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On the Eigenvectors of the 5D Discrete Fourier Transform Number Operator in Newtonian Basis

Natig Atakishiyev

National Autonomous University of Mexico, Institute of Mathematics, Cuernavaca Unit, Cuernavaca, 62210, Morelos, Mexico

ABSTRACT

A simple analytic approach to the evaluation of the eigenvalues and eigenvectors f_n of the 5D discrete number operator $\mathcal{N}_5 = A_5^{\intercal} A_5$ is formulated.

This approach is based on the symmetry of the intertwining operators A_5 and $A_5^{\intercal}$ with respect to the discrete reflection operator. A procedure for the intertwining operators A_5 and $A_5^{\intercal}$ has been developed, which made it possible to establish a discrete analog of the well-known continuous-case formula

$\psi_n(x) = \frac{1}{\sqrt{n!}}(\mathbf{a}^{\dagger})^n \psi_0(x)$. A discrete analog for the eigenvectors f_n of another continuous-case formula $\psi_n(x) = c_n^{-1} H_n(x)\,\psi_0(x)$, $c_n = \sqrt{2^n\,n!}$, is constructed in terms of the Newtonian basis polynomials $\mathcal{P}_n(X_5)$, $n \in \mathbb{Z}_5$, times the lowest eigenvector f_0, as well.

***Corresponding author**

Natig Atakishiyev, National Autonomous University of Mexico, Institute of Mathematics, Cuernavaca Unit, Cuernavaca, 62210, Morelos, Mexico.

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Introduction

First, let me recall that the discrete (finite) Fourier transform (DFT) based on five points is represented by a 5×5 unitary symmetric matrix Φ_5 with entries (see, for example, [1-6])

$$(\Phi_5)_{kl} = 5^{-\frac{1}{2}}\, q^{kl}, \qquad k, l \in \mathbb{Z}_5 := \{0,1,2,3,4\}, \qquad (1)$$

where $q = \exp(2\pi i/5)$ is a primitive 5-th root of unity. Those vectors f_k, which are solutions of the standard equations

$$\sum_{k=0}^{4} (\Phi_5)_{m,n}\,(f_k)_n = \lambda_k\,(f_k)_m, \qquad k \in \mathbb{Z}_5, \qquad (2)$$

then represent five eigenvectors of operator Φ_5, associated with the eigenvalues λ_k. Because the fourth power of Φ_5 is a unit matrix, only four distinct eigenvalues among λ_ks are ±1 and ±i.

In addition, the discrete analog of the reflection operator P (defined on the full real line $x \in \mathbb{R}$ as $Px = -x$), associated with the DFT operator (1) is represented by the 5×5 matrix:

$$P_d := C_5^{\intercal} J_5 \equiv J_5\, C_5\,, \qquad (3)$$

where C_5 is the *5D basic circulant permutation* matrix with entries $(C_5)_{kl} = \delta_{k,l-1}$ and J_5 is the 5×5 'backward identity' permutation matrix with ones on the secondary diagonal (see [7], pages 26 and 28, respectively). It is readily verified that the DFT operator (1) is P_d-symmetric, that is, the commutator $[\Phi_5, P_d] = \Phi_5 P_d - P_d \Phi_5 = 0$. Therefore, similar to the continuous case, governed by the reflection operator P, the eigenvectors of the DFT operator Φ_5 should be either P_d-symmetric or P_d-antisymmetric.

In the present work, additional findings concerning the algebraic properties of two intertwining operators associated with the DFT matrix (1) are discussed. These operators are represented by matrices A_5 and $A_5^{\intercal}$ of the same size 5×5, such that the intertwining relations

$$A_5\,\Phi_5 = i\,\Phi_5 A_5, \qquad A_5^{\intercal}\,\Phi_5 = -i\,\Phi_5\,A_5^{\intercal}, \qquad (4)$$

are valid. Matrices A_5 and $A_5^{\intercal}$ have emerged in a paper [8] devoted to the problem of determining an explicit form for the difference operator that governs the eigenvectors of the DFT matrix Φ_5. They can be interpreted as discrete analogs of the harmonic oscillator lowering and raising operators

$$\mathbf{a} = 2^{-1/2}\left(x + \frac{d}{dx}\right) \text{ and } \mathbf{a}^{\dagger} = 2^{-1/2}\left(x - \frac{d}{dx}\right);$$

their algebraic properties have been studied in detail in [9-11]. In particular, it was shown that the operators A and $A^{\intercal}$ form a cubic algebra $\mathcal{C}_q$ with q a root of unity [11]. The algebra cq is intimately related to the two other well-known realizations of the cubic algebra: the Askey-Wilson algebra [12-15] and the Askey-Wilson-Heun algebra [16]. This particular algebra $\mathcal{C}_q$, associated with operators A and $A^{\intercal}$, is certainly more complicated than Heisenberg-Weyl algebra, generated by the lowering and raising operators a and a^† of the continuous case. Nevertheless, the remarkable fact is that it turns out to be possible to use the same procedure for constructing the eigenvectors of the discrete number operator $\mathcal{N} := A^{\intercal}A$ in the form of the ladder-type hierarchy, as in the linear harmonic oscillator case in quantum mechanics [17]:

$$\mathbf{a}\,\psi_0(x) = 0,\ \psi_n(x) = \frac{1}{\sqrt{n}}\,\mathbf{a}^\dagger\,\psi_{n-1}(x), \quad n = 1,2,3,\ldots. \qquad (5)$$

Note also that from the intertwining relations (4) it follows that the operator $\mathcal{N}_5 = A_5^{\intercal} A_5$ commutes with the DFT operator Φ_5, that is, $[\mathcal{N}_5, \Phi_5] = 0$. The discrete number operator $\mathcal{N}_5$ and DFT operator Φ_5 thus has the same eigenvectors, and the former can be employed to find an explicit form of the eigenvectors of the latter (see [8] for a more detailed discussion of this point).

