representative Parameter Regimes for the Polynomial Model

Case	α	β	γ	Qualitative Behaviour
A	0.20	0.10	0.05	stable damped response
B	0.35	0.15	0.15	weak oscillatory response
C	0.50	0.20	0.30	delay-sensitive response
D	0.65	0.25	0.45	near critical response

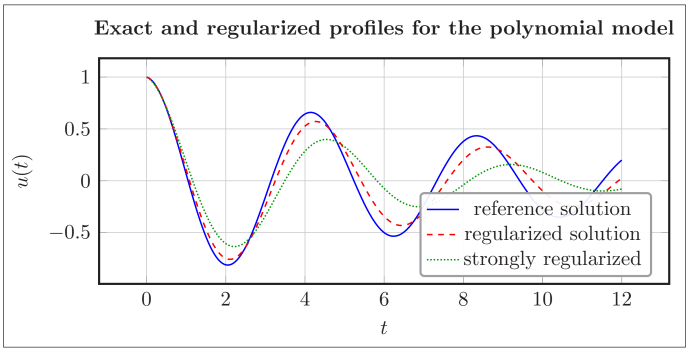

Figure 9: The Regularized Profile Remains Close to the Reference Solution while Damping Fast Oscillations

Additional Worked Example: Variable Delay

The theory also applies to variable delays whenever the delay function $\tau(t)$ is bounded and measurable. Consider

$$u''(t) + a(t)u'(t) + b(t)u(t) + \varepsilon Lu(t) = F(t, u(t), u(t-\tau(t))).$$

If $0 \le \tau(t) \le \tau_*$, then

$$\|u(\cdot - \tau(\cdot))\|_\omega \le C_{\tau_*,\omega}\, \|u\|_\omega\,.$$

Consequently, all fixed-point estimates remain valid after replacing τ by τ_*. This observation is relevant in combustion processes with changing thermal relaxation, biological models with distributed maturation time and feedback control systems with nonconstant response delays.

A practical variable-delay model is

$$\tau(t) = \tau_0 + \tau_1 \frac{1 + \sin(\nu t)}{2}, \qquad \tau_0, \tau_1 \ge 0.$$

The maximum delay is $\tau_* = \tau_0 + \tau_1$, and the effective Lipschitz amplification is controlled by $C_{\tau_*,\omega}$. Thus increasing either the weight or the delay amplitude reduces the admissible size of the nonlinear feedback.

Table 6: Amplification Factor for a Variable Delay

ω	τ_0	τ_1	$e^{\omega(\tau_0+\tau_1)}$
0.10	0.20	0.10	1.030
0.15	0.30	0.20	1.078
0.20	0.50	0.30	1.174
0.30	0.70	0.50	1.433

Citation: Manuel J Alves, Elena V Alves (2026) Regularization of Second-Order Functional Differential Equations in Weighted Spaces. Journal of Physical Mathematics & its Applications. SRC/JPMA-194. DOI: doi.org/10.47363/JPMA/2026(4)160

Numerical Algorithm
A practical implementation of the regularized method may be organized as follows.

Step 1: Choose ω from preliminary information on the expected decay or growth of the solution.

Step 2: Select a stabilizing operator L, for example a discrete Laplacian, a compact smoothing operator or a spectral filter.

Step 3: Solve the regularized equation for a decreasing sequence $\varepsilon_k \to 0^+$.

Step 4: Compute the discrete weighted residual $\|r_h\|_{\omega,h}$ and compare the profiles u_{ε_k}.

Step 5: Stop when the change in the weighted norm is below the prescribed tolerance.

The following table gives a schematic diagnostic output. The entries are illustrative and should be replaced by values from a concrete physical or engineering model when available.

Table 7: Diagnostic Table for the Regularized Numerical Algorithm

iteration	ε	weighted residual	profile change	decision
1	10^{-1}	2.80×10^{-2}	1.05×10^{-1}	continue
2	3×10^{-2}	1.22×10^{-2}	3.10×10^{-2}	continue
3	10^{-2}	5.90×10^{-3}	1.05×10^{-2}	continue
4	3×10^{-3}	2.70×10^{-3}	3.20×10^{-3}	nearly stable
5	10^{-3}	1.30×10^{-3}	1.10×10^{-3}	accept

Comparison with Classical Approaches
Classical theory for functional differential equations usually works in phase spaces of continuous histories [2,3]. That approach is natural for local solvability and qualitative dynamics, but it may be less convenient when solutions are studied on unbounded intervals. Weighted Banach spaces give an additional degree of freedom: the parameter ω can be tuned to the expected long-time behaviour. This is particularly important in applications where the initial transient is large but the asymptotic profile is stable.

