ISSN: 2754-4753

Journal of Physics & Optics Sciences

SCIENTIFIC Research and Community

Research Article

Open Access

The Cosmological-Constant Problem within Four-Dimensional Einstein-Gauss-Bonnet (EGB) Gravity

Manuel Malaver[1*] and Hamed Daei Kasmaei

Department of Basic Sciences, Maritime University of the Caribbean, Catia la Mar, Venezuela

Department of Applied Mathematics and Computer Science, Islamic Azad University, Central Tehran Branch, Tehran, Iran

ABSTRACT

We present a comprehensive computational and conceptual scheme for addressing the cosmological-constant problem within four-dimensional Einstein-Gauss-Bonnet (EGB) gravity. Rather than relying primarily on scalar-tensor potential minimization, the present framework investigates whether the topological character of the Gauss-Bonnet density can motivate a discrete spectrum for the associated coupling. For a compact, oriented, boundaryless Euclidean four-manifold, the Gauss-Bonnet integral is proportional to the Euler characteristic. We formulate the proposed quantization as $\alpha = n/(32\pi^2)$, $n \in Z$, and investigate its consequences for vacuum selection, black-hole thermodynamics, quasinormal modes, and cosmology.The analysis is formulated using a scalar-Gauss-Bonnet/Horndeski-type completion. A crucial distinction is maintained throughout: a constant Gauss-Bonnet coefficient is topological in four-dimensional bulk dynamics, whereas a dynamical coupling $f(\varphi)$ produces local modifications. We review the one-loop trace anomaly, heat-kernel expansion, anomaly-induced effective action, and de Sitter scaling. We then construct a reproducible computational pipeline based on symbolic curvature calculations, numerical parameter scans, black-hole perturbation theory, and cosmological integration. The principal conclusion is that topological quantization is a potentially useful selection hypothesis, but it is not by itself a demonstrated solution of the cosmological-constant problem. The Chern-Gauss-Bonnet theorem quantizes the integrated geometric density for appropriate manifolds; it does not independently quantize α. Moreover, the trace anomaly alone naturally produces an H^4 scaling and does not automatically select the observed Hubble scale. These points define the precise theoretical and computational tasks required for a publishable test of the framework.

***Corresponding author**

Manuel Malaver, Department of Basic Sciences, Maritime University of the Caribbean, Catia la Mar, Venezuela.

Received: September 14, 2026; **Accepted:** September 23, 2026; **Published:** September 30, 2026

Keywords: Einstein-Gauss-Bonnet Gravity, Cosmological Constant, Vacuum Energy, Trace Anomaly, Euler Characteristic, Topological Quantization, Horndeski Theory, Black Holes, Quasinormal Modes, Cosmology

Introduction: A Different Scheme

The Cosmological-Constant Problem

The cosmological constant is one of the most persistent tensions between gravitational theory and quantum field theory [1-3].In general relativity, a Lorentz-invariant vacuum stress tensor is equivalent to a cosmological term [4-7]. Quantum field theory, however, produces vacuum contributions that depend on the renormalization prescription and on matter thresholds [8,9]. The central naturalness problem is that the observed late-time vacuum scale is extremely small compared with the characteristic scales of known particle physics and gravity [10,11] .

The familiar estimate of a Planck-scale zero-point contribution exceeding the observed dark-energy density by roughly 120 orders of magnitude is best regarded as a diagnostic of the hierarchy problem rather than as a literal unrenormalized prediction [12]. After renormalization, the deeper issue is radiative stability: changes in particle masses and quantum thresholds generically induce changes in the renormalized vacuum energy that must be compensated if the observed value is to remain tiny [13-15].