This paper presents a novel analytical method for evaluating the eigenvalues and eigenvectors of the 5D discrete number operator $\mathcal{N}_5 = A_5^{\intercal} A_5$, levering the symmetry of the intertwining operators A_5 and $A_5^{\intercal}$ with respect to the discrete reflection operator P_d. It turned out that to achieve this goal, it is necessary to isolate from the intertwining operators A_5 and $A_5^{\intercal}$ those symmetric and antisymmetric parts that annihilate arbitrary vectors of the same parity. This study develops a procedure for the of operators A_5 and $A_5^{\intercal}$, which enables us to derive a still-missing discrete matrix analog of the well-known formulas

$$\psi_n(x) = \frac{1}{\sqrt{n!}}\,(\mathbf{a}^\dagger)^n \psi_0(x) = c_n^{-1} H_n(x)\,\psi_0(x), \quad c_n = \sqrt{2^n\, n!} \qquad (6)$$

associated with the continuous case for eigenvectors f_k of the DFT operator Φ_5.

The remainder of this paper is organized as follows. In Section 2 an account is given on how to resolve the problem of finding the eigenvectors f_n and eigenvalues λ_n of the $5D$ discrete number operator $\mathcal{N}_5 = A_5^{\intercal} A_5$. A detailed description of how to accomplish this task is provided in Section 3. In Section 4, an explicit form of the ladder-type hierarchy is established, in which the eigenvectors f_n form. Section 4 closes the paper by expressing the eigenvectors f_n in terms of Newtonian basis polynomials times the lowest eigenvector f_0.

5D intertwining operators A_5 and $A_5^{\intercal}$

This section begins by deriving additional symmetry properties of the 5D intertwining operators A_5 and $A_5^{\intercal}$, [8]. The explicit form of the matrices, associated with the operators A_5 and $A_5^{\intercal}$, is

$$\begin{aligned} A_5 &= X_5 + \mathrm{i}Y_5 = X_5 + D_5\,, \\ A_5^{\intercal} &= X_5 - \mathrm{i}Y_5 = X_5 - D_5\,, \end{aligned} \qquad (7)$$

where $X_5 = diag\,(s_0, s_1, s_2, s_3, s_4)$, $s_n := 2\sin(2\pi n/5)$, $n \in \mathbb{Z}_5$,

and $Y_5 = -\,\mathrm{i}\,D_5 = \mathrm{i}\,(C_5^{\intercal} - C_5)$.

The 5D operators X_5 and Y_5 are Hermitian and act as finite-dimensional analogs of the coordinates and momentum operators, respectively, in quantum mechanics. Remarkably, operators X and Y are "classical" operators with good spectral properties [11]. For operator X_5, the spectrum of X_5 is

$$\lambda_n \;=\; s_n = \mathrm{i}(q^{-n} - q^n), \qquad n \in \mathbb{Z}_5. \qquad (8)$$

This indicates that the spectrum (8) belongs to the class of Askey-Wilson spectra of the type

$$\lambda_n \;=\; C_1 q^n + C_2 q^{-n} + C_0\,. \qquad (9)$$

The eigenvectors of operator X_5 are represented by the Euclidean 5-column orthonormal vectors e_k with the components $(e_k)_l = \delta_{kl}$, $k, l \in \mathbb{Z}_5$, that is,

$$X_5\, e_k = s_k\, e_k\,. \qquad (10)$$

The spectrum of the matrix $Y_5 = -\,\mathrm{i}\,D_5$ belongs to the same Askey-Wilson family because the operators X_5 and $Y_5 = -\mathrm{i}\,D_5$ are unitary equivalent, $Y_5 = -\,\mathrm{i}\,D_5 = \Phi_5 X_5 \Phi_5^\dagger$, and hence isospectral [11]. Note that the spectrum of X_5 is simple, that is, it is nondegenerate. In addition, from the unitary equivalence of operators X_5 and $Y_5 = -\mathrm{i}D_5$ it follows that the eigenvectors of the latter operator are of the form:

$$Y_5\,\epsilon_k = -\,\mathrm{i}\,D_5\,\epsilon_k = s_k\,\epsilon_k\,,\ \epsilon_n := \Phi_5\, e_n = 5^{-\frac{1}{2}}(1, q^n, q^{2n}, q^{3n}, q^{4n})^{\intercal}. \qquad (11)$$

Let me draw attention now to the remarkable symmetry between the operators X_5 and $Y_5 = -\mathrm{i}\,D_5$: the operator X_5 is two-diagonal in the eigenbasis of the operator $Y_5 = -\mathrm{i}\,D_5$,

$$X_5\,\epsilon_n = \mathrm{i}\,(\epsilon_{n-1} - \epsilon_{n+1})\,, \qquad (12)$$

whereas operator Y_5 is similarly two-diagonal in the eigenbasis of operator X_5,

$$Y_5\, e_n = -\,\mathrm{i}\,D_5\, e_n = \mathrm{i}\,(e_{n+1} - e_{n-1})\,. \qquad (13)$$

It is also worth mentioning here that the N-column eigenvectors of operator $Y = -\mathrm{i}\,D$ for a general N,

$$\epsilon_n = \Phi_N\, e_n = \sum_{k=0}^{N-1} (\Phi_N)_{kn}\, e_k = N^{-\frac{1}{2}}(1, q^n, q^{2n}, \ldots, q^{(N-1)n})^{\intercal}, \qquad (14)$$

form an orthonormal basis in the N-dimensional complex plane $\mathbb{C}^N$ and are frequently used therefore as building blocks of the discrete Fourier transform in applications (see, for example, p.130 in [18], where the ϵ_n referred to as *discrete trigonometric functions*).

Note also that both operators X_5 and $D_5 = \mathrm{i}Y_5$ are P_d-antisymmetric; that is,

$$P_d\, X_5 + X_5 P_d = 0, \qquad P_d\, D_5 + D_5\, P_d = 0\,. \qquad (15)$$

Moreover, since the 5D intertwining operators A_5 and $A_5^{\intercal}$ under discussion are linear combinations of the operators X_5 and $D_5 = \mathrm{i}Y_5$, from (15) it follows that both operators A_5 and $A_5^{\intercal}$ are also P_d-antisymmetric.

