

Postulates of Special Relativity Need to be Supplemented for Wigner-Thomas Rotation to Exist

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ABSTRACT

The inertial frames are the frames moving with a uniform velocity in any direction. The second postulate of Special Relativity speaks of constancy of light speed (in vacuum) in all inertial frames, with no riders. However, it is taken to implicitly mean only those inertial frames that are moving along the line from origin to the event's location, or of light propagation. For the inertial frames moving otherwise i.e. in directions oblique to the latter, the setup is converted back to the same (i.e. parallel moving observer), by taking up the distance component parallel to observer's motion for transformation, and ignoring the rest of its components.

The Wigner-Thomas rotation arises only on account of this limited interpretation. It disappears when obliquely moving (with respect to direction of event from origin) inertial frames are given recognition. The two non-collinear boosts are equivalent to one boost in the resultant direction that is oblique to the directions of both the boosts. The example presented in the article amply demonstrates it.

Therefore, to give sanctity to the Wigner-Thomas rotation, the second postulate needs to be supplemented by specifying the "Inertial Frames" with a rider "that are moving along the Line of its (light's) motion".

Further, the Lorentz Transformation have not been and also cannot be derived for events other than those of light. However, these are universally applied to such events e.g. those of non-collinearly moving frames in this case. Thus, another (third) postulate is required to be added, and i.e. "The transformation arrived at for light applies to other events also, where the distance-time ratio is not equal to c ".

Addition of the postulate will provide the much needed authorization for working out of Wigner-Thomas rotation, along with numerous other cases such as length contraction and time dilation on moving bodies, though with errors. The error would obviously be proportional to the difference between the distance-time ratio of the event and c .

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Introduction

In Relativity, when two successive non-collinear Lorentz boosts are applied, their resultant is not only a Lorentz boost but also an associated rotation. The rotation obviously emerges again while returning back i.e. when two successive inverse transformations (Lorentz boost) are applied to return to the original state. The total rotation, thus occurring in a pair of onward and inward transformations, is called Wigner-Thomas rotation.

The phenomena go against the very essence of Relativity i.e. reciprocity, which Einstein himself propounded. Einstein's Principle of velocity reciprocity (EPVR) reads as follow.

We postulate that the relation between the coordinates of the two systems is linear. Then the inverse transformation is also linear and the complete non-preference of the one or the other system demands that the transformation shall be identical with the original one, except for a change of v to $-v$.

Contrary to the principle, however, combination of two linear transformations is not linear, and more importantly, one is not able to return back to the original state by inverse transformations, without applying a rotation in addition to the two reverse Lorentz boosts.

Such a disparity arises due to the rule which declares that when the event falls out of line of motion of the observer, only the distance component parallel to the direction of the observer's motion is to be transformed (even if it was zero), **along with the entire time**, leaving untransformed the other (normal) component of the distance.

For example, all the events of an object, moving in a direction perpendicular to that of the observer, are treated as $(0,t)$ for transformation, meaning while the distances along the direction of its motions remain untransformed, the time increases to γt . It leads to a disproportionate reduction in the relativistic velocity of the object, thereby inducing a rotation.

On the other hand, two non-collinear boosts together are equivalent to one oblique boost, without compromising the postulate. In other words, two inertial frames moving successively in different directions are together equivalent to one inertial frame moving in the resultant direction, which is oblique to either of the two. Maintaining conformity to the postulate, the resultant relativistic speed can be worked out in this direction. Such a transformation is complete as only a boost, and does not involve any rotation. Further, the transformation of time is worked out with the entire distance covered, and not any of its components.

Thus, when the two non-collinear velocities are added relativistically with a transformation derived for the resultant oblique direction, the incidence of rotation goes away. Therefore, to provide sanction to the Wigner-Thomas rotation, it is necessary to modify the second postulate of Special Relativity as follows.