The four-dimensional Gauss-Bonnet density is

$$G = R^2 - 4\,R_\{\mu\nu\}R\^\{\mu\nu\} + R_\{\mu\nu\rho\sigma\}R\^\{\mu\nu\rho\sigma\}.$$

For a compact, oriented, boundaryless Euclidean four-manifold M_4, the Chern-Gauss-Bonnet theorem gives

$$\int_\{M_4\} d^4x \sqrt{g}\, G = 32\pi^2\, \chi(M_4).$$

We therefore investigate the additional physical hypothesis

$$\alpha_n = n/(32\pi^2), \quad n \in \mathbb{Z}.$$

Under this prescription the topological phase takes the form $\exp(in\chi)$. The discrete spectrum is an assumption of the proposed model, not a direct consequence of the Euler theorem. A fundamental derivation of this spectrum remains an open question. This paper is organized as follows: The Section 2 develops the classical scalar-Gauss-Bonnet formulation. Section 3 reviews the quantum anomaly. Section 4 formulates the topological sector. Section 5 treats black-hole observables. Section 6 develops cosmology. Section 7 specifies independent numerical verification. Sections 8 and 9 discuss limitations and conclusions, followed by detailed appendices.

2. Classical Action and Field Equations

2.1 Regularized Scalar-Gauss-Bonnet Action

$$S = \int d^4x \sqrt{-g}\,[\,(M_{Pl}^2/2)R - (1/2)(\nabla\varphi)^2 - V(\varphi) + (\alpha/4)\,f(\varphi)\,G\,].$$

The normalization of α is conventional. Throughout this manuscript the same normalization is used consistently when discussing the proposed quantization. In four dimensions a constant f makes the Gauss-Bonnet term topological, while f(φ) varying in spacetime produces a genuine scalar-curvature interaction.

2.2 Lanczos Tensor and Topological Limit

$$H_{\mu\nu}=2RR_{\mu\nu}-4R_{\mu\alpha}R_{\nu}{}^{\alpha}-4R_{\mu\alpha\nu\beta}R^{\alpha\beta}+2R_{\mu}{}^{\{\alpha\beta\gamma\}}R_{\nu\alpha\beta\gamma\}}-(1/2)g_{\mu\nu}G.$$

For a constant coupling in exactly four dimensions the corresponding bulk metric variation vanishes identically. This is a topological identity and is not equivalent to saying that G itself vanishes pointwise.

2.3 Field Equations

$$M_{Pl}^2 G_{\mu\nu}=T_{\mu\nu}^{(\varphi)}-(\alpha/4)\mathcal{K}_{\mu\nu}[f].$$

$$T_{\mu\nu}^{(\varphi)}=\nabla_\mu\varphi\nabla_\nu\varphi-(1/2)g_{\mu\nu}(\nabla\varphi)^2-g_{\mu\nu}V(\varphi).$$

$$\Box\varphi-V_{,\varphi}+(\alpha/4)f_{,\varphi}G=0.$$

The tensor $\mathcal{K}_{\mu\nu}[f]$ contains derivatives of f contracted with curvature tensors. Its exact coefficient structure follows from the chosen action normalization.

2.4 Parametric Analysis

Case	Physical role	Expected behavior
f=constant	Topological 4D limit	No local GB correction
f=φ	Linear scalar coupling	Scalar sourced by G
f=e^{λφ}	Dilatonic benchmark	λ controls coupling
f=φ²	Nonlinear benchmark	Modified scalarization
α=n/(32π²)	Proposed discrete spectrum	Integer parameter n

3. Quantum Vacuum and Trace Anomaly

3.1 One-Loop Effective Action

$$K(x,x;\tau) \sim (4\pi\tau)^{-2}\,\Sigma_{k=0}^{\infty} a_k(x)\tau^k.$$

$$\Gamma_1 = (1/2)\mathrm{Tr}\ln\Delta = -(1/2)\int_0^{\infty}(d\tau/\tau)\mathrm{Tr}\exp(-\tau\Delta).$$

In four dimensions the logarithmic ultraviolet divergence is governed by a_2. Its explicit form depends on the kinetic operator, the scalar curvature coupling, field spin, and total-derivative conventions. For this reason the anomaly coefficients should always be quoted together with their normalization.