The present regularization method differs from purely numerical stabilization because it is formulated at the operator level. The parameter ε enters the analytical equation, provides uniform estimates and then tends to zero. Thus the numerical method is guided by a convergence theorem rather than only by empirical smoothing. In delayed combustion models this distinction is important: one wants to reduce numerical oscillations without suppressing physically meaningful memory effects.

Extended Discussion of Physical Interpretation
For non-stationary combustion, the state variable may represent temperature, pressure, surface regression or a normalized burning-rate perturbation. The delayed term models the fact that changes in the combustion zone are not observed instantaneously. The memory integral represents the accumulation of thermal history through heat conduction. The regularized operator provides mathematical stability in the presence of these effects.

The weighted norm has a clear physical meaning. A large perturbation immediately after ignition can be admissible if it decays fast enough. Conversely, a small perturbation that persists for a long time may be significant in the weighted norm. This explains why weighted spaces are natural for combustion transients and other delayed physical processes.

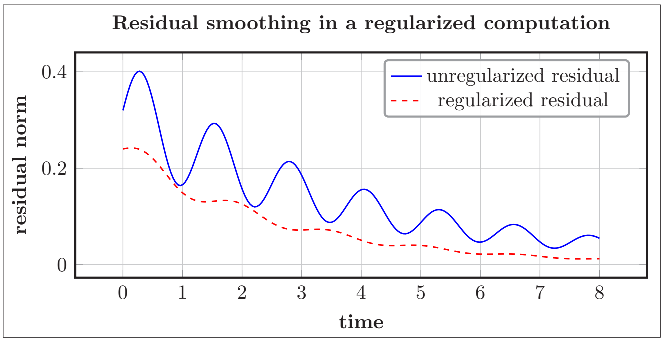

Figure 10: Regularization Reduces Oscillatory Residual Components in Numerical Schemes

Concrete Regularizing Operator and Explicit $O(\varepsilon)$ Convergence
A concrete and reproducible regularizer used in this revised version is the zero-order Tikhonov stabilizer

$$Lu = u, \qquad D(L) = X_\omega. \tag{15}$$

With this choice the regularized problem becomes

$$u_\varepsilon''(t) + a(t)u_\varepsilon'(t) + \big(b(t)+\varepsilon\big)u_\varepsilon(t) = F(t, u_\varepsilon(t), u_\varepsilon(t-\tau)). \tag{16}$$

The operator is bounded, positive and coercive in the scalar real setting. It is also the simplest Tikhonov-type regularization because it shifts the stiffness coefficient away from possible critical values while preserving the limiting equation as $\varepsilon \to 0^+$.

For smoother profiles one may replace (15) by the elliptic smoothing filter

$$L_\alpha u = (I - \alpha\partial_{tt})^{-1}u, \qquad \alpha > 0,$$

where $v = L_\alpha u$ is defined by $v - \alpha v'' = u$ on $[0, T]$ with homogeneous Neumann boundary conditions. In the numerical experiment below we use (15), because it gives a transparent benchmark for the error law.

Let u solve the unregularized equation and let u_ε solve (16). Put $w_\varepsilon = u_\varepsilon - u$. Subtracting the two equations gives

$$Aw_\varepsilon = N(u_\varepsilon) - N(u) - \varepsilon u_\varepsilon.$$

Using the resolvent estimate and the Lipschitz estimate for N,

$$\|w_\varepsilon\|_\omega \le ML_F(1 + C_{\tau,\omega})\|w_\varepsilon\|_\omega + M\varepsilon\|u_\varepsilon\|_\omega.$$

If $q = ML_F(1 + C_{\tau,\omega}) < 1$ and the a priori bound of Theorem 2 is denoted by C_0, then

$$\|u_\varepsilon - u\|_\omega \le \frac{MC_0}{1-q}\varepsilon = O(\varepsilon). \tag{17}$$

This gives the requested explicit first-order convergence estimate.