“The speed of light moving in a given direction in vacuum remains the same in all inertial frames **that are moving along the line of its motion.**”

The rotation worked out currently, as well as its disappearance on recognition and adoption of obliquely moving inertial frames, have been demonstrated with an example. The example is chosen for

addition of two mutually perpendicular velocities, as it simplifies the calculation of Wigner-Thomas Rotation on one hand, and presents the most adverse setup for the proposed action, i.e. oblique transformation, on the other.

Further, the Lorentz Transformation have not been and also cannot be derived for events other than those of light. However, these are applied to such events. For example, in the instant case, the events generated by non-collinearly moving frames are picked up to work out the Wigner-Thomas rotation, by application of the Lorentz Transformation. The practice is universal and not limited to just this case. Thus, another (third) postulate is required to be added, and i.e.

“The Transformation arrived at for light applies to all events, even if their distance-time ratio was not equal to c ”.

Addition of the postulate will provide the much-needed authorization, though with errors proportional to the difference between the distance-time ratio of the event and c .

In absence of any such authorization, the velocities of non-collinearly moving observers have been chosen very close to that of light in vacuum c , in the example taken.

Demonstration with an Example

1. Current Practice - The Wigner-Thomas Rotation:

To keep the exercise simple, let us take up a case of two perpendicular velocities.

The following Figure 1 may be referred, which contains three diagrams denoted as (1), (2) and (3).

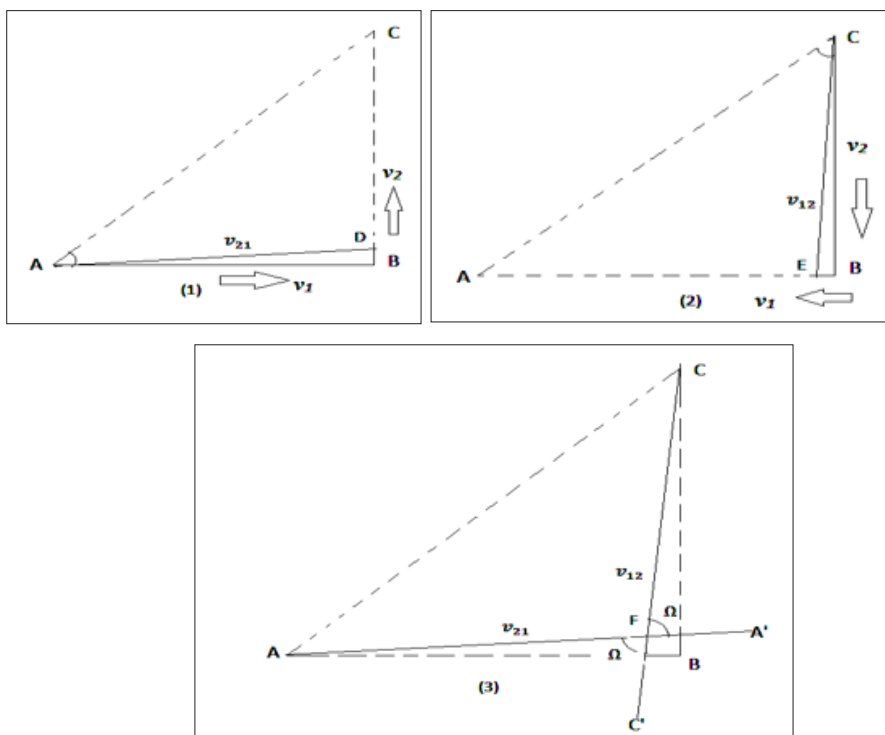


Figure 1

Let there be three observers A, B and C. Observer A is considered stationary. Observer B moves in horizontal direction with a relative velocity v_1 with respect to A, and observer C moves in vertical direction (in plane of paper) with a relative velocity v_2 with respect to B.

Let us set $c=1$. So, the velocities v_1 and v_2 are in terms of speed of light in vacuum.

Let v_{21} be the relativistic velocity of C with respect to A. Similarly, let v_{12} be the relativistic velocity of A with respect to C. The two are shown in Figure 1 above.

Einstein's Principle of velocity reciprocity (EPVR) and common intuition demand that $\vec{v}_{21} = -\vec{v}_{12}$. However, it is not so, though their magnitudes are equal. Their directions are far apart, as can be seen in Figure 1.