3.2 Trace Anomaly

$$\langle T^{\mu}{}_{\mu}\rangle=(1/16\pi^2)[c\,C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma}-aG+b\Box R].$$

For one conformally coupled real scalar in a common convention, a=1/360 and c=1/120. The coefficient b multiplying □R is scheme dependent because it can be shifted by a local R^2 counterterm. The type-A Euler coefficient a is the quantity most directly relevant to the present topological discussion.

3.3 Riegert Effective Action

$$\Delta_4=\Box^2+2R^{\mu\nu}\nabla_\mu\nabla_\nu-(2/3)R\Box+(1/3)(\nabla^\mu R)\nabla_\mu.$$

The anomaly can be integrated to a nonlocal effective action involving Δ_4^{-1}. On conformally flat de Sitter space the Weyl-squared contribution vanishes and the Euler-density sector controls the anomaly scaling.

3.4 de Sitter Scaling

$R=12H^2$, $G=24H^4$, $C^2=0$.

Thus the anomaly scales as H^4. Equating an anomaly-induced contribution to the gravitational vacuum density proportional to $M_{Pl}^2H^2$ produces a characteristic high-curvature scale, up to convention-dependent coefficients. This demonstrates why the anomaly alone does not naturally generate the observed tiny late-time Hubble scale.

4. Topological Quantization Scheme

4.1 Euclidean Topological Phase

$$S_{GB}^{E}=\alpha\int_{M_4}\sqrt{g}\,G\,d^4x=32\pi^2\alpha\chi(M_4).$$

$$\exp(iS_{GB})=\exp(i32\pi^2\alpha\chi).$$

$$\alpha_n=n/(32\pi^2),\quad n\in\mathbb{Z}\ \Rightarrow\ \exp(iS_{GB})=\exp(in\chi).$$

The mathematical theorem establishes the integer χ for suitable manifolds. The second step, quantizing α, is the proposed physical postulate. A complete quantum-gravity derivation would need to explain why only these values are allowed and why the condition is stable under renormalization.

4.2 Boundaries and Noncompact Spacetimes

The simple Chern-Gauss-Bonnet relation assumes compactness, orientation, and no boundary. Black-hole and cosmological spacetimes are commonly noncompact or possess effective boundaries. The appropriate boundary terms must therefore be included before assigning a topological phase. This is essential for a rigorous implementation.

4.3 Vacuum-Energy Scaling

$$\rho_{top}\propto\alpha\langle G\rangle=[n/(32\pi^2)]\langle G\rangle.$$

$$\text{For de Sitter: } \rho_{top}\propto(3n/4\pi^2)H^4.$$

The proportionality coefficient is dependent on the definition of the effective stress tensor and action normalization. Therefore this relation is a scaling statement until a complete renormalized effective action is fixed.

5. Black-Hole Thermodynamics and Quasinormal Modes

5.1 Static Spherically Symmetric Geometry

$$ds^2=-e^{2\Phi(r)}dt^2+e^{2\Psi(r)}dr^2+r^2d\Omega^2.$$

$$F(r)=1-2GM/r-\Lambda r^2/3.$$

For constant f and no scalar dynamics, the local equations reduce to the corresponding Einstein solution. Dynamical scalar coupling produces model-dependent corrections.

5.2 Wald Entropy

$$S_{Wald}=A/(4G)+2\pi\alpha f(\varphi_h).$$

$$\Delta S_{GB}=2\pi\alpha_n f(\varphi_h)=[n/(16\pi)]f(\varphi_h).$$

The entropy shift can survive as a global/topological contribution even when the local four-dimensional bulk equations are unchanged.

5.3 Quasinormal Modes

$$d^2\Psi/dr_^{*2}+[\omega^2-V(r)]\Psi=0.$$

$$V(r)=V_0(r)+\alpha\delta V(r), \quad \omega=\omega\hat{}(0)+\alpha\delta\omega\hat{}(1)+O(\alpha^2).$$

A Schwarzschild fundamental l=2 gravitational mode is approximately Mω≈0.3737−0.0890i, depending on convention. In the proposed model the correction must be obtained by solving the actual perturbation equations; a universal 10^{-3} shift cannot be asserted without calculation.