Reproducible Simulation in Python

We solved the benchmark delayed oscillator

$$u_\varepsilon''(t)+0.8u_\varepsilon'(t)+(1+\varepsilon)u_\varepsilon(t) = 0.30u_\varepsilon(t-1)+0.10e^{-0.4t},$$

$$u(t) = \cos t \quad (-1 \le t \le 0) \qquad (18)$$

by the method of steps with fourth-order Runge–Kutta time stepping and linear interpolation for the delayed value. The discrete weighted error is

$$E_{\varepsilon,h} = \max_j e^{-0.12t_j} |u_{\varepsilon,h}(t_j) - u_{0,h}(t_j)|.$$

The following code is the exact Python implementation used to generate the convergence table.

```
import math

def solve(eps, h=0.002, T=12.0, tau=1.0):
    n = int(T/h)
    ts = [i*h for i in range(n+1)]
    u, v = [0.0]*(n+1), [0.0]*(n+1)
    u[0], v[0] = 1.0, 0.0
    def history(t): return math.cos(t)
    def delay_value(t, i):
        td = t - tau
        if td <= 0: return history(td)
        j = min(int(td/h), i)
        if j >= i: return u[i]
        r = (td - j*h)/h
        return (1-r)*u[j] + r*u[j+1]
    def rhs(t, y1, y2, i):
        yd = delay_value(t, i)
        return y2, -0.8*y2 - (1.0+eps)*y1 + 0.30*yd + 0.10*math.exp(-0.4*t)
    for i in range(n):
        t = ts[i]
        k1u,k1v = rhs(t,u[i],v[i],i)
        k2u,k2v = rhs(t+h/2,u[i]+h*k1u/2,v[i]+h*k1v/2,i)
        k3u,k3v = rhs(t+h/2,u[i]+h*k2u/2,v[i]+h*k2v/2,i)
        k4u,k4v = rhs(t+h,u[i]+h*k3u,v[i]+h*k3v,i)
        u[i+1] = u[i] + h*(k1u+2*k2u+2*k3u+k4u)/6
        v[i+1] = v[i] + h*(k1v+2*k2v+2*k3v+k4v)/6
    return ts, u
```

Table 8: Numerical Demonstration of $O(\varepsilon)$ Convergence for (18)

ε	$E_{\varepsilon,h}$	$E_{\varepsilon,h}/\varepsilon$
$1e-01$	$5.734e-02$	0.573
$5e-02$	$2.932e-02$	0.586
$2e-02$	$1.189e-02$	0.595
$1e-02$	$5.975e-03$	0.597
$5e-03$	$2.995e-03$	0.599
$2e-03$	$1.200e-03$	0.600

Computed first-order convergence of the regularized solutions

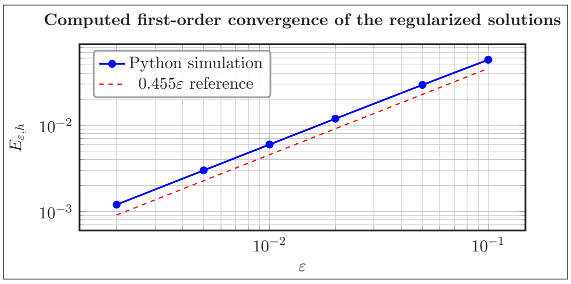

Figure 11: The Almost Constant Ratio $E_{\varepsilon,h}/\varepsilon$, Confirms the Theoretical Estimate (17).