Thus, the observer A finds C moving along AD (diagram 1); however, looking inversely, when the observer C looks at A, (s)he finds A moving along AE (diagram 2). As a result, to reconcile with reciprocity, one has to apply a rotation equal to the angle between the two directions, shown as angle Ω in diagram (3). The rotation is called Wigner-Thomas rotation or simply, Wigner Rotation.

The expressions of v_{21} , v_{12} and Ω can obviously be written as follows[1].

$$\vec{v}_{21} = \vec{v}_1 + \frac{\vec{v}_2}{\gamma_1} = \vec{v}_1 + \vec{v}_2 \sqrt{1 - v_1^2}$$

$$\vec{v}_{12} = \vec{v}_2 + \frac{\vec{v}_1}{\gamma_2} = \vec{v}_2 + \vec{v}_1 \sqrt{1 - v_2^2}$$

$$\|\vec{v}_{21}\| = \|\vec{v}_{12}\| = \sqrt{v_1^2 + v_2^2 - v_1^2 v_2^2}$$

$$\sin \Omega = \frac{\|\vec{v}_{21} \times \vec{v}_{12}\|}{\|\vec{v}_{21}\| \|\vec{v}_{12}\|} = \frac{v_1 v_2 \left(1 - \frac{1}{\gamma_1 \gamma_2}\right)}{v_1^2 + v_2^2 - v_1^2 v_2^2} = \frac{v_1 v_2 \gamma_1 \gamma_2}{1 + \gamma_1 \gamma_2}$$

To see the extent of rotation, let us take velocities v_1 and v_2 very close to c , for reasons mentioned in Introduction.

Let $v_1=0.99997$ and $v_2=0.99995$.

Therefore, $\gamma_1=258.20$ and $\gamma_2=200.00$

From the above-mentioned relations, we get

$$v_{21} = v_{12} = 0.999999$$

$$\Omega = 89.19 \text{ degrees}$$

The rotation angle is as high as 89 degrees, in total disregard of the essence of Relativity i.e. reciprocity.

Disappearance of the Rotation on Adoption of Inertial Frames Moving in Oblique Directions

The frames moving with uniform velocity, but in directions different from that of the light signal under observation, are also inertial frames. Further, such frames too find the speed of light moving in a particular direction, the same as c while measuring it directly from the frame, in an oblique direction. Thus, conformity to the second postulate of Relativity, i.e. constancy of speed of light in all inertial frames, is maintained by such frames.

Ignoring such frames tantamount to deviating from the postulate, which manifests itself in the form of Wigner-Thomas Rotation. On the other hand, working out transformation directly, along the direction from such a frame to the light signal, is free from any rotation.

The derivation is presented below

IMP: Before proceeding further, it is clarified that the derivation ahead is for transformation of events of light in oblique direction. The same relations would be used on the material frames/objects for addition of velocities, as is being done currently for the collinear transformation case. To keep the errors occurring on this account to the minimum, frames moving at velocities very close to speed of light have been considered (refer v_1 and v_2 in the previous section).

The following Figure 2 may be referred.