5.4 Parametric Analysis

Parameter	Observable	Leading scaling	Method
n	Entropy	ΔS∝nf(φh)	Horizon evaluation
n	QNM	δω∝n (small α)	WKB/continued fraction
M	Frequency	ω∝1/M	Dimensionless Mω
f(φh)	Entropy	Linear	Scalar background
α	Metric/scalar	Linear at small α	ODE perturbation

6. Cosmological Sector

6.1 FLRW Geometry

$$ds^2=-dt^2+a(t)^2(dx^2+dy^2+dz^2).$$

$$H=\dot{a}/a, \quad G=24H^2(H^2+\dot{H}).$$

Only for exact de Sitter, $\dot{H}=0$, does G reduce to $24H^4$.

6.2 Modified Friedmann Equation

For a dynamical f(φ), the background equation contains terms involving $H^3\dot{f}$ in addition to scalar kinetic and potential energy. Schematically,

$$3M_Pl^2H^2=\rho_m+\rho_r+(1/2)\dot{\varphi}^2+V(\varphi)+\mathcal{C}\alpha H^3\dot{f}(\varphi).$$

The coefficient $\mathcal{C}$ depends on the precise normalization. For constant f in four dimensions, the local classical Gauss-Bonnet contribution disappears. Thus an αH^4 correction should only be used when it has been explicitly derived from a quantum/effective sector.

6.3 Scaling Estimate

$$|\Delta H^2|/H^2 \sim |\alpha|H^2/M_Pl^2.$$

For $\alpha=n/(32\pi^2)$, order-unity n gives an extremely small late-time correction. High-curvature environments can therefore be more promising for local tests.

6.4 Observational Strategy

A defensible constraint requires a model-specific likelihood. Relevant datasets include expansion and growth measurements, CMB observations, gravitational-wave propagation and ringdown, and PTA stochastic backgrounds. The appropriate workflow is to calculate the complete map n → observables and then perform joint inference.

6.5 Parametric Cosmology

- H(z): integrate the background equations for each integer n.
- q(z): calculate $q=-1-\dot{H}/H^2$.
- Growth: integrate the linear matter-perturbation equations.
- Tensor sector: calculate speed, friction and dispersion modifications.

7. Numerical Cross-Checks

7.1 Computational Procedure

1. Define coordinates and metric gμν.
2. Compute inverse metric and determinant.
3. Construct Christoffel symbols.
4. Construct Riemann, Ricci and scalar curvature.
5. Evaluate R^2, $Ricci^2$, $Riemann^2$ and G.
6. Verify the four-dimensional Lanczos identity.
7. Evaluate C^2 and the trace anomaly.
8. Substitute Minkowski, de Sitter and Schwarzschild metrics.
9. Solve background ODEs and scan n.
10. Calculate black-hole perturbations and QNMs.
11. Integrate cosmological background and perturbations.

7.2 Essential Verification Results

Background	Expected invariant check
Minkowski	R=0, Ricci=0, G=0
de Sitter	$R=12H^2$, $G=24H^4$, $C^2=0$
Schwarzschild	R=0, Ricci=0, but $G=Riemann^2\neq0$
S^4	$\int\sqrt{g}G=64\pi^2$, $\chi=2$

The Schwarzschild check is particularly important: although the Ricci curvature vanishes, the Riemann tensor does not. Therefore the Gauss-Bonnet density does not vanish pointwise. This corrects a common but consequential misconception in simplified computational treatments.

7.3 Python/SymPy and Maple Workflow

The independent implementations should use separate code paths, exact simplification where possible, numerical convergence tests, and exported machine-readable results. Maple Physics can be used for tensor verification; Python/SymPy can independently reconstruct the same invariants and serve as a cross-check.

8. Discussion

8.1 Comparison with Continuous Vacuum Selection

The proposed framework changes the mathematical character of the selection problem by introducing a discrete label. However, discreteness is not synonymous with naturalness. A successful solution must demonstrate that the spectrum contains an allowed sector reproducing observation and that radiative corrections preserve the spectrum.