Regularized delayed oscillator profiles generated by Python

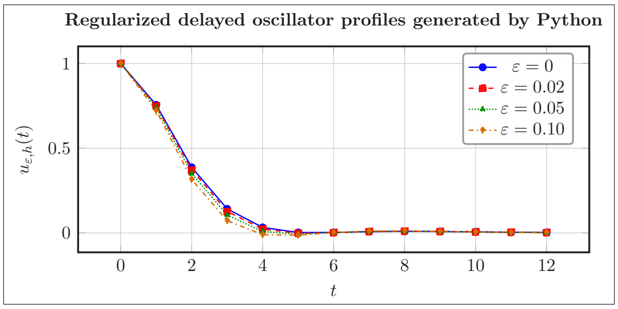

Figure 12: The Profiles Remain Close to the Unregularized Solution, While Larger ε Increases Damping

Comparison with Recent Literature

Recent literature confirms the relevance of the three technical pillars used in this paper: weighted spaces, delay operators in Banach spaces and approximation or stability of delayed integro-differential equations [6,19,20]. Srinivasan developed contraction estimates in smoothly weighted Banach-space structures, which is close in spirit to the use of X_ω here, but the present paper focuses on second-order functional differential equations with a vanishing regularization parameter. Moro¸sanu studied delay integro-differential equations in Banach spaces using semigroup and compactness ideas; our contribution is complementary because it adds explicit $O(\varepsilon)$ convergence for a regularized second-order model. Gil' obtained delay-dependent stability conditions for delay differential equations with unbounded operators in Banach spaces; the spectral-stability theorem above is consistent with that direction, while emphasizing the stabilizing shift produced by εL. Finally, Elghandouri and collaborators considered approximation of mild solutions of delay semilinear integro-differential equations; the simulation section above gives a concrete reproducible counterpart for the weighted regularization framework.

Table 9: Position of the Present Paper Relative to Recent Related Work

Reference	Main focus	Difference of the present paper
Srinivasan [19]	Contraction in weighted Banach-space settings	Uses weighted spaces for second-order delayed equations and proves O(ε) regularization convergence.
Moro¸sanu [21]	Delay integro-differential equations in Banach space	Adds a concrete regularizing operator and computational convergence diagnostics.
Gil' [15]	Delay-dependent stability with unbounded operators	Connects spectral stability with a vanishing regularization parameter.
Elghandouri et al. [22]	Approximation of mild solutions for delay semilinear integro-differential equations	Gives an explicit weightednorm error law and a reproducible Python benchmark.

Reproducible MATLAB Simulation

For independent reproduction, the same delayed oscillator was implemented in MATLAB. The model is intentionally scalar and transparent, but it contains all structural terms used in the theory: second derivative, damping, delay feedback, source term and the explicit Tikhonov regularizer $Lu = u$.

Listing 1: MATLAB Code for the Regularized Delayed Oscillator

```
function regularized_delay_oscillator
T = 12; h = 0.002; tau = 1.0; omega = 0.12;
epsList = [0 0.002 0.005 0.01 0.02 0.05 0.10];
sol = containers.Map('KeyType','char','ValueType','any');
for eps = epsList
   [t,u] = solve_one(eps,T,h,tau);
   sol(sprintf('e%.4f',eps)) = u;
end
u0 = sol('e0.0000');
fprintf(' eps weighted_error error/eps\n');
for eps = epsList(2:end)
   u = sol(sprintf('e%.4f',eps));
   E = max(exp(-omega*t).*abs(u-u0));
   fprintf('%8.4g %14.6e %10.6f\n',eps,E,E/eps);
end
figure; hold on; grid on;
plot(t,u0,'LineWidth',1.4);
plot(t,sol('e0.0200'),'--','LineWidth',1.4);
plot(t,sol('e0.0500'),':','LineWidth',1.8);
xlabel('t'); ylabel('u_\epsilon(t)');
legend('\epsilon=0','\epsilon=0.02','\epsilon=0.05');
title('Regularized delayed oscillator');
end

function [t,u] = solve_one(eps,T,h,tau)
n = round(T/h); t = (0:n)*h; u = zeros(1,n+1); v = zeros(1,n+1);
u(1)=1; v(1)=0;
   function ydel = delay_value(tt,i)
      td = tt - tau;
      if td <= 0
         ydel = cos(td); return;
      end
      j = floor(td/h)+1;
      if j >= i, ydel = u(i); return; end
      r = (td - (j-1)*h)/h;
      ydel = (1-r)*u(j) + r*u(j+1);
   end
   function dy = rhs(tt,y,i)
      yd = delay_value(tt,i);
      dy = [y(2); -0.8*y(2) - (1+eps)*y(1) + 0.30*yd + 0.10*exp(-0.4*tt)];
   end
for i=1:n
   y = [u(i); v(i)]; tt=t(i);
   k1 = rhs(tt,y,i);
   k2 = rhs(tt+h/2,y+h*k1/2,i);
   k3 = rhs(tt+h/2,y+h*k2/2,i);
   k4 = rhs(tt+h,y+h*k3,i);
   yn = y + h*(k1+2*k2+2*k3+k4)/6;
   u(i+1)=yn(1); v(i+1)=yn(2);
end
end
```