It remains only to note that such a detailed discussion of the matrix structure of the operators X_5 and $D_5 = \mathrm{i}Y_5$ is dictated by the fact that it makes it possible to significantly simplify the problem of finding the eigenvectors of the 5D discrete number operator $\mathcal{N}_5 = A_5^{\intercal} A_5$. It is possible to formulate a simpler algorithm for calculating the eigenvectors of the discrete number operator $\mathcal{N}_5 = A_5^{\intercal} A_5$ by separating from the operators A_5 and $A_5^{\intercal}$ their symmetric and antisymmetric parts, which are essentially annihilators. A detailed description of how to achieve this goal is provided in the next section, where it becomes apparent that the key idea of this approach is the essential use of the remarkably symmetric matrix structure of the product $\Phi_5\, X_5$. Therefore, let me close this section with a discussion about a particular P_d-

symmetry property of the product $\Phi_5 X_5$, which will be essentially used in what follows.

Proposition 1: *The product $\Phi_5 X_5$ can be represented as either*

$$\Phi_5 X_5 = s_2^{-1}\,\mathcal{A}^{(s)} + \mathrm{i}\,\mathcal{B}^{(s)}, \qquad (16)$$

where $\mathcal{A}^{(s)}$ is a symmetric annihilator operator that annuls every P_d-symmetric vector $f^{(s)} := (a,b,c,c,b)^{\intercal}$, and $\mathcal{B}^{(s)}$ is a sparse matrix, or

$$\Phi_5 X_5 = s_2^{-1}\left(\mathcal{A}^{(a)} + \mathcal{B}^{(a)}\right), \qquad (17)$$

where $\mathcal{A}^{(a)}$ is an antisymmetric annihilator operator that annuls every P_d-antisymmetric vector $f^{(a)} := (0,b,c,-c,-b)^{\intercal}$, and $\mathcal{B}^{(a)}$ is a sparse matrix.

Proof. The operator $\Phi_5 X_5$ is represented by a traceless matrix

$$\Phi_5 X_5 = \frac{1}{s_2}\begin{bmatrix} 0 & 1 & c_1 & -c_1 & -1 \\ 0 & q & c_1 q^2 & -c_1 q^3 & -q^4 \\ 0 & q^2 & c_1 q^4 & -c_1 q & -q^3 \\ 0 & q^3 & c_1 q & -c_1 q^4 & -q^2 \\ 0 & q^4 & c_1 q^3 & -c_1 q^2 & -q \end{bmatrix}. \qquad (18)$$

From (18) it follows that the first row annihilates an arbitrary P_d-symmetric vector $f^{(s)} := (a,b,c,c,b)^{\intercal}$, that is, $(0,1,c_1,-c_1,-1)\,f^{(s)} = 0$. In addition, with the aid of simple identities $q = q^4 + \mathrm{i}s_1$ and $q^2 = q^3 + \mathrm{i}s_2$, the second and third rows in (18) can be rewritten as

$$(0, q^4 + \mathrm{i}s_1, c_1(q^3 + \mathrm{i}s_2), -c_1 q^3, -q^4) = (0, q^4, c_1 q^3, -c_1 q^3, -q^4) + \mathrm{i}\,(0, s_1, c_1 s_2, 0, 0),$$
$$(0, q^3 + \mathrm{i}s_2, c_1(q - \mathrm{i}s_1), -c_1 q, -q^3) = (0, q^3, c_1 q, -c_1 q, -q^3) + \mathrm{i}\,(0, s_2, -s_2, 0, 0), \qquad (19)$$

respectively.

Similarly, the fourth and fifth rows in (18) can be represented as

$$(0, q^3, c_1 q, -c_1(q - \mathrm{i}s_1), -(q^3 + \mathrm{i}s_2)) = (0, q^3, c_1 q, -c_1 q, -q^3) + \mathrm{i}\,(0,0,0,s_2,-s_2),$$
$$(0, q^4, c_1 q^3, -c_1(q^3 + \mathrm{i}s_2), -(q^4 + \mathrm{i}s_1)) =$$
$$= (0, q^4, c_1 q^3, -c_1 q^3, -q^4) - \mathrm{i}\,(0,0,0,c_1 s_2, s_1), \qquad (20)$$

respectively. Hence the initial matrix (18) can be divided into two parts, $\Phi_5 X_5 = s_2^{-1}\,\mathcal{A}^{(s)} + \mathrm{i}\,\mathcal{B}^{(s)}$, where

$$\mathcal{A}^{(s)} = \begin{bmatrix} 0 & 1 & c_1 & -c_1 & -1 \\ 0 & q^4 & c_1 q^3 & -c_1 q^3 & -q^4 \\ 0 & q^3 & c_1 q & -c_1 q & -q^3 \\ 0 & q^3 & c_1 q & -c_1 q & -q^3 \\ 0 & q^4 & c_1 q^3 & -c_1 q^3 & -q^4 \end{bmatrix}, \qquad \mathcal{B}^{(s)} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & -c_2 & c_1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & -c_1 & c_2 \end{bmatrix}. \qquad (21)$$

From the explicit form of the matrix $\mathcal{A}^{(s)}$ in (21) it is evident that $\mathcal{A}^{(s)} f^{(s)} = 0$. In addition, with only eight non-zero elements, the 5×5 matrix $\mathcal{B}^{(s)}$ is a sparse matrix. Consequently, identity (16) is proved with the identification of the explicit forms of both operators $\mathcal{A}^{(s)}$ and the matrix $\mathcal{B}^{(s)}$.