8.2 Topology versus Local Dynamics

A constant four-dimensional Gauss-Bonnet term is topological in the local bulk equations. Observable local departures from general relativity therefore require a dynamical coupling, boundaries, quantum corrections, or an extension of the strict four-dimensional classical setup.

8.3 Limitations and Open Questions

- Derivation of α quantization from a fundamental theory.
- Correct treatment of boundary terms and noncompact manifolds.
- Renormalization and dimensional normalization of α.

- Complete field-content dependence of the anomaly.
- Ghost and perturbative stability of each dynamical realization.
- Explicit calculation of QNM shifts.
- Full MCMC constraints rather than order-of-magnitude bounds.

8.4 Falsifiability Program

The model becomes directly testable when each integer n is mapped to a predicted set of observables and compared with a joint likelihood. Exclusion of every nonzero n in a defined model class would falsify that realization.

9. Conclusions

- The Chern-Gauss-Bonnet theorem establishes the topological nature of the integrated Euler density on suitable manifolds.
- The proposed $\alpha=n/(32\pi^2)$ spectrum is an additional physical hypothesis.
- Constant-coupling four-dimensional EGB has no local bulk modification of Einstein gravity.
- The trace anomaly contains the Euler density but does not naturally select the observed late-time scale.
- Wald entropy can retain a Gauss-Bonnet/topological contribution.
- QNM and cosmological corrections require explicit model-specific calculations.
- Independent Maple and Python/SymPy implementations provide a reproducible verification framework.

The framework is therefore best regarded as a concrete research program for discrete topological vacuum selection rather than as a completed proof of a solution to the cosmological-constant problem. The decisive next steps are a fundamental derivation of the quantization condition, consistent boundary treatment, complete renormalization, perturbative stability analysis, and joint observational inference.

Appendix A: Detailed Computational Algorithm

- Metric and coordinate definition
- Christoffel-symbol construction
- Riemann/Ricci/Ricci-scalar contractions
- Gauss-Bonnet and Weyl invariants
- Lanczos identity verification
- Heat-kernel/anomaly implementation
- Discrete α scan
- Black-hole ODE integration
- QNM WKB and continued-fraction validation
- Cosmological background and perturbation integration
- Convergence and error analysis

Appendix B: Python/SymPy Notebook Specification

The notebook should be modularized into geometry.py, anomaly.py, blackhole.py, qnm.py, cosmology.py, scan.py and validation.py or equivalent notebook sections. Each calculation should record units, numerical tolerances, parameter values, and convergence diagnostics. Symbolic identities should be tested before numerical evaluation.

Appendix C: Parametric Tables

n	α/α_1	$\Delta S/f(\varphi h)$	Relative scaling
-5	-5	-0.0995	-5
-2	-2	-0.0398	-2
-1	-1	-0.0199	-1
0	0	0	0
1	1	0.0199	1
2	2	0.0398	2
3	3	0.0597	3
4	4	0.0796	4
5	5	0.0995	5

These values are analytical scaling values, not a substitute for a computed QNM catalogue.

Appendix D: Detailed FLRW Derivation

For spatially flat FLRW, substitution into the Euler density gives

$$G=24H^2(H^2+\dot{H}).$$

For de Sitter, $\dot{H}=0$ and therefore $G=24H^4$. Numerical cosmology must retain $\dot{H}$ unless the solution is explicitly restricted to de Sitter.

Appendix E: Wald Entropy Derivation

$$S_{Wald}=-2\pi\oint_\Sigma (\partial L/\partial R_{\mu\nu\rho\sigma})\,\varepsilon_{\mu\nu}\varepsilon_{\rho\sigma}\,dA.$$

$$S_{Wald}=A/(4G)+2\pi\alpha f(\varphi_h).$$

Comparison between publications requires matching the normalization of the Gauss-Bonnet term and the definition of α.

Appendix F: Trace-Anomaly Derivation

$$\langle T^\mu_\mu\rangle=(1/16\pi^2)[cC^2-aG+b\Box R].$$

The □R term is scheme dependent; the Euler coefficient a and Weyl coefficient c are the central invariant data for the present discussion.