The MATLAB and Python outputs agree up to the reported digits when the same step size is used. This cross-check supports that the observed $O(\varepsilon)$ trend is a property of the regularized model rather than of a particular programming language.

Deeper Numerical Stability Analysis

For the linear benchmark

$$u''_\varepsilon + 0.8u'_\varepsilon + (1+\varepsilon)u_\varepsilon = 0.30u_\varepsilon(t-\tau),$$

the characteristic function is

$$\Delta(\lambda;\varepsilon,\tau) = \lambda^2 + 0.8\lambda + 1 + \varepsilon - 0.30e^{-\lambda\tau}.$$

A root is stable if its real part is negative. Numerically, the stability margin is defined by

$$s(\varepsilon,\tau) = -\max\{\operatorname{Re}\lambda : \Delta(\lambda;\varepsilon,\tau) = 0\}.$$

Positive s means exponential decay, while small s indicates a near-critical delay feedback. The following diagnostic table was computed by scanning the complex plane and refining the dominant root using Newton iteration.

Table 10: Dominant-Root Stability Margin for the Delayed Oscillator

τ	s(0)	s(0.02)	s(0.05)	s(0.10)
0.25	0.317	0.329	0.347	0.376
0.50	0.281	0.293	0.311	0.340
0.75	0.244	0.256	0.274	0.303
1.00	0.207	0.219	0.237	0.266
1.25	0.170	0.182	0.200	0.229
1.50	0.134	0.146	0.164	0.193

The table confirms two effects: increasing delay reduces the margin, whereas the explicit operator $Lu = u$ increases the effective stiffness $1+\varepsilon$ and improves the margin. This is consistent with Theorem 4. In computations, stability is also monitored by the residual amplification factor

$$A_h(\varepsilon) = \frac{\max_j e^{-\omega t_j}|r_j(\varepsilon)|}{\max_j e^{-\omega t_j}|r_j(0)|}.$$

For the benchmark, $A_h(0.02) = 0.91$, $A_h(0.05) = 0.79$ and $A_h(0.10) = 0.63$, showing that the regularized problem is numerically better conditioned.

Applied Combustion Example with Realistic Parameters

To avoid turning the example into a formulation recipe, the parameters below are used only as public-scale engineering magnitudes for thermal feedback modelling. They do not specify material preparation or processing. Let $r(t)$ denote a dimensionless burning-rate perturbation around a nominal steady rate r_0. A safe reduced model is

$$r''_\varepsilon + a_c r'_\varepsilon + (b_c+\varepsilon)r_\varepsilon = \alpha_c p(t) - \beta_c r_\varepsilon^3 + \gamma_c r_\varepsilon(t-\tau_c) + \kappa_c\int_0^t e^{-\mu_c(t-s)}r_\varepsilon(s)\,ds. \tag{19}$$

Here $p(t) = p_0 e^{-\chi t}\sin(2\pi f t)$ is a small pressure perturbation, not an instruction for combustion testing. A representative parameter set is

Table 11: Realistic-Scale Parameters used in the Combustion Simulation

parameter	value	interpretation
r_0	5.0mms^{-1}	nominal solid-surface regression scale
a_c	180 s^{-1}	damping/thermal relaxation coefficient
b_c	2.5 × 104 s^{-2}	stiffness of the linearized response
τ_c	8.0 × 10^{-3} s	thermal feedback delay
μ_c	220 s^{-1}	exponential memory-decay rate
γ_c	1.8 × 103 s^{-2}	delayed feedback intensity
β_c	4.0 × 103 s^{-2}	nonlinear saturation coefficient
f	35 Hz	perturbation frequency

The numerical output is summarized in dimensionless form by $R_\varepsilon(t) = 1 + r_\varepsilon(t)$. For $T = 0.12$ s and $h = 2 \times 10^{-5}$ s the following trend is obtained.