Alternatively, the same matrix (18) can be split into two parts: $\Phi_5 X_5 = s_2^{-1}(\mathcal{A}^{(a)} + \mathcal{B}^{(a)})$, where

$$\mathcal{A}^{(a)} = \begin{bmatrix} 0 & -1 & -c_1 & -c_1 & -1 \\ 0 & -q^4 & -c_1 q^3 & -c_1 q^3 & -q^4 \\ 0 & -q^3 & -c_1 q & -c_1 q & -q^3 \\ 0 & q^3 & c_1 q & c_1 q & q^3 \\ 0 & q^4 & c_1 q^3 & c_1 q^3 & q^4 \end{bmatrix}, \qquad \mathcal{B}^{(a)} = \begin{bmatrix} 0 & 2 & 2c_1 & 0 & 0 \\ 0 & c_1 & -1 & 0 & 0 \\ 0 & c_2 & c_1^2 & 0 & 0 \\ 0 & 0 & 0 & -c_1^2 & -c_2 \\ 0 & 0 & 0 & 1 & -c_1 \end{bmatrix}. \qquad (22)$$

From the explicit form of matrix $\mathcal{A}^{(a)}$ in (22) it is evident that $\mathcal{A}^{(a)} f^{(a)} = 0$. The 5×5 matrix 00, with only 10 nonzero elements, is also a sparse matrix. Consequently, identity (17) is also proved, with the identification of the explicit forms of both operators $\mathcal{A}^{(a)}$ and the matrix $\mathcal{B}^{(a)}$. This completes the proof of the proposition.

Eigenvectors and Eigenvalues of the Discrete Number Operator $\mathcal{N}_5$

This section begins with the derivation of an explicit form of the eigenvectors and eigenvalues of discrete number operator $\mathcal{N}_5$.

1. Similar to the continuous case (5), the lowest eigenvector f_0 of the discrete number operator $\mathcal{N}_5$ is obtained by solving the difference equation

$$A_5\, f_0 = (X_5 + D_5)\, f_0 = 0\,. \qquad (23)$$

This vector f_0 must be P_d-symmetric, i.e. it can be written as $f_0 = (x_0, x_1, x_2, x_2, x_1)^{\intercal}$. Therefore the matrix form of (23) is:

$$A_5\, f_0 = \begin{bmatrix} 0 & 1 & 0 & 0 & -1 \\ -1 & s_1 & 1 & 0 & 0 \\ 0 & -1 & s_2 & 1 & 0 \\ 0 & 0 & -1 & -s_2 & 1 \\ 1 & 0 & 0 & -1 & -s_1 \end{bmatrix}\begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_2 \\ x_1 \end{bmatrix} = \begin{bmatrix} 0 \\ s_1 x_1 + x_2 - x_0 \\ s_2 x_2 + x_2 - x_1 \\ x_1 - s_2 x_2 - x_2 \\ x_0 - s_1 x_1 - x_2 \end{bmatrix} = 0\,.$$

Thus, only two linearly independent equations

$$x_0 = s_1 x_1 + x_2\,, \quad x_1 = x_2\,(1 + s_2)\,, \qquad (24)$$

interconnecting components x_0, x_1 and x_2 were obtained. Hence the lowest eigenvector

$$f_0 = x_2\,(\xi_0, \xi_1, 1, 1, \xi_1)^{\intercal}, \qquad \xi_0 = s_1 - 2c_2, \quad \xi_1 = 1 + s_2\,, \qquad (25)$$

is determined up to the multiplicative factor x_2, the explicit form of which is given as follows.

It is directly verified that $\Phi_5 f_0 = f_0$; moreover, from this formula and the second identity in (4) it follows that:

$$\Phi_5 f_k = \mathrm{i}^k f_k\,, \qquad k \in \mathbb{Z}_5\,. \qquad (26)$$

Recall that the functions $\psi_n(x)$ from (6) possess a simple transformation property with respect to the Fourier transform: *they are eigenfunctions of the Fourier transform, associated with the eigenvalues* i^n,

$$(\mathcal{F}\,\psi_n)(x) \equiv \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{\,\mathrm{i}\,x\,y}\,\psi_n(y)\,dy = \mathrm{i}^n\,\psi_n(x)\,. \qquad (27)$$

Thus, (26) is a discrete analog of the continuous case (27).

It should be emphasized that owing to the simplicity of defining Equation (23) for the lowest eigenvector f_0, there was no need to use Proposition 1 in the first step.

2. To define next to the f_0 eigenvector f_1, we evaluate

$$A_5^{\intercal} f_0 = (X_5 - D_5)\, f_0 = \left(X_5 - \mathrm{i}\,\Phi_5 X_5 \Phi_5^{\dagger}\right) f_0 =$$
$$= (X_5 - \mathrm{i}\,\Phi_5 X_5)\, f_0 = \left(X_5 + \mathcal{B}^{(s)}\right) f_0\,, \qquad (28)$$

by using the unitary equivalence of operators $-\mathrm{i}\,D_5$ and X_5 in the first step, the identity $\Phi_5^{\dagger} f_0 = f_0$, which follows from (26), in the second step, and the identity (16) in the third step. Upon rewriting (28) in matrix form, we arrive at the following:

$$A_5^{\mathrm{T}} f_0 = x_2 \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & -c_2\xi_1 & c_1 & 0 & 0 \\ 0 & 1 & \xi_1 - 2 & 0 & 0 \\ 0 & 0 & 0 & 2-\xi_1 & -1 \\ 0 & 0 & 0 & -c_1 & c_2\xi_1 \end{bmatrix} \begin{bmatrix} \xi_0 \\ \xi_1 \\ 1 \\ 1 \\ \xi_1 \end{bmatrix} = 2x_2 s_1 \begin{bmatrix} 0 \\ \xi_1 \\ c_1 \\ -c_1 \\ -\xi_1 \end{bmatrix}, \quad (29)$$

where the readily verified identity $c_2\,\xi_1^2 = c_1 - 2s_1\xi_1$ is considered. Hence, the unit-length P_d-antisymmetric eigenvector f_1 is of the form

$$f_1 = \frac{1}{2\sqrt{s_2\xi_1}}\,(0, \xi_1, c_1, -c_1, -\xi_1)^{\mathrm{T}}. \quad (30)$$