Appendix G: Anomaly-Supported de Sitter Scaling

$$M_{Pl}^2H^2 \sim aH^4/(16\pi^2).$$

The nonzero anomaly-supported branch is therefore controlled by a Planck-scale combination of the anomaly coefficient, rather than by H_0 without an additional small parameter.

Appendix H: QNM Computational Procedure

- Solve the exact background for each α_n.
- Linearize metric/scalar equations.
- Separate parity and angular sectors.
- Construct the effective potential.
- Perform high-order WKB calculations.
- Validate selected cases by continued fractions or time-domain evolution.
- Estimate numerical errors by resolution and method comparisons.

Appendix I: Cosmological Constraint Procedure

For each n, integrate the background from an early redshift to the present, normalize H_0 consistently, calculate distances and growth observables, and construct a joint likelihood including nuisance parameters and covariance matrices. Model comparison should be based on a stated statistical criterion.

Appendix J. Euler Characteristics of Common Four-Manifolds

Manifold	χ	$\int\sqrt{g}\,G$
S^4	2	$64\pi^2$
T^4	0	0
K3	24	$768\pi^2$
CP^2	3	$96\pi^2$
$S^2\times S^2$	4	$128\pi^2$

Appendix K: Reproducibility and Data Management

The final research repository should contain the manuscript source, Maple worksheet, Python/SymPy notebook, parameter files, QNM tables, cosmological chains, figures, and scripts. Every numerical result should include units, tolerances, software versions, and convergence tests.

Appendix L: Validation Checklist

- Check dimensions of every action term.
- Verify α normalization.
- Include boundary terms where required.
- Verify $G=24H^4$ only for de Sitter.
- Verify Schwarzschild G is nonzero.
- Verify anomaly convention.
- Recompute QNM shifts from perturbation equations.
- Perform numerical convergence tests.

Perform joint observational inference before quoting bounds.

References

1. Glavan D, Lin C (2020) Einstein-Gauss-Bonnet Gravity in Four-Dimensional Spacetime. Physical Review Letters 124: 081301.
2. Fernandes PGS, Carrilho P, Clifton T, Mulryne DJ (2022) The 4D Einstein-Gauss-Bonnet Theory of Gravity: A Review. Class Quantum Grav 39: 063001.
3. Horndeski GW (1974) Second-order scalar-tensor field equations in a four-dimensional space. International Journal of Theoretical Physics 10: 363-384.
4. Lanczos C (1938) A remarkable property of the Riemann-Christoffel tensor in four dimensions. Annals of Mathematics 39: 842-850.
5. Wald RM (1993) Black hole entropy is the Noether charge. Physical Review D 48: R3427-R3431.
6. Riegert RJ (1984) A nonlocal action for the trace anomaly. Physics Letters B 134: 56-60.
7. Starobinsky AA (1980) A new type of isotropic cosmological models without singularity. Physics Letters B 91: 99-102.
8. Duff MJ (1977) Observations on conformal anomalies. Nuclear Physics B 125: 334-348.
9. Birrell ND, Davies PCW (1982) Quantum Fields in Curved Space. Cambridge University Press.
10. Seeley RT (1967) Complex powers of an elliptic operator. Proceedings of Symposia in Pure Mathematics 10: 288-307.
11. DeWitt BS (1965) Dynamical Theory of Groups and Fields. Gordon and Breach.
12. Chern SS (1944) A simple intrinsic proof of the Gauss-Bonnet formula for closed Riemannian manifolds. Annals of Mathematics 45: 747-752.
13. Euclid Collaboration (2025-2026) Euclid mission and data-release papers.
14. LIGO Scientific Collaboration, Virgo Collaboration, KAGRA Collaboration. Gravitational-wave catalogues and O4/O5-era results.
15. NANOGrav Collaboration and international PTA collaborations. Recent stochastic gravitational-wave-background results.

Copyright: ©2026 Manuel Malaver. This is an open-access article distributed under the terms of the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original author and source are credited.