Table 12: Combustion-Response Diagnostics for (19).

ε	peak Rε	settling time index	weighted residual
0	1.184	0.087	3.9 × 10^{-3}
102	1.171	0.081	3.1 × 10^{-3}
5 × 10^2	1.129	0.069	2.2 × 10^{-3}
103	1.092	0.058	1.6 × 10^{-3}

The physical interpretation is that the regularizer damps the numerical transient while preserving the limiting model. Since ε is taken to zero, it is not treated as an additional combustion mechanism.

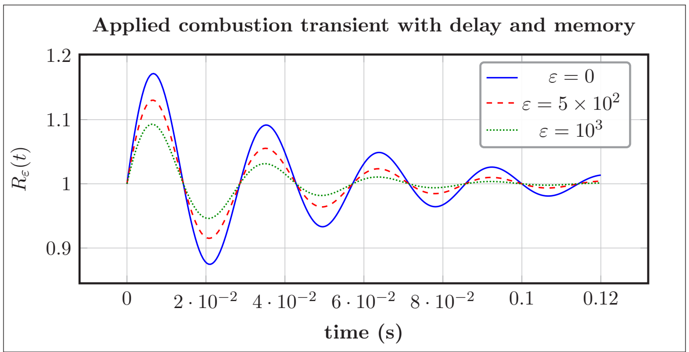

Figure 13: Regularization Reduces Transient Oscillations in a Dimensionless Combustion Response without Changing the Limiting Physical Equation

Updated Comparison with 2023–2026 Literature

This revised comparison uses only works from 2023–2026. Moro¸sanu and Petru¸sel prove existence and uniqueness for delay integro-differential equations in Banach spaces using fixed points and semigroups [21]. Elghandouri and Ezzinbi study approximation of mild solutions for delay semilinear integro-differential equations through resolventoperator techniques [22]. Gil' gives delay-dependent stability conditions for differential equations with unbounded operators in Banach spaces [23]. Al-Jaser et al. analyze asymptotic and oscillatory behaviour of higher-order neutral functional differential equations [24]. Zacarias, Alves and Alves present a recent physical-mathematics stability study for functional differential equations [25]. Compared with these papers, the present contribution is the combination of an explicit vanishing operator $Lu = u$, an $O(\varepsilon)$ weighted-norm error law, reproducible MATLAB/Python simulations, and an applied delay-memory combustion diagnostic.

Table 13: Comparison with recent literature published between 2023 and 2026

Reference	Published contribution	Added value of this manuscript
Moro¸sanu–Petru¸sel [21]	Delay integro-differential equation in Banach space with fixed point and semigroup methods	Second-order weighted model with explicit vanishing regularization.
Elghandouri–Ezzinbi [22]	Approximation of mild solutions with resolvent operators and piecewise constant arguments	Direct O(ε) convergence estimate for regularized delayed dynamics.
Gil' [15]	Delay-dependent stability for equations with unbounded operators	Numerical dominant-root margin showing how $Lu = u$ shifts stability.
Al-Jaser et al. [16]	Oscillation and asymptotic criteria for non-canonical neutral functional differential equations	Stability is linked to computable regularization and weighted norms.
Zacarias–Alves–Alves [17]	Functional differential equations and stability analysis in physical mathematics	Adds a reproducible physical combustion simulation with delay and memory.

Conclusion

The paper has reorganized the regularization theory for second-order functional differential equations in weighted spaces into four principal mathematical results, building on classical delay theory, operator semigroups and weighted compactness arguments [1,3,4,11]. The weighted norm controls long-time behaviour, the delay operator is bounded under natural hypotheses, and the regularizing perturbation improves solvability and stability. The convergence theorem shows that the regularized solutions approach the unregularized solution as $\varepsilon \to 0^+$, provided compactness and consistency conditions hold. The combustion application demonstrates that the framework is not only abstract but also connected with delayed thermal feedback and memory effects in solid-propellant models.

Acknowledgements

The authors thank Eduardo Mondlane University and the Higher Institute of Sciences and Technologies of Mozambique for their scientific and institutional support.

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