To find explicitly the eigenvalue λ_1 of the discrete number operator $\mathcal{N}_5 = A_5^{\mathrm{T}} A_5$, associated with the eigenvector f_1, one evaluates first

$$A_5 f_1 = (X_5 + D_5) f_1 = \left(X_5 + \mathrm{i}\,\Phi_5 X_5 \Phi_5^{\dagger}\right) f_1 = \\ = (X_5 + \Phi_5 X_5) f_1 = \left(X_5 + s_2^{-1}\,\mathcal{B}^{(a)}\right) f_1 = u, \quad (31)$$

$$u = \frac{1}{2\sqrt{s_2\xi_1}}\left(2\xi_1, s_1\xi_1 + c_1, c_2 - c_1^2 s_2, c_2 - c_1^2 s_2, s_1\xi_1 + c_1,\right)^{\mathrm{T}},$$

by using the unitary equivalence of operators $-\mathrm{i}\,D_5$ and X_5 in the first step, the identity $\Phi_5^{\dagger} f_1 = -\mathrm{i} f_1$, which follows from (26), in the second step, and the identity (17) in the third step. It is readily verified that the P_d-symmetric vector u is the eigenvector of the DFT operator Φ_5, that is, $\Phi_5\, u = u$. Therefore, one may use identity (16) to evaluate

$$A_5^{\mathrm{T}} A_5 f_1 = A_5^{\mathrm{T}} u = (X_5 - D_5)\,u = \left(X_5 - \mathrm{i}\,\Phi_5 X_5 \Phi_5^{\dagger}\right) u = \\ = (X_5 - \mathrm{i}\Phi_5 X_5)\,u = \left(X_5 + \mathcal{B}^{(s)}\right) u = \lambda_1 f_1, \quad \lambda_1 = c_1(s_2 - 1) + 7. \quad (32)$$

3. To define the next eigenvector f_2, we evaluate

$$A_5^{\mathrm{T}} f_1 = (X_5 - D_5) f_1 = \left(X_5 - \mathrm{i}\,\Phi_5 X_5 \Phi_5^{\dagger}\right) f_1 = \\ = (X_5 - \Phi_5 X_5) f_1 = \left(X_5 - s_2^{-1}\mathcal{B}^{(a)}\right) f_1, \quad (33)$$

using the identity $\Phi_5^{\dagger} f_1 = -\mathrm{i}\, f_1$ and identity (17). Upon rewriting the right-hand side of (33) in matrix form, we obtain

$$A_5^{\mathrm{T}} f_1 = \frac{1}{2\xi_1^{1/2}\,(s_2)^{3/2}} \begin{bmatrix} 0 & -2 & -2c_1 & 0 & 0 \\ 0 & -c_2 & 1 & 0 & 0 \\ 0 & -c_2 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & c_2 \\ 0 & 0 & 0 & -1 & c_2 \end{bmatrix} \begin{bmatrix} 0 \\ \xi_1 \\ c_1 \\ -c_1 \\ -\xi_1 \end{bmatrix} = \sqrt{s_1(s_1 - c_2)}\, f_2, \quad (34)$$

where the unit-length P_d-symmetric eigenvector f_2 is of the form

$$f_2 = \frac{1}{2\,s_2}\,(-2c_1, 1,1,1,1)^{\mathrm{T}}. \quad (35)$$

To find explicitly the eigenvalue λ_2 of the discrete number operator $\mathcal{N}_5 = A_5^{\mathrm{T}} A_5$, associated with the eigenvector f_2, one evaluates first

$$A_5 f_2 = (X_5 + D_5) f_2 = \left(X_5 + \mathrm{i}\,\Phi_5 X_5 \Phi_5^{\dagger}\right) f_2 = \\ = (X_5 - \mathrm{i}\Phi_5 X_5) f_2 = \left(X_5 + \mathcal{B}^{(s)}\right) f_2 = \sqrt{s_1(s_1 - c_2)}\, f_1, \quad (36)$$

using the identity $\Phi_5^{\dagger} f_2 = -f_2$, which follows from (26), and identity (16). Then, from (34) and (36) it follows that

$$A_5^{\mathrm{T}} A_5 f_2 = \sqrt{s_1(s_1 - c_2)}\, A_5^{\mathrm{T}} f_1 = s_1(s_1 - c_2)\, f_2 = \lambda_2 f_2, \\ \lambda_2 = s_1(s_1 - c_2). \quad (37)$$

4. To define the next eigenvector f_3, we evaluate

$$A_5^{\mathrm{T}} f_2 = (X_5 - D_5) f_2 = \left(X_5 - \mathrm{i}\,\Phi_5 X_5 \Phi_5^{\dagger}\right) f_2 = \\ = (X_5 + \mathrm{i}\,\Phi_5 X_5) f_2 = \left(X_5 - \mathcal{B}^{(s)}\right) f_2, \quad (38)$$

using the identity $\Phi_5^{\dagger} f_2 = -f_2$, which follows from (26), and identity (16). Upon rewriting the right-hand side of (18) in matrix form, we arrive at the following:

$$A_5^{\mathrm{T}} f_2 = \frac{1}{2s_2} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & s_1 + c_2 & -c_1 & 0 & 0 \\ 0 & -1 & \xi_1 & 0 & 0 \\ 0 & 0 & 0 & -\xi_1 & 1 \\ 0 & 0 & 0 & c_1 & -(s_1 + c_2) \end{bmatrix} \begin{bmatrix} -2c_1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{2c_1} \begin{bmatrix} 0 \\ 1 - s_2 \\ c_1 \\ -c_1 \\ s_2 - 1 \end{bmatrix}. \quad (39)$$

Hence the unit-length P_d-antisymmetric eigenvector f_3 is of the form

$$f_3 = \frac{1}{2\sqrt{s_2(s_2 - 1)}}\,(0, 1 - s_2, c_1, -c_1, s_2 - 1)^{\mathrm{T}}. \quad (40)$$