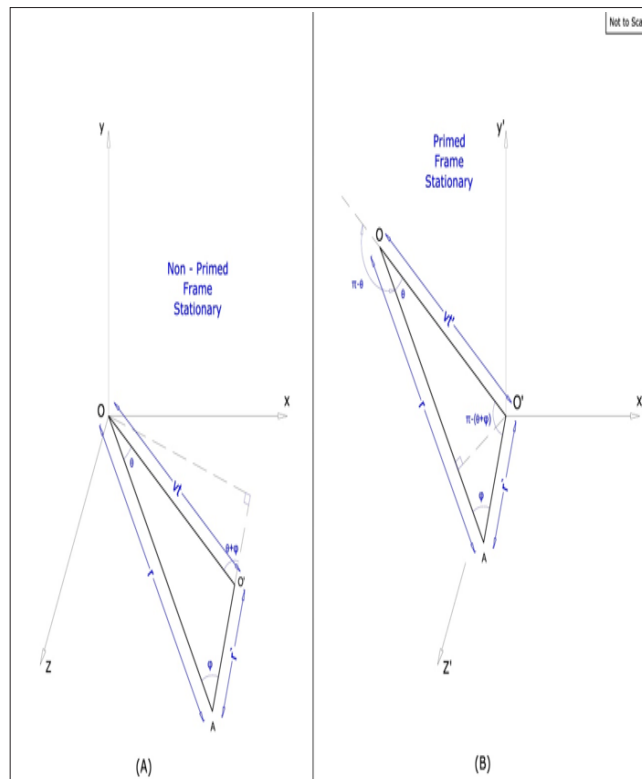


Figure 2

Assumed Setup for Derivations:

The following setup and assumptions are made in a way similar to the "simplified" derivation presented by Einstein [2], but with a more general case of three-dimensional event location as well as frame velocity, instead of a unidirectional setup.

1. The corresponding three axes of frames K and K' are permanently aligned with one another and at the start, their origins are coincident and concurrent i.e. $x=0, y=0, z=0, t=0$ and also $x'=0, y'=0, z'=0, t'=0$.
2. The distance from origin and time of an event at A in three dimensional space in frame K are designated as r and t respectively and those in frame K' for the same event are designated r' and t' .
3. The frame of observation K' is moving with respect to the frame K in three dimensional space with a uniform relative velocity v on the side of the event i.e. the angle between $(\vec{r}$ and $\vec{v})$ is acute.
4. The angle between \vec{r} and \vec{v} is θ and that between \vec{r}' and \vec{r} is ϕ . Consequently, the angle between \vec{r}' and \vec{v} becomes $(\theta+\phi)$.
5. All the events (distance-time sets) are reckoned from the start defined at sr. no. (i) above. Consequently, for events generated

by moving objects, including light, the object is also assumed to start its motion from the aforesaid start.

- The event occurs in the stationary non-primed frame K, and therefore, its location is anchored to the origin of this frame.

NOTE: It may be noted that the values of θ and φ , and also the shape of the triangle of Figure 2(B), would be different when the event occurred in the primed frame K'. In such cases, the event location gets anchored to the origin of the primed frame, and for applying the relations derived below, the primed frame K' should be considered as stationary, and the non-primed frame K as moving with a velocity $-v$.

The event distance triangle OAO' along with the subtended angles is shown in Figure 2(A) and Figure 2(B) considering respectively the non-primed and the primed frame as stationary, to work out the distance and time in the other frame which is considered moving. The origins and axes of the non-primed and the primed frames are O-X-Y-Z and O'-X'-Y'-Z' respectively.

Derivation

From the triangle OAO' of Figure 2 (A), we may write the relation $r' = r \cos \varphi - vt \cos(\theta + \varphi)$ based on classical kinematics. However, this is not true from the concept of relativity, as the distance and time measured in the moving frame change in such a way as to maintain constancy of speed of light. So, let us assume that it will be in some ratio as follows

$$r' = a(r \cos \varphi - vt \cos(\theta + \varphi))$$

Where a is a constant.

By reciprocity of relativity, the relation may as well be stated the reverse way as follows by referring to Figure 2 (B).

$$r = a(r' \cos \varphi + vt' \cos \theta)$$

Now, the above two relations shall be written together and further derivation shall proceed by performing the same action on both the relations so as to maintain reciprocity of relativity.

$$\left. \begin{aligned} r' &= a(r \cos \varphi - vt \cos(\theta + \varphi)) \\ r &= a(r' \cos \varphi + vt' \cos \theta) \end{aligned} \right\} \dots \dots (1)$$

From the above two equations, t' may be expressed in terms of t , r and v ; similarly, t may be expressed in terms of t' , r' and v . By doing so, we get the following relations.