To find explicitly the eigenvalue λ_3 of the discrete number operator $\mathcal{N}_5 = A_5^{\mathrm{T}} A_5$, associated with the eigenvector f_3, one evaluates first

$$A_5 f_3 = (X_5 + D_5) f_3 = \left(X_5 + \mathrm{i}\,\Phi_5 X_5 \Phi_5^{\dagger}\right) f_3 = \\ = (X_5 - \Phi_5 X_5) f_3 = \left(X_5 - s_2^{-1}\,\mathcal{B}^{(a)}\right) f_3 = \sqrt{s_1(s_1 + c_2)}\, f_2, \quad (41)$$

using the identity $\Phi_5^{\dagger} f_3 = \mathrm{i} f_3$, which follows from (26), and identity (17). Then, from (39) and (41), it follows that

$$A_5^{\mathrm{T}} A_5 f_3 = \sqrt{s_1(s_1 + c_2)}\, A_5^{\mathrm{T}} f_2 = s_1(s_1 + c_2)\, f_3 = \lambda_3 f_3, \\ \lambda_3 = s_1(s_1 + c_2). \quad (42)$$

5. Finally, to define the last eigenvector f_4, we evaluate

$$A_5^{\mathrm{T}} f_3 = (X_5 - D_5) f_3 = \left(X_5 - \mathrm{i}\,\Phi_5 X_5 \Phi_5^{\dagger}\right) f_3 = \\ = (X_5 + \Phi_5 X_5) f_3 = \left(X_5 + s_2^{-1}\mathcal{B}^{(a)}\right) f_3, \quad (43)$$

using the identity $\Phi_5^{\dagger} f_3 = \mathrm{i}\, f_3$ and identity (17). Upon rewriting the right-hand side of (43) in matrix form, we obtain

$$A_5^{\mathrm{T}} f_3 = \frac{1}{2c_1 s_2\sqrt{\lambda_3}} \begin{bmatrix} 0 & 2 & 2c_1 & 0 & 0 \\ 0 & 3c_1 + 1 & -1 & 0 & 0 \\ 0 & c_2 & 3 - 2c_1 & 0 & 0 \\ 0 & 0 & 0 & 2c_1 - 3 & -c_2 \\ 0 & 0 & 0 & 1 & -(3c_1 + 1) \end{bmatrix} \begin{bmatrix} 0 \\ 1 - s_2 \\ c_1 \\ -c_1 \\ s_2 - 1 \end{bmatrix} = v, \quad (44)$$

where the P_d-symmetric vector v is of the form

$$v = \frac{\sqrt{\lambda_3}}{2s_2}\,(2c_1, -(2s_2+1), 2s_2+3-2c_1, 2s_2+3-2c_1, -(2s_2+1))^{\intercal}\,. \tag{45}$$

It is readily verified now that the P_d-symmetric vector v is the eigenvector of the DFT operator Φ_5, that is, $\Phi_5 v = v$. To find explicitly the eigenvalue λ_4 of the discrete number operator $\mathcal{N}_5 = A_5^{\intercal} A_5$, associated with the eigenvector v, one evaluates first

$$\begin{gathered} A_5\, v = (X_5 + D_5)\, v = \left(X_5 + \mathrm{i}\,\Phi_5 X_5 \Phi_5^{\dagger}\right) v = \\ = (X_5 + \mathrm{i}\Phi_5 X_5)\, v = \left(X_5 - \mathcal{B}^{(s)}\right) v = (7 - c_1 - c_1 s_2)\, f_3\,. \end{gathered} \tag{46}$$

Then from (44) and (46) it follows at once that

$$\begin{gathered} A_5^{\intercal} A_5\, v = (7 - c_1 - c_1 s_2)\, A_5^{\intercal} f_3 = (7 - c_1 - c_1 s_2)\, v = \lambda_4\, v\,, \\ \lambda_4 = 7 - c_1(1 + s_2)\,. \end{gathered} \tag{47}$$

Therefore, a P_d-symmetric eigenvector f_4 of unit length is simply a properly normalized version of vector v, that is, $v = \sqrt{\lambda_4}\, f_4$ and

$$f_4 = \frac{1}{2\sqrt{\lambda_2\,\lambda_4}}\,(-2, 2s_1 - c_2, 3c_2 + 2 - 2s_1, 3c_2 + 2 - 2s_1, 2s_1 - c_2)^{\intercal}\,. \tag{48}$$

Remark. It should be noted that by defining all the eigenvalues $\lambda_1 = c_1(s_2 - 1) + 7$, $\lambda_2 = s_1(s_1 - c_2)$, $\lambda_3 = s_1(s_1 + c_2)$, and $\lambda_4 = 7 - c_1(1 + s_2)$, it is possible to uniformly write all the preceding f_4 eigenvectors of unit length as

$$\begin{gathered} f_0 = \frac{1}{\sqrt{\lambda_2\lambda_4}}(s_1 - 2c_2, 1 + s_2, 1, 1, 1 + s_2)^{\intercal}, \qquad f_1 = \frac{1}{2\sqrt{\lambda_2}}(0, s_1 - c_2, 1, -1, c_2 - s_1)^{\intercal}, \\ f_2 = \frac{1}{2\sqrt{\lambda_2\,\lambda_3}}(2, c_2, c_2, c_2, c_2)^{\intercal}, \qquad f_3 = \frac{1}{2\sqrt{\lambda_3}}(0, s_1 + c_2, -1, 1, -(s_1 + c_2))^{\intercal}\,. \end{gathered} \tag{49}$$

Note also that the eigenvectors f_k, $k \in \mathbb{Z}_5$, from (48) and (49) are orthonormalized, that is,

$$(f_k, f_l) = \delta_{k,l}, \qquad k, l \in \mathbb{Z}_5\,. \tag{50}$$

Explicit forms of Discrete Analogs

1. Thus, the above formulated approach to finding the eigenvectors of the discrete number operator $\mathcal{N}_5 = A_5^{\intercal} A_5$ really leads to the definition of a certain hierarchy of the ladder type that these eigenvectors form. It only remains to consider that the compact form of this hierarchy still contains another parameter, which can be interpreted as follows.

It is well known that if A and B are $n\times n$ matrices, then AB and BA have the same eigenvalues (see [19], p.54). Hence the operator $\mathcal{N}_5^{(s)} := A_5 A_5^{\intercal}$ has the same 5 eigenvalues λ_n as the DFT operator $\mathcal{N}_5 = A_5^{\intercal} A_5$, that is,

$$\mathcal{N}_5^{(s)} g_n = A_5\, A_5^{\intercal}\, g_n = \lambda_n\, g_n\,, \qquad n \in \mathbb{Z}_5\,. \tag{51}$$

Moreover, it is readily verified that the eigenvectors g_n and f_n of the operators $\mathcal{N}_5^{(s)} = A_5 A_5^{\intercal}$ and $\mathcal{N}_5 = A_5^{\intercal} A_5$, respectively, are interrelated as

$$\begin{gathered} g_0 = \sin\varphi\, f_0 + \cos\varphi\, f_4 = \eta\left(\frac{s_1^2}{4}\, f_0 + f_4\right), \\ g_1 = \cos\varphi\, f_0 - \sin\varphi\, f_4 = \eta\left(f_0 - \frac{s_1^2}{4} f_4\right), \\ g_2 = f_1\,, \qquad g_3 = f_2\,, \qquad g_4 = f_3\,, \end{gathered} \tag{52}$$

$$\cos\varphi = 4/\sqrt{21 - 5c_2} = \eta\,, \qquad \sin\varphi = s_1^2/\sqrt{21 - 5c_2} = \eta\, s_1^2/4\,,$$

$$\varphi = \arctan\left(\frac{s_1^2}{4}\right) = \arctan\left[\frac{5 + \sqrt{5}}{8}\right] = 42{,}13^{\circ}\,,$$

where the parameter $\eta = \cos\varphi$ can also be expressed in terms of eigenvalues λ_k, $1 \le k \le 4$ as $\eta = 4c_1\,(\lambda_2\lambda_3/\lambda_1\lambda_4)^{1/2}$.

2. Having defined the parameter η, the simple geometric interpretation of which is obvious from (51) and (52), it is now easy to show that this parameter also enters the discrete analog of (6) as

$$f_n = (\eta \prod_{k=1}^{n} \lambda_k^{\frac{1}{2}})^{-1} (A_5^{\intercal})^n f_0\,, \qquad n = 1,2,3,4\,. \tag{53}$$

Recall that in the continuous case $N\psi_n = \mathbf{a}^{\dagger}\mathbf{a}\,\psi_n = n\,\psi_n$, hence multiplier n! in (6) can be expressed as $n! = 1\cdot 2\cdots n = \lambda_1\lambda_2\cdots\lambda_n)$ with $\lambda_m := m$, confirming the similarity between (6) and (53).

Moreover, it turns out that the parameter η essentially contributes to a discrete analogue of the three-term recurrence relation [17]

$$\sqrt{2\,(n+1)}\,\psi_{n+1}(x) + \sqrt{2n}\,\psi_{n-1}(x) = 2x\,\psi_n(x)\,, \qquad n = 1,2,\ldots, \tag{54}$$

associated with continuous cases (6). This can be expressed as follows.

From (44), (45) and (48) it follows that $A_5^{\intercal} f_3 = \sqrt{\lambda_4}\, f_4$. Consequently, this can be rewritten as:

$$\sqrt{\lambda_3}\, f_3 = A_5^{\intercal} f_2 = (2X_5 - A_5)\, f_2 = 2X_5 f_2 - \sqrt{\lambda_2}\, f_1\,, \tag{55}$$

where the evident identity $A_5 + A_5^{\intercal} = 2X_5$ has been used first, followed by the identitity $A_5 f_3 = \sqrt{\lambda_3}\, f_2$ from (41) and (42).

Similarly, from(39) and (40) it follows that $A_5^{\intercal} f_2 = \sqrt{\lambda_3}\, f_3$. Therefore,

$$\sqrt{\lambda_3}\, f_3 = A_5^{\intercal} f_2 = (2X_5 - A_5)\, f_2 = 2X_5 f_2 - \sqrt{\lambda_2}\, f_1\,, \tag{56}$$

because $A_5 f_2 = \sqrt{\lambda_2}\, f_1$ by (36). Thus both identities (55) and (56) represent particular cases of the three-term recurrence relation

$$\sqrt{\lambda_{n+1}}\, f_{n+1} + \sqrt{\lambda_n}\, f_{n-1} = 2X_5 f_n \tag{57}$$

for n=3 and n=2, respectively. Note the similarity between the recurrence relations (54) and (57).

Finally, from (34) it follows that $A_5^{\intercal} f_1 = \sqrt{\lambda_2}\, f_2$; hence

$$\sqrt{\lambda_2}\, f_2 = A_5^{\intercal} f_1 = (2X_5 - A_5)\, f_1 = 2X_5 f_1 - \sqrt{\lambda_1}\,\eta\,[f_0 - (s_1^2/4)\, f_4]\,, \tag{58}$$

where the readily verifiable identity

$$A_5 f_1 = \sqrt{\lambda_1}\,(\cos\varphi\, f_0 - \sin\varphi\, f_4) = \sqrt{\lambda_1}\,\eta\,[f_0 - (s_1^2/4)\, f_4] \quad (59)$$

has been used. Thus, (58) represents a four-term recurrence relation

$$\sqrt{\lambda_2}\, f_2 + \sqrt{\lambda_1}\,\eta\,[f_0 - (s_1^2/4)\, f_4] = 2\, X_5 f_1\,, \qquad (60)$$

where a linear combination of the eigenvectors f_0 and f_4 appears on the left side, instead of only f_0 [cf.(59)]. This difference between (57) and (60) is a consequence of the fact that operators X_5 and $Y_5 = -\mathrm{i}D_5$ do not satisfy the Heisenberg commutation relation $[x,p]=\mathrm{i}$ for the standard operators x and p in quantum mechanics (see [11] for a more detailed discussion of this point).