$$\left. \begin{aligned} t' &= a \cos \varphi \frac{\cos(\theta + \varphi)}{\cos \theta} \left[t - \left(1 - \frac{1}{(a \cos \varphi)^2} \right) \frac{r \cos \varphi}{v \cos(\theta + \varphi)} \right] \\ t &= a \cos \varphi \frac{\cos \theta}{\cos(\theta + \varphi)} \left[t' + \left(1 - \frac{1}{(a \cos \varphi)^2} \right) \frac{r' \cos \varphi}{v \cos \theta} \right] \end{aligned} \right\} \dots \dots (2)$$

It may be noted that the angles θ and $(\theta + \varphi)$ in the non-primed frame correspond to angles $(\pi + (\theta + \varphi))$ and $(\pi + \theta)$ respectively in the primed frame, and vice versa.

Further, it is important to note two special cases where one of the relation (1) falls short of the term of time, leading to invalidity of the relations (2). When $\theta = n\pi/2$ or $(\theta + \varphi) = n\pi/2$, RHS of one of the relations (1) loses the term of time – either of t or of t' , and this in turn makes the relations (2) invalid, which also gets reflected in infinite values of t' and t respectively. **In such cases**, the results become extreme and inconsistent for events of light as well as

all others. So, the workaround is to apply an extremely small correction of, say ± 0.000001 , or even lower, to these angle values to keep the relations working.

Going further, the derivation is carried out simultaneously for both the frames to demonstrate that the same results are finally obtained by working in either of the frames. The expressions for the two frames are placed one below the other and enclosed by a curly bracket on right hand side.

The postulate stipulates

$$\left. \begin{aligned} r' &= ct' \\ r &= ct \end{aligned} \right\}$$

On substituting the expressions of t' and t from relations (2) and those of r' and r from relations (1) in the above relations and then replacing r' with ct' and r with ct , we get

$$\left. \begin{aligned} ct \cos \varphi - vt \cos(\theta + \varphi) &= c \cos \varphi \frac{\cos(\theta + \varphi)}{\cos \theta} \left[t - \left(1 - \frac{1}{(a \cos \varphi)^2} \right) \frac{ct \cos \varphi}{v \cos(\theta + \varphi)} \right] \\ ct' \cos \varphi + vt' \cos \theta &= c \cos \varphi \frac{\cos \theta}{\cos(\theta + \varphi)} \left[t' + \left(1 - \frac{1}{(a \cos \varphi)^2} \right) \frac{ct' \cos \varphi}{v \cos \theta} \right] \end{aligned} \right\}$$

On dividing both the sides of either the first equations by $ct \cos(\theta + \varphi)$ or the second one by $ct' \cos \theta$, we get the same resultant as follows, thus maintaining the reciprocity of relative motion we began with.

$$\left. \begin{aligned} \frac{\cos \varphi}{\cos(\theta + \varphi)} - \frac{v}{c} &= \frac{\cos \varphi}{\cos \theta} - \left(1 - \frac{1}{(a \cos \varphi)^2} \right) \frac{c (\cos \varphi)^2}{v \cos \theta \cos(\theta + \varphi)} \\ \frac{\cos \varphi}{\cos \theta} + \frac{v}{c} &= \frac{\cos \varphi}{\cos(\theta + \varphi)} + \left(1 - \frac{1}{(a \cos \varphi)^2} \right) \frac{c (\cos \varphi)^2}{v \cos \theta \cos(\theta + \varphi)} \end{aligned} \right\}$$

This, on rearranging, results into the following

$$1 - \frac{1}{(a \cos \varphi)^2} = \frac{v}{c \cos \varphi} \left[\frac{v \cos(\theta + \varphi) \cos \theta}{c \cos \varphi} - \cos \theta + \cos(\theta + \varphi) \right]$$

Or,

$$a = \frac{1}{\cos \varphi \sqrt{\left(1 - \frac{v \cos(\theta + \varphi)}{c \cos \varphi} \right) \left(1 + \frac{v \cos \theta}{c \cos \varphi} \right)}}$$

Having worked out the value of a, the relations (1) and (2) above become the transformation relations for distance and time of an event, from one frame to the other.