3. Finally, the above formulas for the explicit form of the eigenvectors f_k, $1 \le k \le 4$, allow us to represent them in the form $f_k = d_k^{-1}\,\mathcal{P}_k(X_5) f_0$, where $\mathcal{P}_k(X_5)$ is a polynomial in matrix X_5 of degree k. In this way, one can obtain an explicit form of the discrete analog of the second part of formula (6) associated with the continuous case. This can be ascertained as follows.
From (28)-(30) one readily derives that

$$f_1 = (\eta\sqrt{\lambda_1})^{-1} A_5^{\mathrm{T}} f_0 = (\eta\sqrt{\lambda_1})^{-1}\big(X_5 + \mathcal{B}^{(s)}\big) f_0 = d_1^{-1}\,\mathcal{P}_1(X_5) f_0\,, \qquad (61)$$

where $\mathcal{P}_1(X_5) := X_5 + \mathcal{B}^{(s)}$ and $d_1 = \eta\sqrt{\lambda_1}$.

From (33) and (34) one similarly derives that

$$f_2 = \frac{1}{\sqrt{\lambda_2}} A_5^{\mathrm{T}} f_1 = (\eta\sqrt{\lambda_1\lambda_2})^{-1}\big(X_5 - s_2^{-1}\mathcal{B}^{(a)}\big)\big(X_5 + \mathcal{B}^{(s)}\big) f_0 = d_2^{-1}\,\mathcal{P}_2(X_5) f_0\,, \quad (62)$$

where

$$\mathcal{P}_2(X_5) = \big(X_5 - s_2^{-1}\mathcal{B}^{(a)}\big)\big(X_5 + \mathcal{B}^{(s)}\big), \quad d_2 = \eta\sqrt{\lambda_1\lambda_2}\,. \quad (63)$$

Also, from (38)-(40) one similarly gets that

$$f_3 = \frac{1}{\sqrt{\lambda_3}} A_5^{\mathrm{T}} f_2 = d_3^{-1}\big(X_5 - \mathcal{B}^{(s)}\big)\big(X_5 - s_2^{-1}\mathcal{B}^{(a)}\big)\big(X_5 + \mathcal{B}^{(s)}\big) f_0 = d_3^{-1}\,\mathcal{P}_3(X_5) f_0\,, \quad (64)$$

where

$$\mathcal{P}_3(X_5) = \big(X_5 - \mathcal{B}^{(s)}\big)\big(X_5 - s_2^{-1}\mathcal{B}^{(a)}\big)\big(X_5 + \mathcal{B}^{(s)}\big), \quad d_3 = \eta\sqrt{\lambda_1\lambda_2\lambda_3}. \quad (65)$$

Finally, from (43), (45), and (48), it follows that

$$f_4 = \frac{1}{\sqrt{\lambda_4}} A_5^{\mathrm{T}} f_3 = d_4^{-1}\,\mathcal{P}_4(X_5) f_0\,, \qquad (66)$$

where

$$\mathcal{P}_4(X_5) = \big(X_5 + s_2^{-1}\mathcal{B}^{(a)}\big)\big(X_5 - \mathcal{B}^{(s)}\big)\big(X_5 - s_2^{-1}\mathcal{B}^{(a)}\big)\big(X_5 + \mathcal{B}^{(s)}\big), \quad d_4 = \eta\sqrt{\lambda_1\lambda_2\lambda_3\lambda_4}\,. \quad (67)$$

Thus, the discrete analog of the formula $\psi_n(x) = c_n^{-1} H_n(x)\,\psi_0(x)$, $c_n = \sqrt{2^n\, n!}$, associated with the continuous case, has the form:

$$f_n = d_n^{-1}\,\mathcal{P}_n(X_5) f_0, \qquad n = 1,2,3,4, \qquad (68)$$

where $d_n = \eta\prod_{k=1}^{n}(\lambda_k)^{1/2}$ and $\mathcal{P}_n(X_5)$ are the Newtonian basis matrix polynomials in X_5 (see, e.g.,[20-[23] and relevant references quoted therein on various applications of the Newtonian basis), defined as

$$\mathcal{P}_0(X_5) = 1, \qquad \mathcal{P}_n(X_5) = (X_5 - M_{n-1})\cdots(X_5 - M_1)(X_5 - M_0), \qquad n = 1,2,3,4\,, \quad (69)$$

with the matrices $M_0 = -M_2 = -\mathcal{B}^{(s)}$ and $M_1 = -M_3 = s_2^{-1}\mathcal{B}^{(a)}$ as interpolation nodes at 0,1,2,3. It should be noted that by combining (68) with the orthonormality condition of eigenvectors f_n, it is easy to verify that the polynomials $\mathcal{P}_n(X_5)$ are orthogonal:

$$(\mathcal{P}_k(X_5) f_0, \mathcal{P}_l(X_5) f_0) = d_k^2\,\delta_{kl}\,. \qquad (70)$$

Simultaneously, it is extremely important to emphasize that the matrix polynomials $\mathcal{P}_n(X_5)$ do not belong to the class of hypergeometric-type polynomials on a Newtonian basis, which necessarily satisfies the standard three-term recurrence relations [20].

Conclusions

To summarize, the eigenvalues λ_n and eigenvectors f_n of the 5D discrete number operator $\mathcal{N}_5 = A_5^{\mathrm{T}} A_5$ are evaluated in a systematic way. Because the eigenvalues λ_n are represented by distinct non-negative numbers, the number operator $\mathcal{N}_5$ has been used to classify eigenvectors of the 5D discrete Fourier transform Φ_5, thus resolving the ambiguity caused by the well-known degeneracy of the eigenvalues of the discrete Fourier transform Φ_N. A procedure for the intertwining operators A_5 and A_5^{T} has been formulated, which made it possible to construct the discrete analog (53) of the well-known continuous-case formula $\psi_n(x) = \frac{1}{\sqrt{n!}}(\mathbf{a}^{\dagger})^n\psi_0(x)$.

In addition, a discrete analog for the eigenvectors f_n of the continuous case formula $\psi_n(x) = c_n^{-1} H_n(x)\,\psi_0(x)$, $c_n = \sqrt{2^n\, n!}$, has been established in terms of the Newtonian basis polynomials $\mathcal{P}_n(X_5)$, $n \in \mathbb{Z}_5$, times the lowest eigenvector f_0. The methodology developed not only deepens the understanding of the discrete Fourier transform, but also paves the way for advanced numerical methods in spectral analysis.

Declaration of Competing Interest

There is no competing interest.

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