It is reiterated the **above relations have come out while maintaining reciprocity of velocity and constancy of speed of light in the directions of OA, as well as of O'A which is oblique to the former**. The transformations change the directions of displacement and velocity, and the time transforms with the entire distance, not any of its components.

Thus, no factors, which may cause a rotation, are left.

Addition of Velocities

Moving further, one may work out the relations for addition of velocities also.

It is reiterated that, similar to the current practice for the co-directionally moving inertial frames, the transformation relations derived here too for light, have been used for arbitrary events also, whose distance-time ratio is not equal to c .

Therefore, in working ahead, the ratios r/t and r'/t' are taken as not c but u and u' respectively. To keep the errors occurring on this account to the minimum, however, the velocities considered for addition are chosen very close to speed of light c , as already done in the previous section 1.

With the known expression of a , we introduce a new constant A for brevity as follows.

$$A = 1 - \frac{1}{(a \cos \varphi)^2} = 1 - \left(1 - \frac{v \cos(\theta + \varphi)}{c \cos \varphi}\right) \left(1 + \frac{v \cos \theta}{c \cos \varphi}\right)$$

On substitution of expressions of r and t from equations (1) and (2), we get

$$u = \frac{r}{t} = \frac{a(r' \cos \varphi + vt' \cos \theta)}{a \cos \varphi \frac{\cos \theta}{\cos(\theta + \varphi)} \left[t' + A \frac{r' \cos \varphi}{v \cos \theta} \right]}$$

On substitution of $r'=u't'$, the expression reduces to

$$u = v \left[\frac{1 + \frac{u' \cos \varphi}{v \cos \theta}}{1 + A \frac{u' \cos \varphi}{v \cos \theta}} \right] \frac{\cos(\theta + \varphi)}{\cos \varphi} \dots \dots (3)$$

The relation (3) is used in the exercise below.

Further, on dividing the two relations of (2), we get

$$\frac{t'}{t} = \frac{a \cos \varphi \frac{\cos(\theta + \varphi)}{\cos \theta} \left[t - A \frac{r \cos \varphi}{v \cos(\theta + \varphi)} \right]}{a \cos \varphi \frac{\cos \theta}{\cos(\theta + \varphi)} \left[t' + A \frac{r' \cos \varphi}{v \cos \theta} \right]} = \frac{t' (\cos(\theta + \varphi))^2 \left[1 - A \frac{u \cos \varphi}{v \cos(\theta + \varphi)} \right]}{t' \left(\frac{\cos \theta}{\cos \varphi} \right)^2 \left[1 + A \frac{u' \cos \varphi}{v \cos \theta} \right]}$$

Or,

$$t = \left[\frac{\cos \theta}{\cos(\theta + \varphi)} \sqrt{\frac{1 + A \frac{u' \cos \varphi}{v \cos \theta}}{1 - A \frac{u \cos \varphi}{v \cos(\theta + \varphi)}}} \right] t'$$

On substituting the expression of u from (3), the above relation becomes

$$t = \left[\frac{\cos \theta}{\cos(\theta + \varphi)} \sqrt{\frac{1 + A \frac{u' \cos \varphi}{v \cos \theta}}{1 - A \left[\frac{1 + \frac{u' \cos \varphi}{v \cos \theta}}{1 + A \frac{u' \cos \varphi}{v \cos \theta}} \right]}} \right] t' \dots \dots (4)$$

Reworking of Example of Sr. No. 1

It is pointed out that in all cases of velocity addition below, the setup used in the derivation is strictly followed. Therefore, the first frame/observer, which seeks addition of the velocities of the second and the third frames, in order to know the relative velocity

of the third frame with respect to it, is considered stationary and also as the origin.

In other words, the setup of Figure 2 (A) has to be followed, where the stationary observer at O seeks to know the relative velocity of A with respect to it, with the known velocity (u') of A with respect to O', and that (v) of O' with respect to O.

i) Relative Velocity of C with respect to A, Via B (Figure 1)
In this case, $v=v_1=0.99997$, $u'=v_2=0.99995$ and $u=v_{21}$ in the direction of AC, whose magnitude is to be found out.

Also, the exterior angle between AB and CB = $(\theta + \varphi) = 90^\circ$. However, to avoid any possible division by zero, we will take this angle as 89.99999° . Further, with the lengths of AB and CB being in proportion of v_1 and v_2 respectively, we may work out the interior angles by trigonometry, and those would be as follows.

$$\angle CAB = \theta = 44.99942^\circ \text{ and } \angle ACB = \varphi = 45.00057^\circ$$

On substitution of these values in the above velocity addition formula (3), one gets
 $u=v_{21}=0.009776163$ in the direction of AC.

The resultant velocity falling to such a level should not surprise, as the corresponding time (at A) has dilated to **144.65541** times of that at B, as per relation (4). Multiplication of the resultant velocity with its time, gives the resultant distance as **1.41417** which matches with the non-relativistic vector addition of velocities i.e.

$$\sqrt{(0.99997)^2 + (0.99995)^2} = 1.41417.$$

ii). Relative Velocity of A with respect to C, Via B (Figure 1)
In this case, $v=v_2=0.99995$, $u'=v_1=0.99997$ and $u=v_{12}$ in the direction of CA, whose magnitude is to be found out.

The exterior angle between CB and AB = $(\theta + \varphi) = 90^\circ$. However, maintaining conformity with case (i), we will take this angle also as 89.99999° . Similar to case (i), the interior angles would be as follows.

$$\angle ACB = \theta = 45.00057^\circ \text{ and } \angle BAC = \varphi = 44.99942^\circ$$

As expected, the angle values have got interchanged with each other.

On substitution of these values in the above velocity addition formula (3), one gets

$$u=v_{12}=0.0161881 \text{ in the direction of CA.}$$

As regards its low value comparatively, the reason is similar to case (i) i.e. the corresponding time (at C) has dilated to **87.3585** times of that at B, as per relation (4). Multiplication of the resultant velocity with its time, gives the resultant distance as **1.41417** which matches with the non-relativistic vector addition of velocities i.e.

$$\sqrt{(0.99997)^2 + (0.99995)^2} = 1.41417, \text{ and also with case (i).}$$

Thus, the reciprocity of velocity direction is maintained along AC, and the Wigner-Thomas Rotation goes away. However, the magnitude of the resultant relative relativistic velocity is not the same in both the directions, as the relations used are derived for events of light but applied on moving observers.

The Message

The disappearance of the rotation on recognition of obliquely moving inertial frames within its definition, as shown above, demands that if the Wigner-Thomas rotation has to exist, the specification of inertial frames in the second postulate needs to be joined by a rider to limit them to only those moving collinearly with the line from origin to the event.

Conclusion

The Wigner-Thomas Rotation occurs due to rigidly casting the theory of Relativity to only those inertial frames that moved collinearly with the direction of event from origin, though the postulate did not impose any such restriction. Such a structure is bound to throw up inconsistencies in non-collinear boosts. This has amply been demonstrated with an example. If the scope of inertial frames is expanded to include even those moving in any arbitrary direction, the Wigner-Thomas rotation goes away, as demonstrated in the same example. Thus, a clear-cut supplementation of the second postulate is required, specifying the direction of inertial frames, so as to save the sanctity of Wigner-Thomas rotation.

Further, the Lorentz Transformation have been derived only for events of light, and have not been and cannot be derived for other events, like those of the moving frames considered in working out of the Wigner-Thomas rotation. For this reason, the presented example selects the two non-collinear velocities very close to speed of light in vacuum c , as the degree of correctness depends on how close the distance-time ratio of such an event was with c . Despite the errors, another supplementation is required in the form of a third postulate, validating the use of Lorentz Transformation for events that were not of light e.g. those of fast-moving particles, etc.

References

1. Kane OD, Matt V (2011) Elementary analysis of the special relativistic combination of velocities, Wigner rotation and Thomas precession. European Journal of Physics 31: 1033-1047.
2. Einstein A (1916) Relativity: The Special and The General Theory.

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